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Themistocles M. Rassias *Editor*

Handbook of Functional Equations

Functional Inequalities

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Handbook of Functional Equations

Functional Inequalities

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Preface

Handbook of Functional Equations: Functional Inequalities consists of 20 chapters written by eminent scientists from the international mathematical community who present important research works in the field of mathematical analysis and related subjects with emphasis to functional equations and functional inequalities. As Richard Bellman has so elegantly stated at the second international conference on general inequalities (Oberwolfach 1978), “There are three reasons for the study of inequalities: practical, theoretical, and aesthetic.” On the aesthetic aspects, he said, “As has been pointed out, beauty is in the eye of the beholder. However, it is generally agreed that certain pieces of music, art, or mathematics are beautiful. There is an elegance to inequalities that makes them very attractive.” The chapters of this book focus mainly on both old and recent developments on approximate homomorphisms, on a relation between the Hardy–Hilbert and the Gabriel inequality, generalized Hardy–Hilbert type inequalities on multiple weighted Orlicz spaces, half-discrete Hilbert-type inequalities, affine mappings, contractive operators, multiplicative Ostrowski and trapezoid inequalities, Ostrowski type inequalities for the Riemann–Stieltjes integral, means and related functional inequalities, weighted Gini means, controlled additive relations, Szaz–Mirakyan operators, extremal problems in polynomials and entire functions, applications of functional equations to Dirichlet problem for doubly connected domains, nonlinear elliptic problems depending on parameters, strongly convex functions, as well as applications to some new algorithms for solving general equilibrium problems, inequalities for the Fisher’s information measures, financial networks, mathematical models of mechanical fields in media with inclusions and holes.

It is our pleasure to express our thanks to all the contributors of chapters in this book. I would like to thank Dr. Michael Batsyn and Dr. Dimitrios Dragatogiannis for their invaluable help during the preparation of this publication. Last but not least, I would like to acknowledge the superb assistance that the staff of Springer has provided for the publication of this work.

Athens, Greece Themistocles M. Rassias

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On a Relation Between the Hardy–Hilbert and Gabriel Inequalities

Vandanjav Adiyasuren and Tserendorj Batbold

Abstract In this chapter, we establish some new generalizations of Azar's results, which are relations between the Hardy–Hilbert inequality and the Gabriel inequality. As an application, we obtain a sharper form of the general Hardy-Hilbert inequality. The integral analogues of our main results are also given. Some Gabriel-type inequalities are also considered.

Keywords The Hardy–Hilbert inequality · The Gabriel inequality · The Hölder inequality · Hardy's method

Mathematics Subject Classification (2000): *Primary 26D15, Secondary 05E05*

1 Introduction

The classical Hardy–Hilbert inequality asserts that if $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\pi/(\sin \pi/p)$ is the best possible. Its integral form reads as follows: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0, 0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(x)dx < \infty$, then

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\pi/(\sin\pi/p\ )$ is also the best possible (see e.g. [6]). These two inequalities are important in analysis and its applications. Although classical, they are still of interest to numerous authors, and during subsequent decades numerous generalizations and refinements appeared in the literature (see e.g. [3, 4, 6, 8, 7 10]).

Recently, Das and Sahoo [4], obtained the following discrete version of the Hardy–Hilbert inequality with conjugate parameters p and q, $p > 1$, as

$$\sum_{m=m_0}^\infty \sum_{n=n_0}^\infty \frac{a_m b_n}{(u(m)+v(n))^\lambda} < B(\phi_p,\phi_q) \left\{\sum_{m=m_0}^\infty [u(m)]^{p(1-\phi_q)-1}[u'(m)]^{1-p}a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=n_0}^\infty [v(n)]^{q(1-\phi_p)-1}[v'(n)]^{1-q}b_n^q\right\}^{\frac{1}{q}}, \tag{3}$$

where $a_m, b_n \geq 0$, $\phi_p + \phi_q = \lambda$, $u \in H_{m_0}(1-\phi_q)$, $v \in H_{n_0}(1-\phi_q)$, and the constant $B(\phi_p,\phi_q)$ (B is the usual Beta function) is the best possible. The set of function $H_{m_0}(r)$ is described in the following definition.

Definition 1 Let $r > 0$ and $m_0 \in \mathbb{N}$. We denote by $H_{m_0}(r)$ the set of all non-negative differentiable functions $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(a) u is strictly increasing in (m_0-1,∞).
(b) $u((m_0-1)+) = 0, u(\infty) = \infty$, and $\frac{u'(x)}{[u(x)]^r}$ is decreasing in (m_0-1,∞).

In 2009, Das and Sahoo [3], obtained the following integral version of the inequality (3):

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^\lambda}dxdy < B(\phi_p,\phi_q)\left\{\int_a^b [\varphi(x)]^{p(1-\phi_q)-1}[\varphi'(x)]^{1-p}f^p(x)dx\right\}^{\frac{1}{p}} \times \left\{\int_c^d [\psi(y)]^{q(1-\phi_p)-1}[\psi'(y)]^{1-q}g^q(y)dy\right\}^{\frac{1}{q}}, \tag{4}$$

where $f, g \geq 0$, $\phi_p + \phi_q = \lambda$, $\varphi(x)$ and $\psi(x)$ are differentiable strictly increasing functions on $(a,b)(-\infty \leq a < b \leq \infty)$ and $(c,d)(-\infty \leq c < d \leq \infty)$ respectively, such that $\varphi(a+) = \psi(c+) = 0$ and $\varphi(b-) = \psi(d-) = \infty$. In addition, the constant $B(\phi_p,\phi_q)$ is the best possible. It should be noticed here that we assume the convergence of series and integrals appearing in (3) and (4).

In particular, letting $u(x) \to \alpha u(x), v(x) \to \beta v(x)$, and $\varphi(x) \to \alpha\varphi(x), \psi(y) \to \beta\psi(y)$, $\phi_p = 1 - pA_2, \phi_q = 1 - qA_1, \lambda = \frac{s}{r}(\alpha,\beta > 0)$ in (3) and (4), we have

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(\alpha u(m)+\beta v(n))^{\frac{s}{r}}} < k(pA_2)\left\{\sum_{m=m_0}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=n_0}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}b_n^q\right\}^{\frac{1}{q}}, \quad (5)$$

and

$$\int_a^b\int_c^d\frac{f(x)g(y)}{(\alpha\varphi(m)+\beta\psi(n))^{\frac{s}{r}}}dxdy< k(pA_2)\left\{\int_a^b[\varphi(x)]^{-1+pqA_1}[\varphi'(x)]^{1-p}f^p(x)dx\right\}^{\frac{1}{p}} \times\left\{\int_c^d[\psi(y)]^{-1+pqA_2}[\psi'(y)]^{1-q}g^q(y)dy\right\}^{\frac{1}{q}}, \quad (6)$$

where $k(pA_2)=\frac{B(1-pA_2,1-qA_1)}{\alpha^{1-qA_1}\beta^{1-pA_2}}$, $A_1\in(\max\{\frac{1-\frac{s}{r}}{q},0\},\frac{1}{q})$, $A_2\in(\max\{\frac{1-\frac{s}{r}}{p},0\},\frac{1}{p})$ and $pA_2+qA_1=2-\frac{s}{r}$.

Further, we recall some Carlson-type inequalities. In 1935, Carlson [2], proved the following curious inequality: If $a_1,a_2,\ldots$ are real numbers, not all zero, then

$$\left(\sum_{n=1}^{\infty}a_n\right)^2 < \pi\left(\sum_{n=1}^{\infty}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}n^2a_n^2\right)^{\frac{1}{2}}, \quad (7)$$

where π is the best possible constant. In 1937, Gabriel [5] proved a more general version of the Carlson inequality. In his work, Gabriel used a method similar to Carlson's original proof. However, he mentioned that Hardy's method could also be used. If $p>1$, $a_n\geq 0$ and $0<\delta\leq p-1$, then

$$\left(\sum_{n=1}^{\infty}a_n\right)^p < \frac{2}{(2\delta)^{p-1}}\left(B\left(\frac{1}{2p-2},\frac{1}{2p-2}\right)\right)^{p-1} \times\left(\sum_{n=1}^{\infty}n^{p-1-\delta}a_n^p\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}n^{p-1+\delta}a_n^p\right)^{\frac{1}{2}}, \quad (8)$$

and the constant $\frac{2}{(2\delta)^{p-1}}\left(B\left(\frac{1}{2p-2},\frac{1}{2p-2}\right)\right)^{p-1}$ is the best possible. For more details about the Carlson-type inequalities the reader is referred to [9].

Recently, Azar [1] gave a new discrete inequality with conjugate parameters p and q, $p>1$, which is a relation between the Hardy–Hilbert inequality (1) and the Carlson inequality (7) as

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sigma_{m,n}\right)^2 < L\left\{\sum_{n=1}^{\infty} m^{-1+pqA_1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{-1+pqA_2}b_n^q\right\}^{\frac{1}{q}}$$
$$\times\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\sigma_{m,n}^2}{a_m b_n}\right\}^{pA_2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\sigma_{m,n}^2}{a_m b_n}\right\}^{qA_1}, \tag{9}$$

where $a_m, b_n, \sigma_{m,n} > 0$, $A_1 \in \left(0, \frac{1}{q}\right)$, $A_2 \in \left(0, \frac{1}{p}\right)$, $pA_2+qA_1 = 1$ and the constant $L = \frac{B(pA_2,1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}$ is the best possible.

In this chapter, we establish a new inequality with the best constant factor, which is a relation between the Hardy–Hilbert and the Gabriel inequalities. It is a generalization of Azar's result (9). We employ Hardy's method to prove our main results. As an application we obtain a sharper form of the general Hardy–Hilbert inequality. The integral analogues of our main results are also given and some Gabriel-type inequalities are also considered.

Throughout this chapter, all the functions are assumed to be non-negative and measurable. Also, all series and integrals are assumed to be convergent.

2 Main Results

In order to prove our results, we shall utilize the following simple property of the usual Beta function:

$$B(t+1,s) = B(s,t+1) = \frac{t}{s+t}B(s,t), \quad s,t > 0. \tag{10}$$

2.1 A New Discrete Inequality

Theorem 1 *Let* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, r > 1, \frac{1}{r}+\frac{1}{s} = 1$ *and* $m_0, n_0 \in \mathbb{N}$. *Suppose that* $A_1 \in (\max\{\frac{1-\frac{s}{r}}{q}, 0\}, \frac{1}{q})$, $A_2 \in (\max\{\frac{1-\frac{s}{r}}{p}, 0\}, \frac{1}{p})$, $pA_2 + qA_1 = 2 - \frac{s}{r} > 0$, $u \in H_{m_0}(qA_1)$ *and* $v \in H_{n_0}(pA_2)$. *If* $\{a_m\}, \{b_n\}$ *and* $\{\sigma_{m,n}\}$ *are positive sequences, then*

$$\left(\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\sigma_{m,n}\right)^r < C\left\{\sum_{m=m_0}^{\infty} w_1(m)a_m^p\right\}^{\frac{r}{ps}}\left\{\sum_{n=n_0}^{\infty} w_2(n)b_n^q\right\}^{\frac{r}{qs}}$$
$$\times\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}\right\}^{\frac{r(1-qA_1)}{s}}\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}\right\}^{\frac{r(1-pA_2)}{s}}, \tag{11}$$

where $w_1(x) = [u(x)]^{-1+pqA_1}[u'(x)]^{1-p}, w_2(x) = [v(x)]^{-1+pqA_2}[v'(x)]^{1-q}$. *In addition, the constant*

$$C = \frac{s[B(1-pA_2, 1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}}$$

is the best possible.

Proof Let $\alpha, \beta > 0$. Utilizing the Hölder inequality and then, applying (5), we have

$$\begin{aligned}
&\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\sigma_{m,n}\right\}^r \\
&= \left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\left(\frac{(a_m b_n)^{\frac{1}{s}}}{(\alpha u(m)+\beta v(n))^{\frac{1}{r}}}\right)\left(\frac{(\alpha u(m)+\beta v(n))^{\frac{1}{r}}}{(a_m b_n)^{\frac{1}{s}}}\sigma_{m,n}\right)\right\}^r \\
&\le \left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(\alpha u(m)+\beta v(n))^{\frac{s}{r}}}\right\}^{\frac{r}{s}}\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{\alpha u(m)+\beta v(n)}{(a_m b_n)^{\frac{r}{s}}}\sigma_{m,n}^r\right\} \\
&< \frac{[B(1-pA_2, 1-qA_1)]^{\frac{r}{s}}}{\alpha^{\frac{r(1-qA_1)}{s}}\beta^{\frac{r(1-pA_2)}{s}}}\left\{\sum_{m=m_0}^{\infty}w_1(m)a_m^p\right\}^{\frac{r}{ps}}\left\{\sum_{n=n_0}^{\infty}w_2(n)b_n^q\right\}^{\frac{r}{qs}} \\
&\quad\times\left\{\alpha\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}+\beta\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}\right\} \\
&< [B(1-pA_2, 1-qA_1)]^{\frac{r}{s}}\left\{\sum_{m=m_0}^{\infty}w_1(m)a_m^p\right\}^{\frac{r}{ps}}\left\{\sum_{n=n_0}^{\infty}w_2(n)b_n^q\right\}^{\frac{r}{qs}} \\
&\quad\times\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{r(1-pA_2)}{s}}\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}+\left(\frac{\beta}{\alpha}\right)^{\frac{r(1-qA_1)}{s}}\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}\right\}.
\end{aligned}$$

Now, set $S = \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}, T = \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}}, t = \frac{\alpha}{\beta}$ and consider the function $h(t) = t^{\frac{r(1-pA_2)}{s}}S + t^{\frac{r(qA_1-1)}{s}}T$. Since

$$h'(t) = \frac{r(1-pA_2)S}{s}t^{\frac{r(1-pA_2)}{s}-2}\left(t - \frac{(1-qA_1)T}{(1-pA_2)S}\right),$$

it follows that h attains its minimum for $t = \frac{(1-qA_1)T}{(1-pA_2)S}$. Thus, letting $\alpha = (1-qA_1)T$ and $\beta = (1-pA_2)S$, we obtain (11).

Now, in order to prove that C is the best constant, suppose that $\varepsilon > 0$ is sufficiently small, $\widetilde{a}_m = [u(m)]^{-qA_1-\frac{\varepsilon}{p}}u'(m), \widetilde{b}_n = [v(n)]^{-pA_2-\frac{\varepsilon}{q}}v'(n)(m \ge m_0, n \ge n_0)$, and $\widetilde{\sigma}_{m,n} = \frac{\widetilde{a}_m\widetilde{b}_n}{(u(m)+v(n))^{\frac{s}{r}}}$. Then, considering the integral sums, we have

$$
\begin{aligned}
\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} &= \int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}d[u(x)] < \sum_{m=m_0}^{\infty}[u(m)]^{-1-\varepsilon}u'(m)\\
&= \sum_{m=m_0}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p\\
&< [u(m_0)]^{-1-\varepsilon}u'(m_0) + \int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}d[u(x)]\\
&= [u(m_0)]^{-1-\varepsilon}u'(m_0) + \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}},
\end{aligned}
$$

and so $\sum_{m=m_0}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p = \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} + O(1)$. Similarly,

$$
\sum_{n=n_0}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}\widetilde{b}_n^q = \frac{1}{\varepsilon[v(n_0)]^{\varepsilon}} + O(1).
$$

In addition, substituting the above defined sequences $\widetilde{a}_m, \widetilde{b}_n$, and $\widetilde{\sigma}_{m,n}$ in the left-hand side of (11), we get the inequality

$$
\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(u(m)+v(n))^{\frac{s}{r}}}\\
&> \int_{m_0}^{\infty}[u(x)]^{-qA_1-\frac{\varepsilon}{p}}\left(\int_{n_0}^{\infty}\frac{[v(y)]^{-pA_2-\frac{\varepsilon}{q}}}{(u(x)+v(y))^{\frac{s}{r}}}v'(y)dy\right)u'(x)dx\\
&= \int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}\left(\int_{\frac{v(n_0)}{u(x)}}^{\infty}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{\frac{s}{r}}}dt\right)u'(x)dx\\
&= \int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}\left(\int_{0}^{\infty}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{\frac{s}{r}}}dt - \int_{0}^{\frac{v(n_0)}{u(x)}}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{\frac{s}{r}}}dt\right)u'(x)dx\\
&> \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}}B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right)\\
&- \int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}u'(x)\int_{0}^{\frac{v(n_0)}{u(x)}}t^{-pA_2-\frac{\varepsilon}{q}}dtdx\\
&= \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}}B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right)\\
&- \frac{1}{(1-pA_2-\frac{\varepsilon}{q})(1-pA_2+\frac{\varepsilon}{p})}\cdot\frac{[v(n_0)]^{1-pA_2-\frac{\varepsilon}{q}}}{[u(m_0)]^{1-pA_2+\frac{\varepsilon}{p}}}\\
&= \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}}B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right) - \bigcirc(1).
\end{aligned}
$$

In the same way, utilizing (10), we have

$$\begin{aligned}
\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\widetilde{\sigma}_{m,n}^{r}}{(a_m b_n)^{\frac{r}{s}}} &= \sum_{m=m_0}^{\infty}[u(m)]^{1-qA_1-\frac{\varepsilon}{p}}u'(m)\sum_{n=n_0}^{\infty}\frac{[v(n)]^{-pA_2-\frac{\varepsilon}{q}}v'(n)}{(u(m)+v(n))^{s}} \\
&< \sum_{m=m_0}^{\infty}[u(m)]^{1-qA_1-\frac{\varepsilon}{p}}u'(m)\int_0^{\infty}\frac{[v(x)]^{-pA_2-\frac{\varepsilon}{q}}v'(x)}{(u(m)+v(x))^{s}}dx \\
&= \sum_{m=m_0}^{\infty}[u(m)]^{-1-\varepsilon}u'(m)\int_0^{\infty}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{s}}dt \\
&= \frac{1+\varepsilon[u(m_0)]^{\varepsilon}O(1)}{\varepsilon[u(m_0)]^{\varepsilon}}B\left(s+pA_2+\frac{\varepsilon}{q}-1,1-pA_2-\frac{\varepsilon}{q}\right) \\
&= \frac{1+\varepsilon[u(m_0)]^{\varepsilon}O(1)}{\varepsilon[u(m_0)]^{\varepsilon}}B\left(2-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right) \\
&= \frac{1+\varepsilon[u(m_0)]^{\varepsilon}O(1)}{\varepsilon[u(m_0)]^{\varepsilon}}\cdot\frac{r(1-qA_1+\frac{\varepsilon}{q})}{s} \\
&\quad\times B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right),
\end{aligned}$$

and similarly,

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\widetilde{\sigma}_{m,n}^{r}}{(a_m b_n)^{\frac{r}{s}}} < \frac{1+\varepsilon[v(n_0)]^{\varepsilon}O(1)}{\varepsilon[v(n_0)]^{\varepsilon}}\cdot\frac{r(1-pA_2+\frac{\varepsilon}{p})}{s}B\left(1-pA_2+\frac{\varepsilon}{p},1-qA_1-\frac{\varepsilon}{p}\right).$$

If the constant factor C in (11) is not the best possible, then there exists a positive constant $\widetilde{C}$(with $\widetilde{C} < C$), such that (11) is still valid when replacing C by $\widetilde{C}$. In particular, utilizing the derived inequalities, we have

$$\begin{aligned}
&\left(\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}}B\left(1-qA_1-\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)-\bigcirc(1)\right)^{r} \\
&< \widetilde{C}\left\{\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}}+O(1)\right\}^{\frac{r}{ps}}\left\{\frac{1}{\varepsilon[v(n_0)]^{\varepsilon}}+O(1)\right\}^{\frac{r}{qs}} \\
&\quad\times\left\{\frac{1+\varepsilon[u(m_0)]^{\varepsilon}O(1)}{\varepsilon[u(m_0)]^{\varepsilon}}\cdot\frac{r(1-qA_1+\frac{\varepsilon}{q})}{s}B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)\right\}^{\frac{r(1-qA_1)}{s}} \\
&\quad\times\left\{\frac{1+\varepsilon[v(n_0)]^{\varepsilon}O(1)}{\varepsilon[v(n_0)]^{\varepsilon}}\cdot\frac{r(1-pA_2+\frac{\varepsilon}{p})}{s}B\left(1-pA_2+\frac{\varepsilon}{p},1-qA_1-\frac{\varepsilon}{p}\right)\right\}^{\frac{r(1-pA_2)}{s}}.
\end{aligned}$$

Multiplying the above inequality by ε^{r} and then, letting $\varepsilon\to 0^{+}$, it follows that

$$C=\frac{s[B(1-pA_2,1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}}\le\widetilde{C},$$

which contradicts with the fact that $\widetilde{C} < C$. Hence, the constant factor C in (11) is the best possible. This completes the proof.

Considering Theorem 1 with $\sigma_{m,n} = \frac{a_m b_n}{(u(m)+v(n))^{\frac{s}{r}}}$, $S = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{u(m) a_m b_n}{(u(m)+v(n))^s}$, $T = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{v(n) a_m b_n}{(u(m)+v(n))^s}$ and $S + T = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_m b_n}{(u(m)+v(n))^{\frac{s}{r}}}$, we obtain the following consequence:

Corollary 1 *Suppose the parameters p, q, r, s, A_1, A_2, and the functions $u, v : \mathbb{R}_+ \to \mathbb{R}$ are defined as in the statement of Theorem 1. If $\{a_m\}$, and $\{b_n\}$ are positive sequences, then the following inequality holds:*

$$\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_m b_n}{(u(m)+v(n))^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=m_0}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=n_0}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}}. \quad (12)$$

In addition, the constant factor

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot B(1 - pA_2, 1 - qA_1)$$

is the best possible and

$$R = \frac{\left(\frac{S}{1-qA_1}\right)^{\frac{r(1-qA_1)}{s}} \left(\frac{T}{1-pA_2}\right)^{\frac{r(1-pA_2)}{s}}}{S+T},$$

$w_1(x) = [u(x)]^{-1+pqA_1} [u'(x)]^{1-p}$, $w_2(x) = [v(x)]^{-1+pqA_2} [v'(x)]^{1-q}$.

In particular, (**I**) for $A, B, \alpha, \beta > 0$, setting $u(x) = Ax^{\alpha}, v(x) = Bx^{\beta}, m_0 = n_0 = 1$, we have the inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha} + Bn^{\beta})^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=1}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}},$$

where the constant

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot \frac{B(1 - pA_2, 1 - qA_1)}{A^{1-qA_1} B^{1-pA_2} \alpha^{\frac{1}{q}} \beta^{\frac{1}{q}}},$$

is the best possible and $w_1(m) = m^{p(\alpha q A_1 - \alpha + 1) - 1}$, $w_2(n) = n^{q(\beta p A_2 - \beta + 1) - 1}$.

(**II**) If $\alpha, \beta > 0$, putting $u(x) = \alpha \ln x, v(x) = \beta \ln x, m_0 = n_0 = 2$, we have

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\alpha \ln m + \beta \ln n)^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=2}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}},$$

where

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot \frac{B(1 - pA_2, 1 - qA_1)}{\alpha^{1-qA_1} \beta^{1-pA_2}},$$

is the best constant and $w_1(m) = (\ln m)^{-1+pqA_1} m^{p-1}, w_2(n) = (\ln n)^{-1+pqA_2} n^{q-1}$.

(**III**) For $\alpha, \beta > 0$, set $u(x) = \alpha \ln x, v(x) = \beta x, m_0 = 2, n_0 = 1$. Then,

$$\sum_{m=2}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(\alpha \ln m + \beta n)^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=2}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}},$$

where

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot \frac{B(1 - pA_2, 1 - qA_1)}{\alpha^{1-qA_1} \beta^{1-pA_2}}$$

is the best constant and $w_1(m) = (\ln m)^{-1+pqA_1} m^{p-1}, w_2(n) = n^{-1+pqA_2}$.

Theorem 2 *Inequality* (12) *is a refinement of inequality* (5).

Proof Utilizing the well-known Young inequality, we have

$$\begin{aligned} R &= \frac{\left(\frac{S}{1-qA_1}\right)^{\frac{r(1-qA_1)}{s}} \left(\frac{T}{1-pA_2}\right)^{\frac{r(1-pA_2)}{s}}}{S+T} \\ &\le \frac{\frac{r(1-qA_1)}{s} \cdot \frac{S}{1-qA_1} + \frac{r(1-pA_2)}{s} \cdot \frac{T}{1-pA_2}}{S+T} = \frac{r}{s}. \end{aligned}$$

Now, the inequality (5) follows from (12), which completes the proof.

Setting $u(x) = v(x) = x^{\alpha}, \alpha = \frac{p-q}{pq(qA_1-pA_2)} > 0, a_m = m^{\frac{k}{p}}, k = \alpha p(1 - qA_1) - 1 - p, b_n = n^{\frac{l}{q}}, l = \alpha q(1 - pA_2) - 1 - q$ and $\sigma_{m,n} = c_m c_n$ in Theorem 1, we obtain the following Gabriel-type inequality:

Corollary 2 *Suppose the parameters* p, q, r, s, A_1, *and* A_2, *are defined as in the statement of Theorem 1. If* $\{c_m\}$ *is a positive sequence, then*

$$\left(\sum_{m=1}^{\infty} c_m\right)^r < C^* \left\{ \sum_{m=1}^{\infty} m^{\alpha - \frac{rk}{sp}} c_m^r \right\}^{\frac{1}{2}} \left\{ \sum_{m=1}^{\infty} m^{-\frac{rl}{sq}} c_m^r \right\}^{\frac{1}{2}},$$

where the constant $C^* = \sqrt{C} \cdot \left(\frac{\pi^2}{6\alpha}\right)^{\frac{r}{2\,s}}$ *is the best possible.*

2.2 An Associated Integral Form

Theorem 3 *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *and* $r > 1, \frac{1}{r} + \frac{1}{s} = 1$. *Suppose that* $A_1 \in (\max\{\frac{1-\frac{s}{r}}{q}, 0\}, \frac{1}{q})$, $A_2 \in (\max\{\frac{1-\frac{s}{r}}{p}, 0\}, \frac{1}{p})$, $pA_2 + qA_1 = 2 - \frac{s}{r} > 0$, $\varphi(x)$ *and* $\psi(y)$ *are differentiable strictly increasing functions on* (a, b) $(-\infty \le a < b \le \infty)$ *and* (c, d) $(-\infty \le c < d \le \infty)$ *respectively, such that* $\varphi(a+) = \psi(c+) = 0$

and $\varphi(b-)=\psi(d-)=\infty$. If $f(x), g(y)$ and $G(x,y)$ are positive functions on $(a,b),(c,d)$ and $(a,b)\times(c,d)$ respectively, then the following inequality holds:

$$\left(\int_a^b\int_c^d G(x,y)dxdy\right)^r < C\left\{\int_a^b w_1(x)f^p(x)dx\right\}^{\frac{r}{ps}}\left\{\int_c^d w_2(y)g^q(y)dy\right\}^{\frac{r}{qs}}$$
$$\times\left\{\int_a^b\int_c^d \frac{\varphi(x)G^r(x,y)}{(f(x)g(y))^{\frac{r}{s}}}dxdy\right\}^{\frac{r(1-qA_1)}{s}}$$
$$\times\left\{\int_a^b\int_c^d \frac{\psi(y)G^r(x,y)}{(f(x)g(y))^{\frac{r}{s}}}dxdy\right\}^{\frac{r(1-pA_2)}{s}}. \tag{13}$$

Here, $w_1(x)=[\varphi(x)]^{-1+pqA_1}[\varphi'(x)]^{1-p}, w_2(y)=[\psi(y)]^{-1+pqA_2}[\psi'(y)]^{1-q}$ and the constant

$$C=\frac{s[B(1-pA_2,1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}}$$

is the best possible.

Proof Using the Hölder inequality, the Hilbert-type inequality (6) and proceeding as in the proof of Theorem 1, we have that (13) holds. Now, to prove the part with the best constant, suppose that $\varepsilon>0$ is sufficiently small, and let

$$\widetilde{f}(x)=\begin{cases}0, & \text{if } x\in(a,a_1)(a_1=\varphi^{-1}(1))\\ [\varphi(x)]^{-qA_1-\frac{\varepsilon}{p}}\varphi'(x), & \text{if } x\in[a_1,b)\end{cases},$$

$$\widetilde{g}(y)=\begin{cases}0, & \text{if } y\in(c,c_1)(c_1=\psi^{-1}(1))\\ [\psi(y)]^{-pA_2-\frac{\varepsilon}{q}}\psi'(x), & \text{if } y\in[c_1,d)\end{cases},$$

and $\widetilde{G}(x,y)=\frac{\widetilde{f}(x)\widetilde{g}(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}}$. Then we have

$$\left\{\int_a^b w_1(x)\widetilde{f}^p(x)dx\right\}^{\frac{r}{ps}}\left\{\int_c^d w_2(y)\widetilde{g}^q(y)dy\right\}^{\frac{r}{qs}}=\left(\frac{1}{\varepsilon}\right)^{\frac{r}{s}},$$

and

$$\int_a^b\int_c^d\widetilde{G}(x,y)dxdy=\int_a^b\int_c^d\frac{\widetilde{f}(x)\widetilde{g}(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}}dxdy$$
$$=\int_{a_1}^b[\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_{1/\varphi(x)}^\infty\frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{\frac{s}{r}}}dudx$$
$$=\int_{a_1}^b[\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_0^\infty\frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{\frac{s}{r}}}dudx$$

$$
\begin{aligned}
&- \int_{a_1}^{b} [\varphi(x)]^{-1-\varepsilon} \varphi'(x) \int_0^{1/\varphi(x)} \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{\frac{s}{r}}} du dx \\
&> \frac{1}{\varepsilon} B\left(1 - qA_1 + \frac{\varepsilon}{q}, 1 - pA_2 - \frac{\varepsilon}{q}\right) \\
&- \int_{a_1}^{b} [\varphi(x)]^{-1-\varepsilon} \varphi'(x) \int_0^{1/\varphi(x)} u^{-pA_2-\frac{\varepsilon}{q}} du dx \\
&= \frac{1}{\varepsilon} B\left(1 - qA_1 + \frac{\varepsilon}{q}, 1 - pA_2 - \frac{\varepsilon}{q}\right) \\
&- \frac{1}{(1 - pA_2 - \frac{\varepsilon}{q})(1 - pA_2 + \frac{\varepsilon}{p})} \\
&= \frac{1}{\varepsilon} B\left(1 - qA_1 + \frac{\varepsilon}{q}, 1 - pA_2 - \frac{\varepsilon}{q}\right) - \bigcirc(1).
\end{aligned}
$$

On the other hand, employing (10), it follows that

$$
\begin{aligned}
\int_a^b \int_c^d \frac{\varphi(x)\widetilde{G}^r(x,y)}{(\widetilde{f}(x)\widetilde{g}(y))^{\frac{r}{s}}} dx dy &= \int_{a_1}^{b} [\varphi(x)]^{-1-\varepsilon} \varphi'(x) \int_{1/\varphi(x)}^{\infty} \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{s}} du dx \\
&< \int_{a_1}^{b} [\varphi(x)]^{-1-\varepsilon} \varphi'(x) \int_0^{\infty} \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{s}} du dx \\
&= \frac{1}{\varepsilon} B\left(2 - qA_1 + \frac{\varepsilon}{q}, 1 - pA_2 - \frac{\varepsilon}{q}\right) \\
&= \frac{1}{\varepsilon} \frac{r(1-qA_1+\frac{\varepsilon}{q})}{s} B\left(1 - qA_1 + \frac{\varepsilon}{q}, 1 - pA_2 - \frac{\varepsilon}{q}\right),
\end{aligned}
$$

and similarly,

$$
\int_a^b \int_c^d \frac{\psi(y)\widetilde{G}^r(x,y)}{(\widetilde{f}(x)\widetilde{g}(y))^{\frac{r}{s}}} dx dy < \frac{1}{\varepsilon} \frac{r(1 - pA_2 + \frac{\varepsilon}{p})}{s} B\left(1 - pA_2 + \frac{\varepsilon}{p}, 1 - qA_1 - \frac{\varepsilon}{p}\right).
$$

Assuming that the constant C in (13) is not the best possible, then there exists a positive constant $\widetilde{C} < C$, such that (13) is still valid when we replace C by $\widetilde{C}$. In particular, utilizing the above inequalities, we have

$$
\begin{aligned}
&\left(\frac{1}{\varepsilon} B\left(1 - qA_1 - \frac{\varepsilon}{q}, 1 - pA_2 - \frac{\varepsilon}{q}\right) - \bigcirc(1)\right)^r \\
&< \widetilde{C} \left(\frac{1}{\varepsilon}\right)^{\frac{r}{s}} \left\{\frac{1+\varepsilon O(1)}{\varepsilon} \cdot \frac{r(1-qA_1+\frac{\varepsilon}{q})}{s} B\left(1-qA_1+\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right)\right\}^{\frac{r(1-qA_1)}{s}} \\
&\times \left\{\frac{1+\varepsilon O(1)}{\varepsilon} \cdot \frac{r(1 - pA_2 + \frac{\varepsilon}{p})}{s} B\left(1 - pA_2 + \frac{\varepsilon}{p}, 1 - qA_1 - \frac{\varepsilon}{p}\right)\right\}^{\frac{r(1-pA_2)}{s}}.
\end{aligned}
$$

Now, multiplying the above inequality by ε^r and then, letting $\varepsilon \to 0^+$, it follows that

$$C = \frac{s[B(1-pA_2, 1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}} \le \widetilde{C},$$

which is in contrast to $\widetilde{C} < C$. The proof is now complete.

Similarly to the discrete case, if $G(x, y) = \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}}$, then, setting $S = \int_a^b \int_c^d \frac{\varphi(x)f(x)g(y)}{(\varphi(x)+\psi(y))^s} dxdy$, $T = \int_a^b \int_c^d \frac{\psi(y)f(x)g(y)}{(\varphi(x)+\psi(y))^s} dxdy$, we easily obtain that $S + T = \int_a^b \int_c^d \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}} dxdy$, and the Theorem 3 yields the following consequence:

Corollary 3 *Suppose the parameters* p, q, r, s, A_1, A_2, *and the functions* $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}$ *are defined as in the statement of Theorem 3. If* $f(x)$ *and* $g(x)$ *are positive functions on* $(0, \infty)$*, then the following inequality holds:*

$$\int_a^b \int_c^d \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}} dxdy < C_1 \left\{ \int_a^b w_1(x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_c^d w_2(y) g^q(y) dy \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}}. \tag{14}$$

In addition, the constant

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot B(1-pA_2, 1-qA_1)$$

is the best possible and

$$R = \frac{\left(\frac{S}{1-qA_1}\right)^{\frac{r(1-qA_1)}{s}} \left(\frac{T}{1-pA_2}\right)^{\frac{r(1-pA_2)}{s}}}{S+T},$$

$w_1(x) = [\varphi(x)]^{-1+pqA_1}[\varphi'(x)]^{1-p}$, $w_2(y) = [\psi(y)]^{-1+pqA_2}[\psi'(y)]^{1-q}$.

It should be noticed here that the inequality (14) is more accurate than the inequality (6).

Theorem 4 *Inequality (14) is a refinement of inequality (6).*

Proof The proof follows the lines of the proof of Theorem 2.

If $\varphi(x) = \psi(x) = x^\alpha, 0 < \alpha < \min\left\{\frac{1}{1-qA_1}, \frac{1}{1-pA_2}\right\}$, $f(x) = g(x) = e^{-x}$ and $G(x, y) = \omega(x)\omega(y)$, the Theorem 3 yields the following integral Gabriel-type inequality:

Corollary 4 *Suppose the parameters* p, q, r, s, A_1, *and* A_2, *are defined as in the statement of Theorem 3. If* $\omega(x)$ *is a positive function on* $(0, \infty)$*, then*

$$\left(\int_0^\infty \omega(x)dx\right)^r < C^* \left\{\int_0^\infty x^\alpha e^{\frac{rx}{s}} [\omega(x)]^r dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty e^{\frac{rx}{s}} [\omega(x)]^r dx\right\}^{\frac{1}{2}},$$

where the constant factor $C^{*} = \sqrt{C}\left(\frac{1}{\alpha}\right)^{\frac{r}{2s}}\left(\frac{\Gamma(\mu)}{p^{\mu}}\right)^{\frac{r}{2ps}}\left(\frac{\Gamma(\nu)}{p^{\nu}}\right)^{\frac{r}{2qs}}$ *is the best possible and* $\mu = p + \alpha p(qA_1 - 1), \nu = q + \alpha q(pA_2 - 1)$.

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Mathematical Models of Mechanical Fields in Media with Inclusions and Holes

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Abstract Various problems of mechanics described by two-dimensional harmonic and biharmonic functions are investigated by application of the generalized alternating method of Schwarz (GMS). It is demonstrated that the GMS in zeroth approximation coincides with the principle of superposition. Iterative schemes for the $\mathbb{R}$-linear problem on harmonic functions for multiply connected domains are constructed and compared to the GMS. The method is applied in symbolic form to the case when inclusions have elliptical shape. Two-dimensional problems for biharmonic functions by application of the Kolosov–Muskhelishvili formulae are considered by the principle of superposition to describe gas flows in rigid bodies. Viscoelastic problems in porous media are solved by use of the method of finite elements.

Keywords Alternating method of Schwarz · Functional equations for analytic functions · Superposition principle · Elastic half plane with cavities

1 Introduction to the Generalized Alternating Method of Schwarz (GMS)

Mechanical fields considered in this paper are described by two-dimensional harmonic and biharmonic functions. Many problems of the mechanics and of composites are stated as boundary value problems for domains with holes and inclusions when a condition of the contact between the components is written as a conjugation condition for the limit values of the unknown functions and their derivatives [12, 13]. Such problems have been the subject of research interest in porous media and composites

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(e.g. [1, 2, 3, 5, 14, 15, 16]. In the present chapter, attention is paid to the problem of interactions of inclusions and its investigation by the generalized alternating method of Schwarz (GMS) [19, 22, 26].

The main idea of the method can be presented by the $\mathbb{R}$–linear problem on harmonic functions for multiply connected domains. Let D_k be mutually disjoint simply connected domains in the complex plane $\mathbb{C}$ bounded by smooth curves L_k $(k = 1, 2, \ldots, n)$ and D be the complement of all closures of D_k to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Below, the domains D_k are called by inclusions. Denote by D^+ the union of all inclusions D_k, i.e., the domain D^+ consists of n connected components. Let L_k are orientated in a counterclockwise direction. Let ρ be a constant and $c(t)$ be given Hölder continuous functions on $L = \cup_{k=1}^{n} L_k$, the boundary of D^+.

The $\mathbb{R}$–linear conjugation problem with constant coefficients is stated as follows [22]. To find a function $\varphi(z)$ analytic in D and in all components of D^+, continuous by differentiable in the closures of the considered domains with the following conjugation condition:

$$\varphi^+(t) = \varphi^-(t) - \rho\overline{\varphi^-(t)} + c(t), \quad t \in L. \tag{1}$$

Here $\varphi^{\pm}(t)$ denotes the limit values of $\varphi(z)$, as z tends to a point $t \in L$ from D^+ and from D, respectively. Moreover, $\varphi(z)$ vanishes at infinity. If $|\rho| < 1$, the problem has a unique solution. This follows from a more general result obtained by Bojarski [6].

In order to describe the GMS we first recall the Sochocki–Plemelj formulae. The curve $L := \cup_{k=1}^{n} \partial D_k$ divides the complex plane onto domains D^+ and D. Here, each curve ∂D_k is orientated in the clockwise sense. Let $\mu(t)$ be a Hölder continuous function on L. Introduce the function

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\mu(t)}{t - z} dt \tag{2}$$

It is continuous on the complex plane except L where its limit boundary values $\Phi^+(t) = \lim_{z \to t \in D^+} \Phi(z)$ and $\Phi^-(t) = \lim_{z \to t \in D} \Phi(z)$ satisfy the jump condition [11]

$$\Phi^+(t) - \Phi^-(t) = \mu(t), \quad t \in L. \tag{3}$$

The condition (1) can by written in the form (3) with $\Phi^+(t) = \varphi_k(t) - f^+(t)$, $\Phi^-(t) = \varphi(t) - f^-(t)$, $\mu(t) = \rho\overline{\varphi_k(t)}$, where the Cauchy integral

$$f(z) = \frac{1}{2\pi i} \int_L \frac{c(t)}{t - z}\, dt \tag{4}$$

determines the function $f(z)$ analytic outside of L. Then (1) yields

$$\varphi_k(z) = \rho \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m(t)}}{t - z} dt + f(z). \; z \in D_k, \quad k = 1, 2, \ldots, n. \tag{5}$$

The function $\varphi(z)$ is calculated by $\varphi_m(z)$ as follows:

$$\varphi(z) = \rho \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m(t)}}{t-z} dt + f(z), \quad z \in D. \tag{6}$$

One can consider (5) as a system of integral equations on $\varphi_k(z)$ analytic in D_k and continuously differentiable in its closure. It is worth noting that the equations (5) are not the classic integral equations of the potential theory. They correspond to integral equations which are be deduced from the GMS. In order to analyse (5) we rewrite them in the form

$$\varphi_k(z) - \frac{\rho}{2\pi i} \int_{\partial D_k} \frac{\overline{\varphi_k(t)}}{t-z} dt = \rho \sum_{m \neq k} \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m(t)}}{t-z} dt + f(z), z \in D_k,\ k = 1, 2, \ldots, n. \tag{7}$$

The equations (5) can be solved by the following two iterative schemes. First, the direct iterations can be applied to (5)

$$\varphi_k^{(0)}(w) = f(z),$$

$$\varphi_k^{(p+1)}(z) = \rho \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m^{(p)}(t)}}{t-z} dt + f(z),\ z \in D_k,\ k = 1, 2, \ldots, n,\ p = 0, 1, 2, \ldots, \tag{8}$$

where $\varphi_k^{(p)}(w)$ denotes the pth approximation of $\varphi_k(w)$. As it is proved in [21], the iterations (8) uniformly converge for all $|\rho| \leq 1$.

The second iterative scheme is constructed on the basis of the equations (7). The zeroth approximation can be written in the form of the separate equations for each $k = 1, 2, \ldots, n$

$$\varphi_k^{(0)}(z) - \frac{\rho}{2\pi i} \int_{\partial D_k} \frac{\overline{\varphi_k^{(0)}(t)}}{t-z} dt = f(z),\ z \in D_k. \tag{9}$$

According to Bojarski [6], Eq. (9) has a unique solution. The pth approximation has also the form of the equation on $\varphi_k^{(p+1)}(z)$ for each $k = 1, 2, \ldots, n$

$$\varphi_k^{(p+1)}(z) - \frac{\rho}{2\pi i} \int_{\partial D_k} \frac{\overline{\varphi_k^{(p+1)}(t)}}{t-z} dt = \rho \sum_{m \neq k} \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m^{(p)}(t)}}{t-z} dt + f(z), z \in D_k. \tag{10}$$

Contrary to the first algorithm (8), convergence results for the second algorithm (9)–(10) are unknown.

The integrals from (10) for $m \neq k$ and $z \in D_k$ can be estimated as follows:

$$\Big| \int_{\partial D_m} \frac{\overline{\varphi_m^{(p)}(t)}}{t - z} dt \Big| \leq max_{t \in \partial D_m} |\varphi_m^{(p)}(t)| \frac{diam(D_k)}{d_{km}},$$

where $d_{km} = \inf_{t \in \partial D_m \ z \in D_k} |t - z|$, $diam(D_k) = \sup_{z_{1,2} \in D_k} |z_1 - z_2|$.

The values d_{km} and $diam(D_k)$ characterize the distance between D_k and D_m, and the linear size of D_k. If the sum of the ratios

$$\sum_{k=1}^{n} \sum_{m \neq k} \frac{diam(D_k)}{d_{km}} \tag{11}$$

is sufficiently small, the zeroth approximation $\varphi_k^0(z)$ can be accepted as an approximate solution of (5). Then, the approximation for $\varphi(z)$ from (6) becomes

$$\varphi^{(0)}(z) = \rho \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m^{(0)}(t)}}{t - z} dt + f(z), z \in D. \tag{12}$$

Formula (12) expresses the *superposition principle* used in physics when the field in D is approximated by a sum of the separate fields induced by the inclusions D_m. Therefore, the GMS applied within the zeroth approximation yields the superposition principle. In Sect. 3, this principle is applied to complicated mechanical fields.

2 R–Linear Problem with Elliptical Inclusions

The present section is devoted to application of the GMS to the $\mathbb{R}$–linear problem with many inclusions of elliptic shapes. We follow Sect. 1 and the paper [20] where this problem was considered in the case when all the ellipses have the same shape. In this section, we consider the general case when each ellipse can have arbitrary semi-axes and arbitrary size.

2.1 Statement of the Problem and Reduction to Integral Equations

Suppose that the elliptical inclusions $D_m (m = 1, 2, \ldots, n)$ do not overlap. For convenience, put the semiaxes equal to $r_m(1 + \alpha_m)$ and $r_m(1 - \alpha_m)$, respectively. The parameter r_m is positive and characterizes the size of inclusion, and α_m is the shape of the mth ellipse ($0 < \alpha_m < 1$). The case $\alpha_m = \alpha$ ($m = 1, 2, \ldots, n$) was considered in [20]. Let an inclusion D_m be centred at (x_m, y_m) and the angle between the major semiaxis of the ellipse and the x-axis be equal to θ_m. In accordance with Mityushev [20], introduce the local coordinates (X, Y) for a fixed inclusion D_m as follows:

$$X = \frac{1}{r_m} [(x - x_m) \cos \theta_m + (y - y_m) \sin \theta_m], \tag{13}$$

$$Y = \frac{1}{r_m}[(x - x_m)\sin\theta_m + (y - y_m)\cos\theta_m]. \tag{14}$$

The local equation of the ellipse ∂D_m has the form

$$\frac{X^2}{(1+\alpha_m)^2} + \frac{Y^2}{(1-\alpha_m)^2} = 1. \tag{15}$$

The foci of the ellipse ∂D_m in the local coordinates are located at $(\pm 2\sqrt{\alpha_m}, 0)$.

Let $Z = X + iY$ be the local complex coordinate, $z = x + iy$ and $w = \xi + i\zeta$ be global complex coordinates, where i denotes the imaginary unit. The Joukowsky conformal mapping

$$Z = w + \frac{\alpha_m}{w} \tag{16}$$

transforms the annulus $\sqrt{\alpha_m} < |w| < 1$ onto $D_m - \Gamma_m$, where Γ_m denotes the slit $(-2\sqrt{\alpha_m}, 2\sqrt{\alpha_m})$ along the X-axis. The inverse mapping to (16) has the form

$$w = \frac{1}{2}\left(Z + \sqrt{Z^2 - 4\alpha_m}\right) \tag{17}$$

where the branch of the square root is chosen in such a way that

$$\lim_{X \to \pm i0} \sqrt{Z^2 - 4\alpha_m} = \pm i\sqrt{4\alpha_m - X^2} \tag{18}$$

for $-2\sqrt{\alpha_m} < X < 2\sqrt{\alpha_m}$. Formulae (16)–(17) in the global coordinates become

$$z = s_m(w + \frac{\alpha_m}{w} + a_m) \tag{19}$$

$$w = \frac{1}{2}\left[\frac{z - a_m}{s_m} + \sqrt{\left(\frac{z - a_m}{s_m}\right)^2 - 4\alpha_m}\right], \tag{20}$$

where $s_m = r_m e^{i\theta_m}$.

Let D denote the complement of the closures of all domains D_m to the extended complex plane. We study the conductivity of the two-dimensional composite, when the domains D and D_m are occupied by materials of unit and λ conductivity, respectively, where $0 < \lambda < \infty$. Then, the potentials $u(z)$ and $u_m(z)$ are harmonic in D and D_m $(m = 1, 2, \ldots n)$ and satisfies the conjugation (transmission) conditions

$$u = u_m, \; \frac{\partial u}{\partial n} = \lambda \frac{\partial u_m}{\partial n}, \text{ on } \partial D_m, \; m = 1, 2, \ldots, n, \tag{21}$$

where $\partial/\partial n$ denotes the outward normal derivative to the ellipses. For simplicity, it is assumed that the potential $u(z)$ has singularities only in the domain D described by a function $Ref(z)$, where $f(z)$ is analytic in all inclusions D_k, Re stands for the real part of a complex number.

Following Mityushev and Rogosin [22], introduce complex potentials $\varphi(z)$ and $\varphi_m(z)$ analytic in D and D_m, respectively, in such a way that $u(z)$ and $u_m(z)$ are related to the complex potentials by

$$u(z) = Re[\varphi(z) + f(z)],\, u_m(z) = \frac{2}{1+\lambda} Re\varphi_m(z). \tag{22}$$

Then the conditions (21) can by reduced to the $\mathbb{R}$-linear problem (1), where $c(t) = f(z)$ and ρ denotes the contrast parameter

$$\rho = \frac{\lambda - 1}{\lambda + 1}. \tag{23}$$

2.2 Solution to Integral Equations

It follows from Sect. 1 that the $\mathbb{R}$-linear problem (1) is reduced to the integral equations (5). We now reduce these equations for elliptic inclusions to a system of functional equations (without integral terms).

Let k be fixed in (5). The doubly connected domain $D_k - \Gamma_k$ is mapped onto the annulus $\sqrt{\alpha_k} < |w| < 1$ by the conformal mapping (20); D_k is transformed onto the unit circle $|w| = 1$, Γ_k onto the circle $|w| = \sqrt{\alpha_k}$. Introduce the functions

$$\Phi_k(w) = \varphi_k(z) = \varphi_k\left[s_k\left(w + \frac{\alpha_k}{w}\right) + a_k\right] \tag{24}$$

analytic in $\sqrt{\alpha_k} < |w| < 1$ and continuous in $\sqrt{\alpha_k} \le |w| \le 1$. Substitute (24) in (5) and change the variables in the integrals as follows:

$$t = s_k\left(\tau + \frac{\alpha_k}{\tau}\right) + a_k. \tag{25}$$

Then (5) becomes

$$\Phi_k(w) = \rho \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\Phi_m(\tau)}(1 - \frac{\alpha_m}{\tau^2})d\tau}{\tau + \frac{\alpha_m}{\tau} - \frac{s_k}{s_m}(w + \frac{\alpha_k}{w}) + \frac{a_m - a_k}{s_m}} + F(w), \tag{26}$$

$$\sqrt{\alpha_k} < |w| < 1, \quad k = 1, 2, \ldots, n,$$

where $F(w) = f(z)$. Moreover, it follows from the continuity of $\varphi_k(z)$ when z passes the slit Γ_k that

$$\Phi_k(\tau) = \Phi_k\left(\frac{\alpha_k}{\tau}\right), \quad |\tau| = \sqrt{\alpha_k}. \tag{27}$$

Equation (27) implies that $\Phi_k(w)$ is represented in the form

$$\Phi_k(w) = \phi_k(w) + \phi_k\left(\frac{\alpha_k}{w}\right), \quad \alpha_k \le |w| \le 1, \tag{28}$$

where $\phi_k(w)$ is analytic in the unit disk $|w| < 1$. Equation (28) follows from the representation of $\Phi_k(w)$ in the form of the Laurent series in the annulus $\alpha_k \le |w| \le 1$ and form (27). The same arguments yield the representation of $F(w)$ in the form $F(w) = g_k(w) + g_k(\frac{\alpha_k}{w})$, where $g_k(w)$ is analytic in the unit disk. Substitution of (28) into (26) yields

$$\phi_k(w) + \phi_k\left(\frac{\alpha_k}{w}\right) = \rho \sum_{m=1}^{n} \Big[P_{km}(w) + Q_{km}(w)\Big] + g_k(w) + g_k\left(\frac{\alpha_k}{w}\right), \tag{29}$$

$$\alpha_k < |w| < 1, \quad k = 1, 2, \ldots, n,$$

where

$$P_{km}(w) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\overline{\phi_m(\frac{1}{\overline{\tau}})}(1 - \frac{\alpha_m}{\tau^2})d\tau}{\tau + \frac{\alpha_m}{\tau} - \frac{s_k}{s_m}(w + \frac{\alpha_k}{w}) + \frac{a_m - a_k}{s_m}}, \tag{30}$$

$$Q_{km}(w) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\overline{\phi_m(\alpha_m \overline{\tau})}(1 - \frac{\alpha_m}{\tau^2})d\tau}{\tau + \frac{\alpha_m}{\tau} - \frac{s_k}{s_m}(w + \frac{\alpha_k}{w}) + \frac{a_m - a_k}{s_m}}. \tag{31}$$

Here, the relation $\overline{\tau} = \frac{1}{\tau}$ on the unit circle is used.

The integrals (30)–(31) are analytically calculated by residues in [20]. Following [20] consider the quadratic equation with respect to τ

$$\tau^2 - s_m^{-1}\left[s_k\left(w + \frac{\alpha_k}{w}\right) + a_k - a_m\right]\tau + \alpha_m = 0. \tag{32}$$

The cases of equal and non equal k and m have to be separately investigated.
a) Let $k = m$. Then Eq. (32) becomes

$$\tau^2 - \left(w + \frac{\alpha_k}{w}\right)\tau + \alpha_k = 0. \tag{33}$$

Its two solutions have the form

$$\tau_1 = w,\ \tau_2 = \frac{\alpha_k}{w}. \tag{34}$$

b) Let $k \ne m$. In order to avoid a confusion with (34), the roots of (32) in this case are denoted by w_1 and w_2

$$w_1 = \frac{1}{2}\left\{s_m^{-1}\left[\left(w + \frac{\alpha_k}{w}\right) + a_k - a_m\right] - \sqrt{\left[\frac{s_k(w + \frac{\alpha_k}{w}) + a_k - a_m}{s_m}\right]^2 - 4\alpha_m}\right\} \tag{35}$$

$$w_2 = \frac{1}{2}\left\{s_m^{-1}\left[s_k\left(w + \frac{\alpha_k}{w}\right) + a_k - a_m\right] + \sqrt{\left[\frac{s_k(w + \frac{\alpha_k}{w}) + a_k - a_m}{s_m}\right]^2 - 4\alpha_m}\right\}$$

The branch of the square root is chosen in accordance with (18). It was proved in [20] that $|w_1| < 1$ and $|w_2| > 1$.

The integrals (30)–(31) were calculated in [20]. From $m \neq k$ it can be written in the form

$$P_{km}(w) = \overline{\phi_m(0)} - \overline{\phi_m\left(\frac{\overline{w_1}}{\alpha_m}\right)}, \quad (m \neq k), \quad P_{kk}(w) = \overline{\phi_k(0)},$$

$$Q_{km}(w) = \overline{\phi_m(0)} - \overline{\phi_m(\alpha_m w_1)} \quad (m \neq k), Q_{kk}(w) = \overline{-\phi_k(0)} + \overline{\phi_k(\alpha_k \overline{w})} + \overline{\phi_k\left(\frac{\alpha_k^2}{\overline{w}}\right)},$$

where w_1 and w_2 are given by (35).

2.3 Functional Equations

Substituting (34)–(35) into (29) we transform the integral equations (29) to the following functional equations:

$$\phi_k(w) + \phi_k\left(\frac{\alpha_k}{w}\right) = \rho\Bigg\{\overline{\phi_k(\alpha_k\overline{w})} + \overline{\phi_k\left(\frac{\alpha_k^2}{\overline{w}}\right)} - \tag{36}$$

$$-\sum_{m\neq k}\left[-2\overline{\phi_m(0)} + \overline{\phi_m(\alpha_m \overline{\beta_{km}(w)})} + \overline{\phi_m\left(\frac{\overline{\beta_{km}(w)}}{\alpha_m}\right)}\right] + g_k(w) + g_k\left(\frac{\alpha_m(w)}{w}\right)\Bigg\},$$

$$\sqrt{\alpha_k} < |w| < 1, \quad k = 1, 2, \ldots, n.$$

Here, for convenience the root w_1 is written as the function of w

$$\beta_{km}(w) = \frac{1}{2}\left\{s_m^{-1}\left[s_k\left(w + \frac{\alpha_k}{w}\right) + a_k - a_m\right] - \sqrt{h_{km}(w)}\right\}, \tag{37}$$

where

$$h_{km}(w) = \left[\frac{s_k(w + \frac{\alpha_k}{w}) + a_k - a_m}{s_m}\right]^2 - 4\alpha_m. \tag{38}$$

The right hand part of (36) consists of the functions $\phi_k(w)$ and $\phi_k\left(\frac{\alpha_m}{w}\right)$ analytic in $|w| < 1$ and $|w| > \sqrt{\alpha_m}$, respectively. Denote by P^+ the project operator which transforms a function analytic in $\sqrt{\alpha_m} < |w| < 1$ to its part analytic in the unit disk. This operator can by considered as taking the regular part of the Laurent series or as the integral operator $\frac{1}{2\pi i}\int_{|w|=1} \frac{\bullet\, dw}{w-\zeta}$ with $|\zeta| < 1$. Application of P^+ to (36) yields

$$\phi_k(w) + \phi_k(0) = \rho \Big\{ \overline{\phi_k(\alpha_m \overline{w})} + \overline{\phi_k(0)} - \tag{39}$$

$$\sum_{m \neq k} \left[-2\overline{\phi_m(0)} + P^+ \overline{\phi_m(\alpha_m \overline{\beta_{km}(w)})} + P^+ \overline{\phi_m \left(\frac{\overline{\beta_{km}(w)}}{\alpha_m} \right)} \right] + g_k(w) + g_k(0) \Big\},$$

$$|w| \leq 1, \; k = 1, 2, \ldots, n.$$

Here, the following relation is used:

$$P^+ \overline{\phi_k \left(\frac{\alpha_k^2}{\overline{w}} \right)} = \overline{\phi_k(0)}.$$

One can consider (39) as a system of functional equations on the functions $\phi_k(w)$ analytic in the unit disk and continuous in its closure. The solution of (39) can be found by the method of successive approximations corresponding to the algorithm (8). The equations (39) can by considered as iterative functional equations with shift into domain [20, 22], since $|\beta_{km}(w)| < 1$. It is worth noting that the equations (39) do not contain integral terms and can be solved by use of the symbolic computations, hence, the obtained results can be obtained in the form of approximate analytical formulae.

3 Some Model Problems of Gas Flows in Rigid Bodies

3.1 Stress–Strain State of the Elastic Half Plane with Holes Filled by Gas

One of the mathematical model approaches to creation describe the stress–strain state of an elastic half plane with cavities that can contain gas, is discussed in this section. We consider an elastic isotropic half plane with two holes which are far away from the half-plane boundary and each other. This assumption allows us to apply the GMS in the zeroth approximation (the method of superposition discussed in Sect. 1). All the problems are considered in the plain strain condition. One of the holes is a circle with radius R and centre at the origin. The second hole is an ellipse with semiaxes a and b. The centre of the ellipse is placed at the point $O_1(x_{01}, y_{01})$. The x-axis forms the angle ϵ with $O_1 O$ (see Fig. 1). Let the real axis and the boundary of holes be denoted by L_0, L_1, L_2, respectively, and the distance from the centre of the circle to L_0 be H. Let the homogeneous pressure p_0 be given on the boundary L_1 and boundaries L_0 and L_2 be free.

The Kolosov–Muskhelishvili method will be used to solve this problem. Let S^* be a domain bounded by contours L_0, L_1, L_2 and S a domain bounded by L_1 and L_2. The problem is described by the following equilibrium equations:

$$\frac{\partial \sigma_x^{(1)}}{\partial x} + \frac{\partial \tau_{xy}^{(1)}}{\partial y} = 0,$$

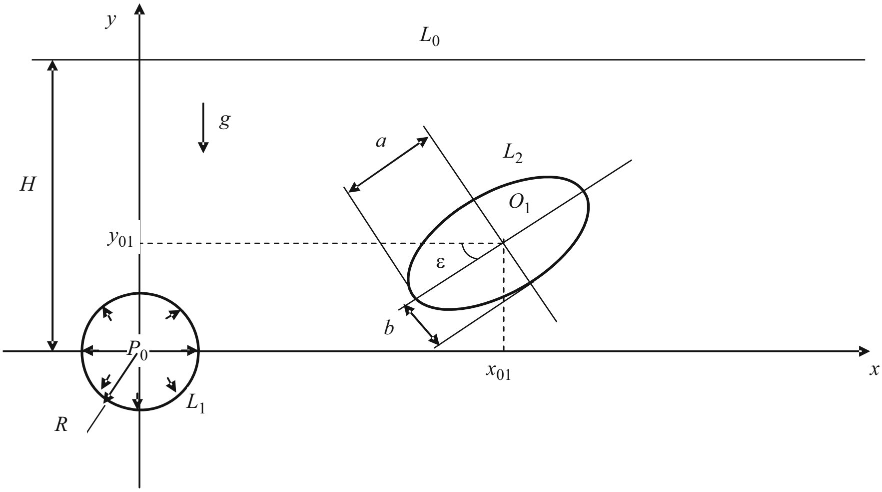


Fig. 1 Scheme of model problem

$$\frac{\partial \tau_{xy}^{(1)}}{\partial x} + \frac{\partial \sigma_y^{(1)}}{\partial y} - \rho g = 0,$$

$$\Delta(\sigma_x^{(1)} + \sigma_y^{(1)}) = 0 \tag{40}$$

and the boundary conditions

$$\sigma_x^{(1)} \cos(n, x) + \tau_{xy}^{(1)} \cos(n, y) = 0,\ \tau_{xy}^{(1)} \cos(n, x) + \sigma_y^{(1)} \cos(n, y) = 0$$
$$\text{on } L_j\ (j = 0, 2), \tag{41}$$

$$\sigma_x^{(1)} \cos(n, x) + \tau_{xy}^{(1)} \cos(n, y) = -P_0 \cos(n, x),$$
$$\tau_{xy}^{(1)} \cos(n, x) + \sigma_y^{(1)} \cos(n, y) = -P_0 \cos(n, y) \text{ on } L_1, \tag{42}$$

where σ_{ij} denote the stress components, ρ the media density, g the gravity acceleration, and n the outward normal to the boundaries L_j.

Using the superposition principle we can represent the stress components as follows:

$$\sigma_x^{(1)} = \sigma_x^{(0)} + \sigma_x, \tau_{xy}^{(1)} = \tau_{xy}^0 + \tau_{xy}, \sigma_y^{(1)} = \sigma_y^{(0)} + \sigma_y, \tag{43}$$

The sizes of holes are small in comparison with plane sizes. Therefore stresses σ_{ij} are negligible at large distance from holes, hence, σ_{ij} vanish at infinity. It is evident that the additional stresses satisfy homogeneous equilibrium equations. If the boundaries L_1 and L_2 are far away from the boundary L_0, we can consider an infinite plane with holes. The formulae for initial stresses are well known

$$\sigma_x^{(0)} = \rho g(y - H), \tau_{xy}^{(0)} = 0, \sigma_y^{(0)} = \lambda \rho g(y - H),$$

where λ is the ratio of horizontal to vertical stress.

The boundary conditions for the additional stresses have the form

$$\sigma_y = \tau_{xy} = 0 \text{ on } L_0; \tag{44}$$

$$\sigma_x \cos(n,x) + \tau_{xy} \cos(n,y) = f_i \cos(n,x),$$

$$\tau_{xy} \cos(n,x) + \sigma_y \cos(n,y) = g_i \cos(n,y) \text{ on } L_j,\ j = 1,2,$$

where

$$f_1 = p\rho g H - P_0,\ g_1 = \rho g H - P_0,\ f_2 = p\rho g(H - y_{01}),\ g_2 = \rho g(H - y_{01}).$$

Following the Kolosov–Mushelishvili method we use the complex potentials $\Phi(z)$ and $\Psi(z)$

$$\sigma_x + \sigma_y = 2(\Phi(z) - \overline{\Phi(z)}), \sigma_y - \sigma_x + 2\tau_{xy} = 2(\overline{z}\Phi'(z) - \Psi(z)).$$

Using the superposition principle and the well-known solutions for infinite plane with elliptic and circular holes [23, 30] (with $p = 1$), we obtain

$$\sigma_x = Re[-\Psi_1(x) + 2\Phi_2(z_1) - K(z_1)],$$

$$\tau_{xy} = Im[\Psi_1(z) + K(z_1)],$$

$$\sigma_y = Re[\Psi_1(z) + 2\Phi_2(z_1) + K(z_1)],$$

where

$\Psi_1(z) = \frac{P_1 R^2}{z^2}$, $z = x + iy$, $\Phi_2(z_1) = \frac{\varphi_2'(\varsigma)}{\omega'(\varsigma)}$, $z_1 = e^{-is}(z - z_{01})$, $K(z_1) = e^{-2is}(\overline{z_1}\Phi_2'(z_1) + \Psi_2(z_1))$, $P_1 = P_0 - \rho g h$, $z_{01} = x_{01} + iy_{01}$, $\varphi'(\varsigma) = \frac{\omega'(\varsigma)\varphi_2''(\varsigma) - \varphi_2'(\varsigma)\omega''(\varsigma)}{(\omega'(\varsigma))^2}$, $\varphi_2(\varsigma) = \frac{P_2 E s}{\varsigma}$, $E = \frac{a+b}{2}$, $s = \frac{a-b}{a+b}$, $\omega(\varsigma) = E(\varsigma + s/\varsigma)$, $\varsigma = \frac{z_1 + \sqrt{z_1^2 - 4E^2\ s}}{2E}$, $\Psi_2(z_1) = \frac{\psi_2'(\varsigma)}{\omega'(\varsigma)}$, $\psi_2 = \frac{P_2 E}{\varsigma} + \frac{P_2 E s}{\varsigma}\frac{1+s\varsigma^2}{\varsigma^2 - s}$, $P_2 = \rho g(H - y_{01})$, $\Phi_2'(z_1) = \frac{\varphi'(\varsigma)}{\omega'(\varsigma)}$.

The main stresses become

$$\sigma_1 = \frac{\sigma_x^{(1)} + \sigma_y^{(1)}}{2} + \frac{\sigma_x^{(1)} - \sigma_y^{(1)}}{2}\cos(2\theta) + \tau_{xy}^{(1)}\sin(2\theta),$$

$$\sigma_2 = \frac{\sigma_x^{(1)} + \sigma_y^{(1)}}{2} - \frac{\sigma_x^{(1)} - \sigma_y^{(1)}}{2}\cos(2\theta) - \tau_{xy}^{(1)}\sin(2\theta),$$

$$\sigma_3 = \nu(\sigma_x^{(1)} + \sigma_y^{(1)}), \tag{45}$$

where $\theta = \frac{1}{2}\arctan\frac{2\tau_{xy}^{(1)}}{\sigma_x^{(1)} - \sigma_y^{(1)}}$ and ν denotes Poisson's ratio. Solutions to these problems by other methods are described in [7, 18, 28, 29].

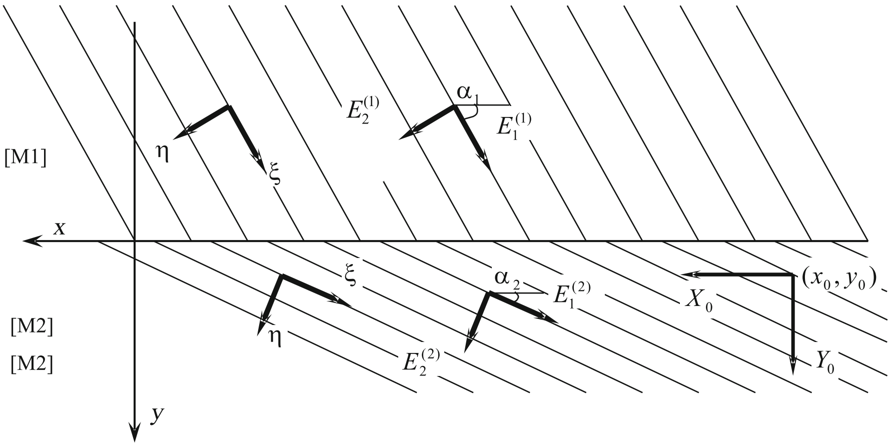


Fig. 2 Scheme of model problem for two orthotropic half planes

3.2 Stress–Strain State of the Rock Massif and Applications to Gas Dynamic Phenomena

We consider an orthotropic elastic body which consists of two half planes D_j $(j = 1, 2)$ with different elastic constants. The line $y = 0$ be the boundary between the half planes and let the y-axis be directed downward. Each half plane is orthotropic in the local coordinate system (ξ, η) as displayed in Fig. 2.

Hooke's law in the local coordinate system becomes

$$\begin{aligned}
\frac{\partial U^{(j)}}{\partial \xi} &= \beta_{11}^{(j)} \sigma_{\xi}^{(j)} + \beta_{12}^{(j)} \sigma_{\eta}^{(j)}, \\
\frac{\partial V^{(j)}}{\partial \eta} &= \beta_{12}^{(j)} \sigma_{\xi}^{(j)} + \beta_{22}^{(j)} \sigma_{\eta}^{(j)}, \\
\frac{\partial U^{(j)}}{\partial \eta} + \frac{\partial V^{(j)}}{\partial \xi} &= \beta_{66}^{(j)} \tau_{\xi\eta}^{(j)}.
\end{aligned} \tag{46}$$

where

$$\beta_{11}^{(j)} = \frac{1 - \nu_{31}^{(j)} \nu_{13}^{(j)}}{E_1^{(j)}}, \beta_{12}^{(j)} = -\frac{\nu_{21}^{(j)} + \nu_{31}^{(j)} \nu_{23}^{(j)}}{E_2^{(j)}}, \beta_{22}^{(j)} = \frac{1 - \nu_{23}^{(j)} \nu_{32}^{(j)}}{E_2^{(j)}}, \beta_{66}^{(j)} = \frac{1}{G_{12}^{(j)}}.$$

E_1^j and E_2^j are Young's modules in the principal directions ξ and η, respectively. Here, G_{12}^j denotes the shear modulus in the plain (ξ, η), ν_{kl}^j Poisson's coefficients. The angle between the local and global coordinate systems is denoted by α_j.

Hooke's law in the main coordinate system can be written in the form

$$\frac{\partial U^{(j)}}{\partial x} = c_{11}^{(j)}\sigma_x^{(j)} + c_{12}^{(j)}\sigma_y^{(j)} + c_{16}^{(j)}\tau_{xy}^{(j)},$$
$$\frac{\partial V^{(j)}}{\partial y} = c_{12}^{(j)}\sigma_x^{(j)} + c_{22}^{(j)}\sigma_y^{(j)} + c_{26}^{(j)}\tau_{xy}^{(j)},$$
$$\frac{\partial U^{(j)}}{\partial y} + \frac{\partial V^{(j)}}{\partial x} = c_{16}^{(j)}\sigma_x^{(j)} + c_{26}^{(j)}\sigma_y^{(j)} + c_{66}^{(j)}\tau_{xy}^{(j)}. \tag{47}$$

The coefficients c_{mn} linearly depend on the coefficients β_{kl} as follows:

$$c_{11}^{(j)} = \beta_{11}^{(j)}\cos^4(\alpha_j) + B^{(j)}\sin^2(\alpha_j)\cos^2(\alpha_j) + \beta_{22}^{(j)}\sin^4(\alpha_j),$$
$$c_{22}^{(j)} = \beta_{11}^{(j)}\sin^4(\alpha_j) + B^{(j)}\sin^2(\alpha_j)\cos^2(\alpha_j) + \beta_{22}^{(j)}\cos^4(\alpha_j),$$
$$c_{12}^{(j)} = \beta_{12}^{(j)} + (\beta_{11}^{(j)} + \beta_{22}^{(j)} - B^{(j)})\sin^2(\alpha_j)\cos^2(\alpha_j),$$
$$c_{66}^{(j)} = \beta_{66}^{(j)} + 4(\beta_{11}^{(j)} + \beta_{22}^{(j)} - B^{(j)})\sin^2(\alpha_j)\cos^2(\alpha_j),$$
$$c_{16}^{(j)} = (2\beta_{22}^{(j)}\sin^2(\alpha_j) - 2\beta_{11}^{(j)}\cos^2(\alpha_j) + B^{(j)}\cos(2\alpha_j))\sin(\alpha_j)\cos(\alpha_j),$$
$$c_{26}^{(j)} = (2\beta_{22}^{(j)}\cos^2(\alpha_j) - 2\beta_{11}^{(j)}\sin^2(\alpha_j) - B^{(j)}\cos(2\alpha_j))\sin(\alpha_j)\cos(\alpha_j),$$

where $B^{(j)} = 2\beta_{12}^{(j)} + \beta_{66}^{(j)}$.

We will solve the problem when the body forces are absent. Then, the equilibrium equations become

$$\frac{\partial \sigma_x^{(j)}}{\partial x} + \frac{\partial \tau_{xy}^{(j)}}{\partial y} = 0,$$
$$\frac{\partial \tau_{xy}^{(j)}}{\partial x} + \frac{\partial \sigma_y^{(j)}}{\partial y} = 0. \tag{48}$$

The stress tensor components can be written in the form

$$\sigma_x^{(j)} = \frac{\partial^2 W^{(j)}}{\partial y^2},\ \sigma_y^{(j)} = \frac{\partial^2 W^{(j)}}{\partial x^2},\ \tau_{xy}^{(j)} = -\frac{\partial^2 W^{(j)}}{\partial y \partial x}, \tag{49}$$

where $W^{(j)}$ denotes the Airy function. Then, the equilibrium equations are satisfied and the compatibility equation becomes

$$c_{22}^{(j)}\frac{\partial^4 W^{(j)}}{\partial x^4} - 2c_{26}^{(j)}\frac{\partial^4 W^{(j)}}{\partial x^3 \partial y} + (2c_{12}^{(j)} + c_{66}^{(j)})\frac{\partial^4 W^{(j)}}{\partial x^2 \partial y^2} - 2c_{16}^{(j)}\frac{\partial^4 W^{(j)}}{\partial x \partial y^3} + c_{11}^{(j)}\frac{\partial^4 W^{(j)}}{\partial y^4} = 0. \tag{50}$$

We use the following representation for the function W [9]

$$W^{(j)} = 2Re[F_1^{(j)}(z_1^{(j)}) + F_2^{(j)}(z_2^{(j)})], \tag{51}$$

where $F_i^{(j)}$ are analytical functions of the complex argument $z_k^j = x + \mu_k^j y (k = 1, 2)$. The constants μ_k^j will be defined below.

We introduce the functions

$$\frac{dF_1^{(j)}(z)}{dz} = \varphi_1^{(j)}(z), \frac{dF_2^{(j)}(z)}{dz} = \varphi_2^{(j)}(z),$$
$$\frac{d\varphi_1^{(j)}(z)}{dz} = \Phi_1^{(j)}(z), \frac{d\varphi_2^{(j)}(z)}{dz} = \Phi_2^{(j)}(z).$$

Then, the stress components are calculated by the following formulae:

$$\sigma_x^{(j)} = 2Re[(\mu_1^{(j)})^2\Phi_1^{(j)}(z_1^{(j)}) + (\mu_2^{(j)})^2\Phi_2^{(j)}(z_2^{(j)})],$$
$$\sigma_y^{(j)} = 2Re[\Phi_1^{(j)}(z_1^{(j)}) + \Phi_2^{(j)}(z_2^{(j)})],$$
$$\tau_{xy}^{(j)} = -2Re[\mu_1^{(j)}\Phi_1^{(j)}(z_1^{(j)}) + \mu_2^{(j)}\Phi_2^{(j)}(z_2^{(j)})]. \tag{52}$$

The displacements components become

$$U^{(j)} = 2Re[p_1^{(j)}\varphi_1^{(j)}(z_1^{(j)}) + p_2^{(j)}\varphi_2^{(j)}(z_2^{(j)})],$$
$$V^{(j)} = 2Re[q_1^{(j)}\varphi_1^{(j)}(z_1^{(j)}) + q_2^{(j)}\varphi_2^{(j)}(z_2^{(j)})], \tag{53}$$

where $p_k^{(j)} = c_{11}^{(j)}(\mu_k^{(j)})^2 + c_{12}^{(j)} - c_{16}^{(j)}\mu_k^{(j)}$, $\mu_k^{(j)}q_k^{(j)} = c_{11}^{(j)}(\mu_k^{(j)})^2 + c_{22}^{(j)} - c_{26}^{(j)}\mu_k^{(j)}$.

The compatibility equation (50) with (52) and (53) yields

$$c_{11}^{(j)}\mu^4 - 2c_{16}^{(j)}\mu^3 + (2c_{12}^{(j)} + c_{66}^{(j)})\mu^2 - 2c_{26}^{(j)}\mu + c_{22}^{(j)} = 0. \tag{54}$$

As shown in [17] this equation has two pairs of complex conjugate roots.

Let a concentrated force be applied at a point $M_0(X_0, Y_0)$ of the domain D_j. Then, the complex potentials in a neighbourhood of this point become

$$\varphi_1^{(j)}(z_1^{(j)}) = a_0^{(j)}\ln(z_1^{(j)} - \tau_1^{(j)}) + \varphi_*^{(j)}(z_1^{(j)}), z_1^{(j)} \to \tau_1^{(j)},$$
$$\varphi_2^{(j)}(z_2^{(j)}) = b_0^{(j)}\ln(z_2^{(j)} - \tau_2^{(j)}) + \psi_*^{(j)}(z_2^{(j)}), z_2^{(j)} \to \tau_2^{(j)}, \tag{55}$$

where $\varphi_*^{(j)}(z_1^{(j)})$ and $\psi_*^{(j)}(z_2^{(j)})$ are holomorphic functions in a vicinity of the point M_0.

The coefficients $a_0^{(j)}, b_0^{(j)}$ are calculated by formulae [24]

$$a_0^{(j)} = \frac{i(X_0 + \mu_2^{(j)}Y_0) + m^{(j)} - n^{(j)}\mu_2^{(j)}}{4\pi(\mu_1^{(j)} - \mu_2^{(j)})},$$
$$b_0^{(j)} = -\frac{i(X_0 + \mu_1^{(j)}Y_0) + m^{(j)} - n^{(j)}\mu_1^{(j)}}{4\pi(\mu_1^{(j)} - \mu_2^{(j)})}, \tag{56}$$

where $m^{(j)} = \frac{k_0^{(j)}(\delta_1^{(j)}X_0 - \delta_3^{(j)}Y_0)}{(\delta_1^{(j)})^2 + \delta_2^{(j)}\delta_3^{(j)}}$, $n^{(j)} = \frac{k_0^{(j)}(\delta_2^{(j)}X_0 + \delta_1^{(j)}Y_0)}{(\delta_1^{(j)})^2 + \delta_2^{(j)}\delta_3^{(j)}}$, $\delta_1^{(j)} = Im[\mu_1^{(j)}\mu_2^{(j)}]$, $\delta_2^{(j)} = Im[\mu_1^{(j)} + \mu_2^{(j)}]$, $\delta_3^{(j)} = Im[(\mu_1^{(j)} + \mu_2^{(j)})\overline{\mu_1^{(j)}\mu_2^{(j)}}]$, $k_0^{(j)} = Re[\mu_1^{(j)}\mu_2^{(j)}] - \frac{c_{12}^{(j)}}{c_{11}^{(j)}}$.

3.2.1 Fundamental Solution

We determine the stress–strain state of the described body loaded by a concentrated force P applied at the point $M_0(x_0, y_0)$. It is assumed that the condition of the ideal contact on the line $y = 0$ takes place

$$\sigma_y^{(1)} = \sigma_y^{(2)},\ \tau_{xy}^{(1)} = \tau_{xy}^{(2)},\ U^{(1)} = U^{(2)},\ V^{(1)} = V^{(2)} \tag{57}$$

and all the stresses vanish at infinity.

The following stress functions are used:

$$\begin{aligned}
\Phi_1^{(1)}(z_1^{(1)}) &= \frac{s_1}{z_1^{(1)} - \tau_1^{(2)}} + \frac{s_2}{z_1^{(1)} - \tau_2^{(2)}},\\
\Phi_2^{(1)}(z_2^{(1)}) &= \frac{l_1}{z_2^{(2)} - \tau_1^{(2)}} + \frac{l_2}{z_2^{(2)} - \tau_2^{(2)}},\\
\Phi_1^{(2)}(z_1^{(2)}) &= \frac{a_0^{(2)}}{z_1^{(2)} - \tau_1^{(2)}} + \frac{n_1}{z_1^{(2)} - \overline{\tau_1^{(2)}}} + \frac{n_2}{z_1^{(2)} - \overline{\tau_2^{(2)}}},\\
\Phi_2^{(2)}(z_2^{(2)}) &= \frac{b_0^{(2)}}{z_2^{(2)} - \tau_2^{(2)}} + \frac{m_1}{z_2^{(2)} - \overline{\tau_1^{(2)}}} + \frac{m_2}{z_2^{(2)} - \overline{\tau_2^{(2)}}},
\end{aligned} \tag{58}$$

where s_1, s_2, l_1, l_2, n_1, n_2, m_1, and m_2 are arbitrary coefficients. The coefficients s_i, l_i, n_i, and m_i are defined by the equations (57). Consider the case when $\sigma_y^{(1)} = \sigma_y^{(2)}$. For $y = 0$, we have

$$\sigma_y^{(1)} = 2Re\left[\frac{s_1}{x - \tau_1^{(2)}} + \frac{s_2}{x - \tau_2^{(2)}} + \frac{l_1}{x - \tau_1^{(2)}} + \frac{l_2}{x - \tau_2^{(2)}}\right],$$

$$\sigma_y^{(2)} = Re\left[\frac{a_0^{(2)}}{x - \tau_1^{(2)}} + \frac{n_1}{x - \overline{\tau_1^{(2)}}} + \frac{n_2}{x - \overline{\tau_2^{(2)}}} + \frac{b_0^{(2)}}{x - \tau_2^{(2)}} + \frac{m_1}{x - \overline{\tau_1^{(2)}}} + \frac{m_2}{x - \overline{\tau_2^{(2)}}}\right].$$

Comparing the coefficients at $\frac{1}{1-\tau_1^2}$ and $\frac{1}{1-\tau_2^2}$ we obtain, respectively

$$s_1 + l_1 - \overline{n_1} - \overline{m_1} = a_0^{(2)}, \quad s_2 + l_2 - \overline{n_2} - \overline{m_2} = b_0^{(2)}.$$

Thus, the coefficients satisfy the following system of equations:

$$\begin{aligned}
s_1 + l_1 - \overline{n_1} - \overline{m_1} &= a_0^{(2)},\\
\mu_1^{(1)} s_1 + \mu_2^{(1)} l_1 - \overline{\mu_1^{(2)}}\,\overline{n_1} - \overline{\mu_2^{(2)}}\,\overline{m_1} &= \mu_1^{(2)} a_0^{(2)},\\
p_1^{(1)} s_1 + p_2^{(1)} l_1 - \overline{p_1^{(2)}}\,\overline{n_1} - \overline{p_2^{(2)}}\,\overline{m_1} &= p_1^{(2)} a_0^{(2)},\\
q_1^{(1)} s_1 + q_2^{(1)} l_1 - \overline{q_1^{(2)}}\,\overline{n_1} - \overline{q_2^{(2)}}\,\overline{m_1} &= q_1^{(2)} a_0^{(2)},
\end{aligned} \tag{59}$$

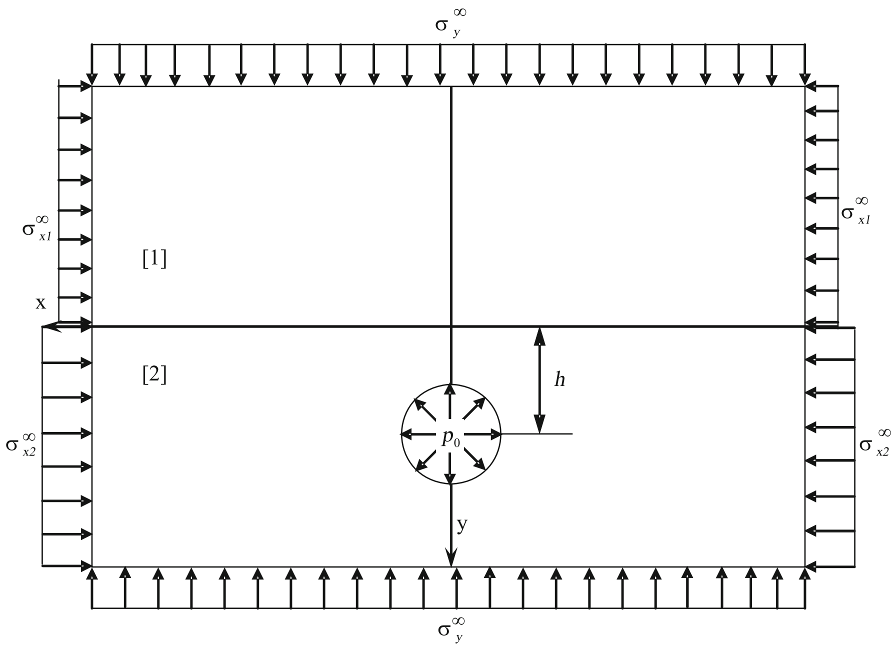


Fig. 3 Scheme of model problem with circular hole in an orthotropic sectionally homogeneous plane

$$
\begin{aligned}
s_2 + l_2 - \overline{n_2} - \overline{m_2} &= b_0^{(2)}, \\
\mu_1^{(1)} s_2 + \mu_2^{(1)} l_2 - \overline{\mu_1^{(2)}}\,\overline{n_2} - \overline{\mu_2^{(2)}}\,\overline{m_2} &= \mu_2^{(2)} b_0^{(2)}, \\
p_1^{(1)} s_2 + p_2^{(1)} l_2 - \overline{p_1^{(2)}}\,\overline{n_2} - \overline{p_2^{(2)}}\,\overline{m_2} &= p_2^{(2)} b_0^{(2)}, \\
q_1^{(1)} s_2 + q_2^{(1)} l_2 - \overline{q_1^{(2)}}\,\overline{n_2} - \overline{q_2^{(2)}}\,\overline{m_2} &= q_2^{(2)} b_0^{(2)}.
\end{aligned}
\tag{60}
$$

Let this system be solved. Then, the stress functions would be given by (58) and the stresses would be given by (52) and (53). These solutions were also obtained by other methods [4, 8, 10, 27].

3.2.2 Example

We consider sectionally homogeneous infinite media with a circular hole when the surface homogeneous pressure is applied as shown in the Fig. 3. It is assumed that the stresses at infinity take the following values:

$$\sigma_{xj}^{\infty}, \quad \sigma_y^{\infty} = \rho g H, \quad \tau_{xy}^{\infty} = 0.$$

The complex potentials have the following asymptotic far away from the contact surface:

$$\Phi_1^{(j)}(z_1) = \Gamma_1, \quad \Phi_2^{(j)}(z_2) = \Gamma_2,$$

where $\Gamma_i^{(j)}$ are constants to be defined.

The equations (60) yield the following system of equations:

$$\begin{aligned} 2Re[(\mu_1^{(j)})^2\Gamma_1 + (\mu_2^{(j)})^2\Gamma_2] &= \sigma_{xj}^{\infty}, \\ 2Re[\Gamma_1 + \Gamma_2] &= \sigma_y^{\infty}, \\ 2Re[\mu_1^{(j)}\Gamma_1 + \mu_2^{(j)}\Gamma_2] &= 0. \end{aligned} \tag{61}$$

The last equation of (61) holds if

$$2(\mu_1^{(j)}\Gamma_1 + \mu_2^{(j)}\Gamma_2) = ir_0^{(j)},$$

where $r_0^{(j)}$ is an arbitrary real constant. The system (61) can be easily solved and

$$\begin{aligned} \Gamma_1 &= i(r_0^{(1)}\mu_2^{(2)} - r_0^{(2)}\mu_2^{(1)})/\Delta, \\ \Gamma_2 &= i(r_0^{(2)}\mu_1^{(1)} - r_0^{(1)}\mu_1^{(2)})/\Delta, \end{aligned} \tag{62}$$

where $\Delta = \mu_1^{(1)}\mu_2^{(2)} - \mu_2^{(1)}\mu_1^{(2)}$. Substitution of (62) into the first equation of (61) yields

$$\begin{aligned} \delta_1 r_0^{(1)} + \delta_2 r_0^{(2)} &= -\frac{\sigma_{x_1}^{\infty}}{2}, \\ \delta_3 r_0^{(1)} + \delta_4 r_0^{(2)} &= -\frac{\sigma_{x_2}^{\infty}}{2}, \end{aligned} \tag{63}$$

where $\delta_1 = Im[((\mu_1^{(1)})^2\mu_2^{(2)} - (\mu_2^{(1)})^2\mu_1^{(2)})/\Delta]$, $\delta_2 = Im[(\mu_1^{(1)}\mu_2^{(1)}(\mu_2^{(1)} - \mu_1^{(1)})/\Delta]$, $\delta_3 = Im[(\mu_1^{(2)}\mu_2^{(2)}(\mu_1^{(2)} - \mu_2^{(2)})/\Delta]$, $\delta_4 = Im[((\mu_2^{(2)})^2\mu_1^{(1)} - (\mu_1^{(2)})^2\mu_2^{(1)})/\Delta]$.

Therefore,

$$r_0^{(1)} = \frac{\delta_1\sigma_{x_2}^{\infty} - \delta_4\sigma_{x_1}^{\infty}}{2(\delta_1\delta_4 - \delta_2\delta_3)}, r_0^{(2)} = \frac{\delta_3\sigma_{x_1}^{\infty} - \delta_1\sigma_{x_2}^{\infty}}{2(\delta_1\delta_4 - \delta_2\delta_3)}.$$

The second equation of (61) yields

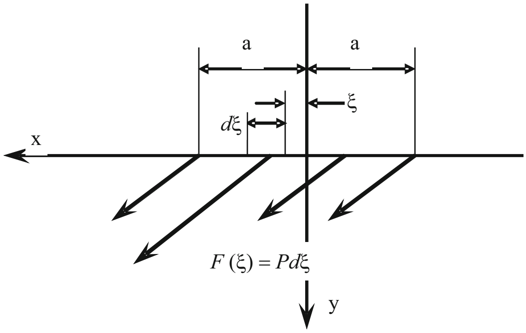
$$(\delta_4\delta_5 - \delta_3\delta_6)\sigma_{x_1}^{\infty} + (\delta_1\delta_6 - \delta_2\delta_5)\sigma_{x_2}^{\infty} = (\delta_1\delta_4 - \delta_2\delta_3)\sigma_y^{\infty}, \tag{64}$$

where

$$\delta_5 = Im[(\mu_2^{(2)} - \mu_1^{(2)})/\Delta], \ \delta_6 = Im[(\mu_1^{(1)} - \mu_2^{(1)})/\Delta].$$

Let the condition $\sigma_{x1}^{\infty} = \sigma_{x2}^{\infty}$ hold at infinity. Then the value σ_x^{∞} is defined through σ_y^{∞}.

Fig. 4 Constant force acting on the segment



We will solve the source problem by the superposition principle for the following two stress states. The first one is the stress state in the media without a hole and the second one is characterized by the vanishing stresses at infinity and by the following boundary conditions on the hole surface:

$$\sigma_s = (-p_0 + \sigma_{x_2}^{\infty})\cos(n, y) - (\sigma_y^{\infty} - p_0)\cos(n, x),$$
$$\sigma_n = (-p_0 + \sigma_{x_2}^{\infty})\cos^2(n, x) + (\sigma_y^{\infty} - p_0)\cos^2(n, y). \tag{65}$$

Then, the full stresses are defined by formulae at the upper and lower half planes

$$\sigma_{x_2}^{(1)} = -\sigma_{x_1}^{\infty} + \sigma_x^{(1)}, \sigma_{y_2}^{(1)} = -\sigma_y^{\infty} + \sigma_y^{(1)}, \tau_{xy_2}^{(1)} = \tau_{xy}^{(1)},$$
$$\sigma_{x_2}^{(2)} = -\sigma_{x_2}^{\infty} + \sigma_x^{(2)}, \sigma_{y_2}^{(2)} = -\sigma_y^{\infty} + \sigma_y^{(2)}, \tau_{xy_2}^{(2)} = \tau_{xy}^{(2)}. \tag{66}$$

3.2.3 Method of Unknown Loads and its Numerical Realization

We consider a problem of the load uniformly distributed on the segment $|x| \leq a$ as shown in the Fig. 4. Let it be solved by the method presented in the previous section. Then, we define the stresses near the point (X_0, Y_0) as the functions of (x, y)

$$\sigma_x^{(j)} = X_0 A_x^{(j)}(x, y) + Y_0 B_x^{(j)}(x, y),$$
$$\sigma_y^{(j)} = X_0 A_y^{(j)}(x, y) + Y_0 B_y^{(j)}(x, y),$$
$$\tau_{xy}^{(j)} = X_0 A_{xy}^{(j)}(x, y) + Y_0 B_{xy}^{(j)}(x, y). \tag{67}$$

The following expressions for the stresses take place on the segment

$$\sigma_x^{(j)} = P_{X_0} I A_x^{(j)}(x, y) + P_{Y_0} I B_x^{(j)}(x, y),$$
$$\sigma_y^{(j)} = P_{X_0} I A_y^{(j)}(x, y) + P_{Y_0} I B_y^{(j)}(x, y),$$
$$\tau_{xy}^{(j)} = P_{X_0} I A_{xy}^{(j)}(x, y) + P_{Y_0} I B_{xy}^{(j)}(x, y), \tag{68}$$

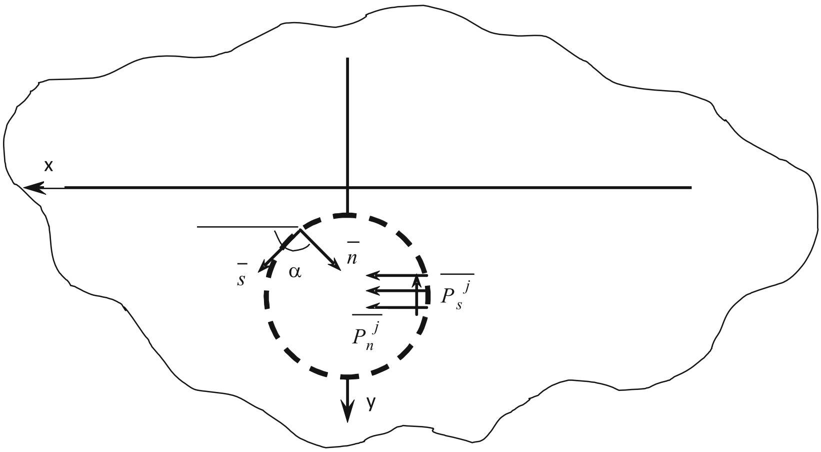


Fig. 5 Scheme of model problem solving

where

$$IA_x(j)(x,y) = \int_{-a}^{a} A_x^{(j)}(x-\xi,y)d\xi, \quad IA_y(j)(x,y) = \int_{-a}^{a} A_y^{(j)}(x-\xi,y)d\xi,$$

$$IB_x^{(j)}(x,y) = \int_{-a}^{a} B_x^{(j)}(x-\xi,y)d\xi, \quad IB_y^{(j)}(x,y) = \int_{-a}^{a} B_y^{(j)}(x-\xi,y)d\xi$$

$$IA_{xy}^{(j)}(x,y) = \int_{-a}^{a} A_{xy}^{(j)}(x-\xi,y)d\xi, \quad IB_{xy}^{(j)}(x,y) = \int_{-a}^{a} B_{xy}^{(j)}(x-\xi,y)d\xi,$$

$$P_{X_0} = \int_{-a}^{a} X_0 d\xi, \quad P_{Y_0} = \int_{-a}^{a} Y_0 d\xi.$$

The method of solution near the circular hole (see Fig. 5) can be presented as follows. First, the circle is divided onto N segments. Unknown constant shear and normal loads P_s^j and P_n^j are applied (to each small segment). Using (67) and (68) we can calculate the stresses at the middle points of each segment

$$\sigma_s^i = \Sigma_{k=1}^N A_{ss}^{ik} P_s^k + \Sigma_{k=1}^N A_{sn}^{ik} P_n^k,$$
$$\sigma_n^i = \Sigma_{k=1}^N A_{ns}^{ik} P_s^k + \Sigma_{k=1}^N A_{nn}^{ik} P_n^k, i = \overline{1,N}. \tag{69}$$

The values P_n^j and P_s^j can be found from the conditions at the centres of each element. As a result we obtain the following system of equations:

$$(-p_0 + \sigma_{x_2}^\infty)\cos(n,y) - (\sigma_y^\infty - p_0)\cos(n,x) = \Sigma_{k=1}^N A_{ss}^{ik} P_s^k + \Sigma_{k=1}^N A_{sn}^{ik} P_n^k,$$
$$(-p_0 + \sigma_{x_2}^\infty)\cos^2(n,x) + (\sigma_y^\infty - p_0)\cos^2(n,y) = \Sigma_{k=1}^N A_{ns}^{ik} P_s^k + \Sigma_{k=1}^N A_{nn}^{ik} P_n^k,$$
$$i = \overline{1,N}. \tag{70}$$

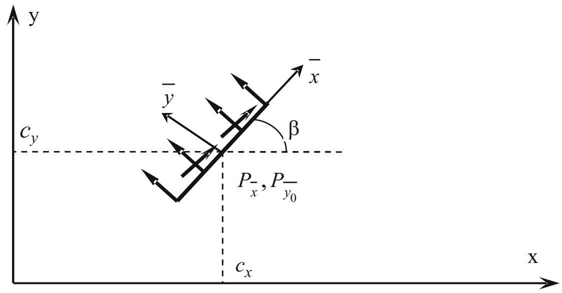


Fig. 6 Stresses caused by load of arbitrary orientation

Consider an example of the distributed force on the segment in the local coordinate system $(\overline{x}, \overline{y})$ shown in Fig. 6. The segment is defined by equations: $|\overline{x}| \leq a, y = 0$. The coordinate systems are related by equations

$$\overline{x} = (x - c_x)\cos(\beta) + (y - c_y)\sin(\beta), \overline{y} = -(x - c_x)\sin(\beta) + (y - c_y)\cos(\beta). \tag{71}$$

The stresses in the global coordinates have the form

$$\begin{aligned} \sigma_x &= \sigma_{\overline{x}}\cos^2(\beta) - 2\tau_{\overline{xy}}\sin(\beta)\cos(\beta) + \sigma_{\overline{y}}\sin^2(\beta), \\ \sigma_y &= \sigma_{\overline{x}}\sin^2(\beta) + 2\tau_{\overline{xy}}\sin(\beta)\cos(\beta) + \sigma_{\overline{y}}\cos^2(\beta), \\ \tau_{xy} &= (\sigma_{\overline{x}} - \sigma_{\overline{y}})\sin(\beta)\cos(\beta) + \tau_{\overline{xy}}\cos(2\beta). \end{aligned} \tag{72}$$

Moreover, we have

$$\begin{aligned} \sigma_x^{(j)} = {} & P_{\overline{X_0}}(IA_x^{(j)}(\overline{x},\overline{y})\cos^2(\beta) - 2IA_{xy}^{(j)}(\overline{x},\overline{y})\cos(\beta)\sin(\beta) + \\ & IA_y^{(j)}(\overline{x},\overline{y})\sin^2(\beta)) + P_{\overline{Y_0}}(IB_x^{(j)}(\overline{x},\overline{y})\cos^2(\beta) - \\ & 2IB_{xy}^{(j)}(\overline{x},\overline{y})\cos(\beta)\sin(\beta) + IB_y^{(j)}(\overline{x},\overline{y})\sin^2(\beta)), \\ \sigma_y^{(j)} = {} & P_{\overline{X_0}}(IA_x^{(j)}(\overline{x},\overline{y})\sin^2(\beta) + 2IA_{xy}^{(j)}(\overline{x},\overline{y})\cos(\beta)\sin(\beta) + \\ & IA_y^{(j)}(\overline{x},\overline{y})\cos^2(\beta)) + P_{\overline{Y_0}}(IB_x^{(j)}(\overline{x},\overline{y})\sin^2(\beta) + \\ & 2IB_{xy}^{(j)}(\overline{x},\overline{y})\cos(\beta)\sin(\beta) + IB_y^{(j)}(\overline{x},\overline{y})\cos^2(\beta)), \\ \tau_{xy}^{(j)} = {} & P_{\overline{X_0}}((IA_x^{(j)}(\overline{x},\overline{y}) - IA_y^{(j)}(\overline{x},\overline{y}))\sin(\beta)\cos(\beta) + \\ & IA_{xy}^{(j)}(\overline{x},\overline{y})(\cos^2(\beta) - \sin^2(\beta))) + P_{\overline{Y_0}}((IB_x^{(j)}(\overline{x},\overline{y}) - \\ & IB_y^{(j)}(\overline{x},\overline{y}))\sin(\beta)\cos(\beta) + IB_{xy}^{(j)}(\overline{x},\overline{y})(\cos^2(\beta) - \sin^2(\beta))). \end{aligned} \tag{73}$$

In order to obtain the influence coefficients $IA_{xy}^{(j)}, IA_{xy}^{(j)}, \ldots$ we choose the point (x, y) as the centre of the jth element. The scheme for boundary elements is shown in Fig. 7. The local coordinates of the ith point relative to the jth point have the form

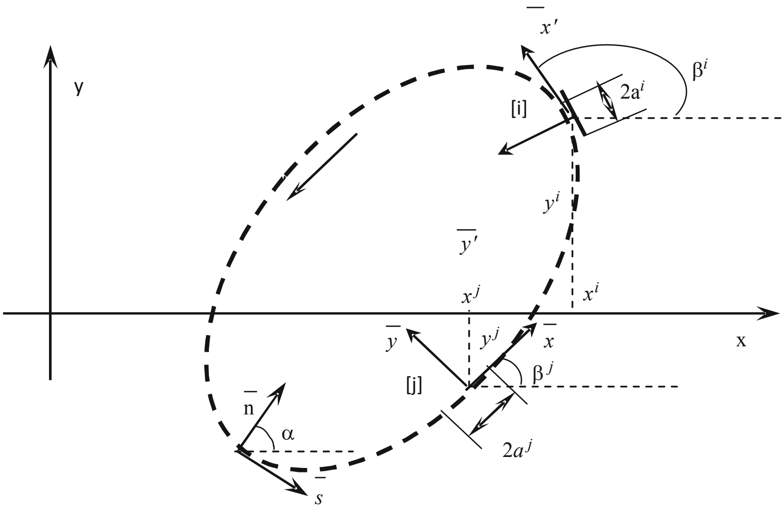


Fig. 7 Scheme for boundary elements

$$\begin{aligned}\overline{x} &= (x^i - x^j)\cos(\beta^j) + (y^i - y^j)\sin(\beta^j),\\ \overline{y} &= -(x^i - x^j)\sin(\beta^j) + (y^i - y^j)\cos(\beta^j)\end{aligned} \tag{74}$$

The stress components at the ith point relative to the jth point can be obtained by (68)

$$\begin{aligned}\sigma_{\overline{x}}^{(k)i} &= P_{\overline{x}}^{j} I A_{x}^{(k)}(\overline{x},\overline{y}) + P_{\overline{y}}^{j} I B_{x}^{(k)}(\overline{x},\overline{y}),\\ \sigma_{\overline{y}}^{(k)i} &= P_{\overline{x}}^{j} I A_{y}^{(k)}(\overline{x},\overline{y}) + P_{\overline{y}}^{j} I B_{y}^{(k)}(\overline{x},\overline{y}),\\ \tau_{\overline{xy}}^{(k)i} &= P_{\overline{x}}^{j} I A_{xy}^{(k)}(\overline{x},\overline{y}) + P_{\overline{y}}^{j} I B_{xy}^{(k)}(\overline{x},\overline{y}),\end{aligned} \tag{75}$$

where k is the number of the half-plane; i, j elements numbers. Ultimately, we have

$$\begin{aligned}\sigma_n^{i(k)} = {} & P_s^j (I A_x^{(k)}(\overline{x},\overline{y})\sin^2(\gamma) - I A_{xy}^{(k)}(\overline{x},\overline{y})\sin(2\gamma) + I A_y^{(k)}(\overline{x},\overline{y})\cos^2(\gamma)) + \\ & P_n^j (I B_x^{(k)}(\overline{x},\overline{y})\sin^2(\gamma) - I B_{xy}^{(k)}(\overline{x},\overline{y})\sin(2\gamma) + I B_y^{(k)}(\overline{x},\overline{y})\cos^2(\gamma)),\\ \sigma_s^{i(k)} = {} & P_s^j ((I A_y^{(j)}(\overline{x},\overline{y}) - I A_x^{(j)}(\overline{x},\overline{y}))\frac{\sin(2\gamma)}{2} + I A_{xy}^{(j)}(\overline{x},\overline{y})\cos(2\gamma)) + \\ & P_n^j ((I B_y^{(j)}(\overline{x},\overline{y}) - I B_x^{(j)}(\overline{x},\overline{y}))\frac{\sin(2\gamma)}{2} + I B_{xy}^{(j)}(\overline{x},\overline{y})\cos(2\gamma)),\end{aligned} \tag{76}$$

where $\gamma = \beta_i - \beta_j$. So we can find the influence coefficients expressed through P_s^j and P_n^j in (76). Substituting them in (70) we arrive at a linear system. After its solution the stress–strain state can be explicitly determined.

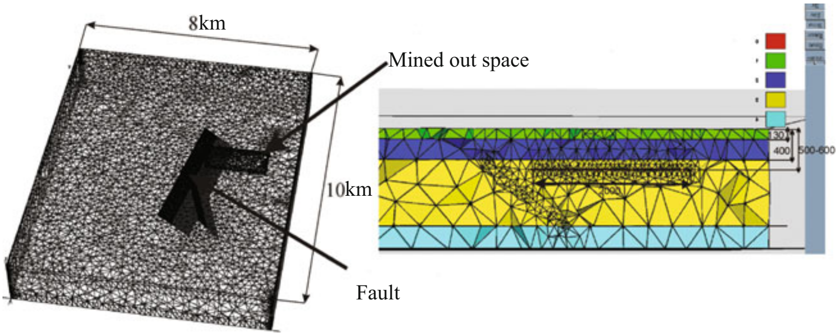


Fig. 8 Four layered massive

3.3 *Three-dimensional Models of Conjugative Processes in Porous Media by Use of the Finite Element Method*

We consider transversely an isotropic viscoelastic four-layered massif with intersection faults. There is also a mined out space. The scheme of the problem is displayed in Fig. 8. The second layer from the top has porous liquid in its skeleton. We investigate the flow in the massif skeleton when a mined external space is moved to the fault. The problem is described by the following equation:

1) Equilibrium equations with the fluid pressure have the form

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - \frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} - \frac{\partial p}{\partial y} = 0,$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - \frac{\partial p}{\partial z} + \rho g = 0.$$

2) Storage equation with the pressure terms [25]

$$\frac{\partial p}{\partial t} = a\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) - \alpha_p \frac{\partial}{\partial t}\left(\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}\right),$$

where $a = \frac{k(1+\varepsilon)}{\gamma(a_\nu + \varepsilon\beta)}$, $\alpha_p = \frac{a_\nu}{a_\nu + \varepsilon\beta}$, ε is the porosity coefficient, β is the fluid compressibility, a_ν is the rock hardening coefficient, k is the filtration coefficient, and t is time.

3) The physical law yields

$$\sigma_{xx} = \frac{1 - \nu_{pz}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{xx} + \frac{\nu_p + \nu_{zp}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{yy} + \frac{\nu_{zp} + \nu_{zp}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{zz} + 2D(\dot{\varepsilon}_{xx} - \dot{\varepsilon}_0),$$

$$\sigma_{yy} = \frac{\nu_p + \nu_{zp}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{xx} + \frac{1 - \nu_{pz}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{yy} + \frac{\nu_{zp} + \nu_{zp}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{zz} + 2D(\dot{\varepsilon}_{yy} - \dot{\varepsilon}_0),$$

$$\sigma_{zz} = \frac{\nu_{zp} + \nu_{zp}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{xx} + \frac{\nu_{zp} + \nu_{zp}\nu_{zp}}{E_p E_z \Delta}\varepsilon_{yy} + \frac{1-\nu_p^2}{E_p^2 \Delta}\varepsilon_{zz} + 2D(\dot{\varepsilon}_{zz} - \dot{\varepsilon}_0),$$

$$\sigma_{xz} = 2G_{zp}\varepsilon_{xz} + 2D\dot{\varepsilon}_{xz}, \sigma_{yz} = 2G_{zp}\varepsilon_{yz} + 2D\dot{\varepsilon}_{yz}, \sigma_{xy} = \frac{E_p}{1+\nu_p}\varepsilon_{xy} + 2D\dot{\varepsilon}_{xy},$$

where E_p, and ν_p are Young's modulus and Poisson's ratio respectively in the horizontal plane; E_z, ν_{zp}, and G_{zp} are Young's modulus, Poisson's ratio, and the shear modulus in the vertical plane; $\nu_{pz} = \frac{E_p}{E_z}\nu_{zp}$;$\Delta = \frac{(1+\nu_p)(1-2\nu_{zp}\nu pz)}{E_p^2 Ez}$; and D is the viscosity coefficient. Here, the Kelvin model is used.

4) Compatibility equations

$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y},$$

$$\frac{\partial^2\varepsilon_{xx}}{\partial z^2} + \frac{\partial^2\varepsilon_{zz}}{\partial x^2} = 2\frac{\partial^2\varepsilon_{xz}}{\partial x\partial z},$$

$$\frac{\partial^2\varepsilon_{zz}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial z^2} = 2\frac{\partial^2\varepsilon_{yz}}{\partial z\partial y},$$

$$\frac{\partial^2\varepsilon_{xx}}{\partial y\partial z} - \frac{\partial^2\varepsilon_{xy}}{\partial x\partial z} - \frac{\partial^2\varepsilon_{xz}}{\partial x\partial y} + \frac{\partial^2\varepsilon_{yz}}{\partial x^2} = 0,$$

$$\frac{\partial^2\varepsilon_{zz}}{\partial x\partial y} - \frac{\partial^2\varepsilon_{xz}}{\partial y\partial z} - \frac{\partial^2\varepsilon_{yz}}{\partial x\partial z} + \frac{\partial^2\varepsilon_{xy}}{\partial z^2} = 0,$$

$$\frac{\partial^2\varepsilon_{yy}}{\partial x\partial z} - \frac{\partial^2\varepsilon_{xy}}{\partial y\partial z} - \frac{\partial^2\varepsilon_{yz}}{\partial x\partial y} + \frac{\partial^2\varepsilon_{xz}}{\partial y^2} = 0.$$

5) Cauchy's relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

6) Boundary and initial conditions:
   a) On the left and right edges, i.e. with $y = Y_s$, $y = Y_n$: $\sigma_{yz} = \sigma_{xy} = 0, u_y = 0, p = 0$
   b) On the front and back edges, i.e. with $x = X_w$, $x = X_e$: $\sigma_{xz} = \sigma_{yx} = 0, u_x = 0, p = 0$
   c) At the bottom: $u_x = u_y = u_z = 0$
   d) At the upper surface: $\sigma_{xz} = \sigma_{yz} = \sigma_{zz}$
   e) At the boundary of the water layer with the massif–impermeability condition $\frac{\partial p}{\partial n} = 0$
   f) $p = 0$ for $t = 0$
   g) Contact conditions on the faults surfaces

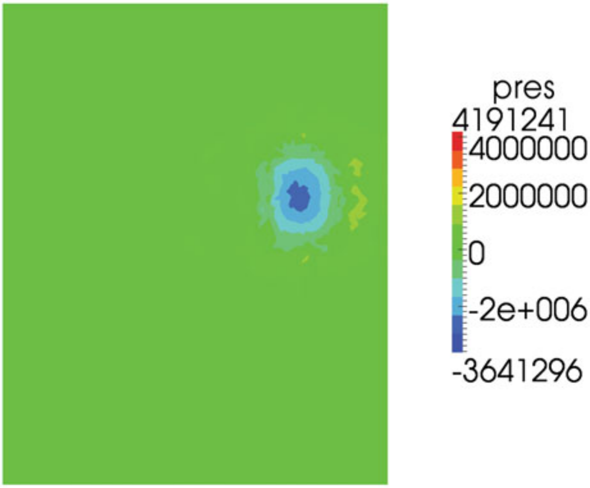


Fig. 9 Pressure distribution in horizontal plane after third step of deleting elements

$$
\begin{aligned}
\sigma_{n1} &= \sigma_{n2}, \\
u_{n1} &= u_{n2}, \\
\sigma_{\tau 1} = \sigma_{\tau 2}, \ |\sigma_\tau| &< f\sigma_n, \\
u_{\tau 1} = u_{\tau 2}, \ |\sigma_\tau| &< f\sigma_n, \\
\sigma_{\tau 1} = \sigma_{\tau 2} = f\sigma_n, \ |\sigma_\tau| &> f\sigma_n.
\end{aligned}
$$

Here, σ_n and σ_τ are the normal and shear stresses and f is the friction coefficient. We are interested only in the additional pressure, which is caused by mining works. Hence, we impose vanishing boundary and initial conditions for the pressure.

The problem is solved by a finite element package using the following scheme:

1) Calculate the initial stress–strain state of the massif caused by gravity. Hydromechanical processes are not considered at this stage.
2) Step-by-step deletion of the elements which model the mined out space. Dimensions of the mined out space are $2000 \times 2000 \times 10$ m^3. And on each step we delete elements of size $l \times 2000 \times 10$ m^3. Hence, the mined out space is in movement to the fault (on the first step $l = 1200$ m and on the following steps $l = 200$ m). The velocity of movement is 1 km/year.
3) Calculation of the moment when the steady state of the massif is reached. We used the following as the physical, mechanical, and geometry parameters of the layers:
   First layer: $E_p = 0.3(GPa)$, $E_z = 1(GPa)$, $\nu_p = \nu_{zp} = 0.3$, $G_{zp} = 0.0577(GPa)$, $top = 0$, $bottom = -130$ m,
   Second layer: $E_p = 5(GPa)$, $E_z = 5(GPa)$, $\nu_p = \nu_{zp} = 0.3$, $G_{zp} = 0.288(GPa)$, $top = -130$, $bottom = -400$ m,
   Third layer: $E_p = 14(GPa)$, $E_z = 14(GPa)$, $\nu_p = \nu_{zp} = 0.3$, $G_{zp} = 0.8(GPa)$, $top = -400$, $bottom = -1800$ m,

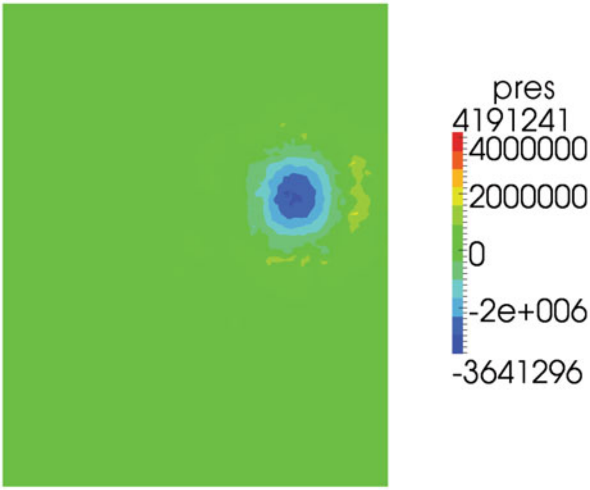


Fig. 10 Pressure distribution in horizontal plane after fifth step of deleting elements

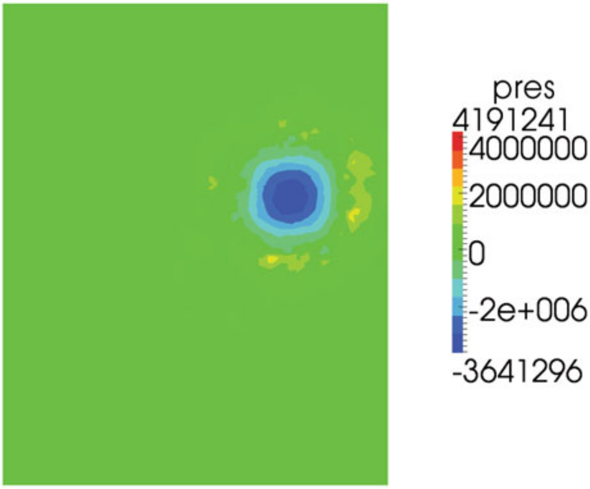


Fig. 11 Pressure distribution in horizontal plane after last step of deleting elements

Fourth layer: $E_p = 14(GPa)$, $E_z = 14(GPa)$, $\nu_p = \nu_{zp} = 0.3$, $G_{zp} = 0.8(GPa)$, $top = -1800$, $bottom = -2200$ m.

The fluid properties of the second layer are expressed by the parameters

$$k = 10^{-9}\,(m/s),\ \beta = 10^{-10}\,(Pa^{-1}),\ \varepsilon = 0.11,\ a_\nu = 10^{-9}\,(Pa^{-1})$$

Distribution of fluid pressure is shown In Figs. 9, 10, 11, 12, 13 and 14 at the horizontal plane on the depth 250 m and at the vertical plane the middle of the mined out space perpendicular to the x direction.

Remark: The fluid pressure has the same sign as the stresses, hence, at the compressible pressure is negative.

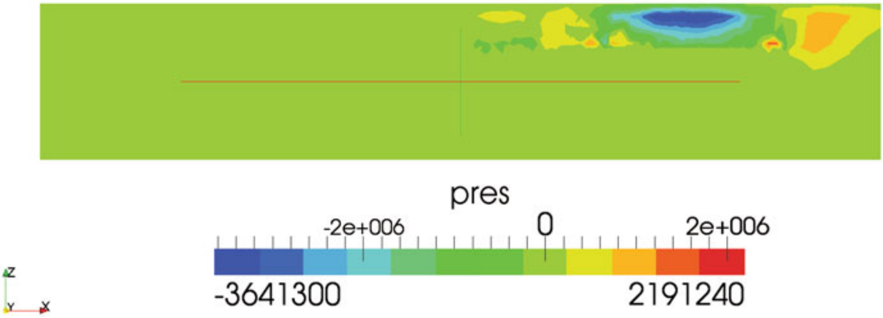


Fig. 12 Pressure distribution in vertical plane in the third step of deleting elements

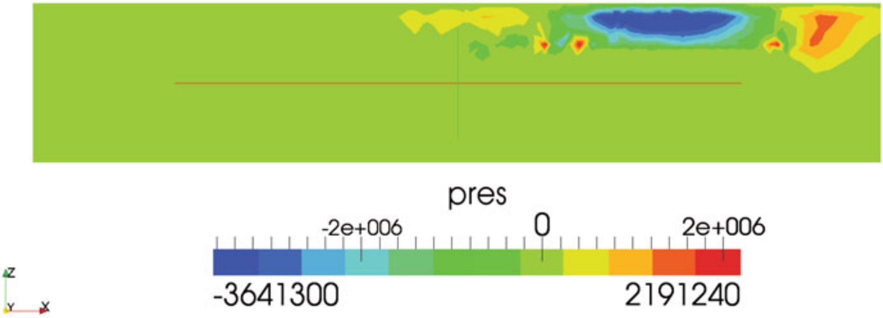


Fig. 13 Pressure distribution in horizontal plane after fifth step of deleting elements

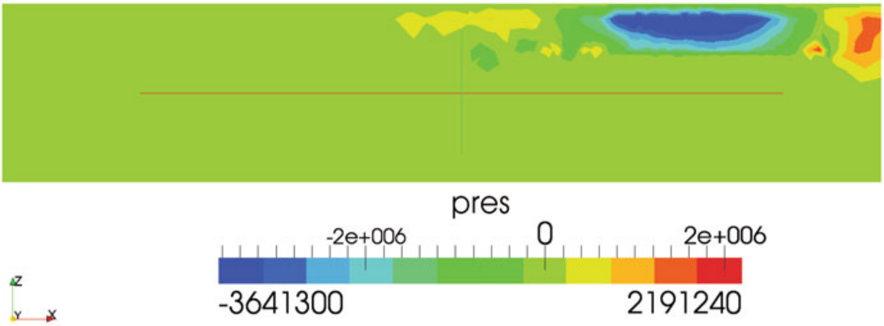


Fig. 14 Pressure distribution in horizontal plane after last step of deleting elements

One can see from the graphics that the water layer is in zones with high horizontal stresses, because large sizes of the mined out space of compressible horizontal stresses are greater than the tension of vertical stresses. Therefore, the additional pressure of fluid is positive in the usual sense. We can also see the influence of the

fault on the pressure distribution because the pressure distribution is different on both sides of the fault.

The problem is solved by use of the finite difference approximation in time; a finite element method is used at each time step for spacial variables. The package $Tochnog^{©}$ is used for implementation of the finite element method.

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A Note on the Functions that Are Approximately p-Wright Affine

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Mathematics Subject Classification (2010) 39B52, 39B82.

Abstract Let W be a Banach space, $(V, +)$ be a commutative group, p be an endomorphism of V, and $\overline{p} : V \to V$ be defined by $\overline{p}(x) := x - p(x)$ for $x \in V$. We present some results on the Hyers–Ulam type stability for the following functional equation

$$f(p(x) + \overline{p}(x)) + f(\overline{p}(x) + p(y)) = f(x) + f(y),$$

in the class of functions $f : V \to W$.

Keywords Hyers–Ulam stability · p-Wright affine function · Polynomial function

1 Introduction

Let $0 < p < 1$ be a fixed real number and P be a nonempty subset of a real linear space X. Assume that P is p-convex, i.e., $px + (1 - p)y \in P$ for $x, y \in P$. We say that a function f mapping P into the set of reals $\mathbb{R}$ is p-Wright convex (see, e.g., [7, 8, 14, 17, 26]) if it satisfies the inequality

$$f(px + (1 - p)y) + f((1 - p)x + py) \le f(x) + f(y) \qquad x, y \in P. \tag{1}$$

Note that we obtain (1) by adding the following usual p-convexity inequality

$$f(px + (1 - p)y) \le pf(x) + (1 - p)f(y) \qquad x, y \in P \tag{2}$$

to its corresponding version (with x and y interchanged)

$$f(py + (1 - p)x) \le pf(y) + (1 - p)f(x) \qquad x, y \in P. \tag{3}$$

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Analogously, we say that $g : P \to \mathbb{R}$ is p-Wright concave provided the subsequent inequality holds:

$$f(px + (1-p)y) + f((1-p)x + py) \geq f(x) + f(y) \qquad x, y \in P.$$

The functions that are simultaneously p-Wright convex and p-Wright concave, i.e., satisfy the functional equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y), \tag{4}$$

are called p-Wright affine (see [7]).

Note that for $p = 1/2$, Eq. (4) is just the well-known Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

If $p = 1/3$, then Eq. (4) can be written in the form

$$f(x + 2y) + f(2x + y) = f(3x) + f(3y). \tag{5}$$

Solutions and stability of the latter equation have been investigated in [16] (cf. [5]) in connection with a generalized notion of the Jordan derivations on Banach algebras. Solutions and stability of Eq. (4), for more arbitrary p, have been studied in [4, 6, 7] (see also [13, 23]). (For further information and references on stability of functional equations, we refer to, e.g., [3, 10, 11, 15, 18–22, 25]). In particular, the following results have been obtained in [4] ($\mathbb{C}$ denotes the set of complex numbers).

Theorem 1 *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a Banach space, $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^k + |1-p|^k < 1$, and $g : X \to Y$ satisfy*

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq A(\|x\|^k + \|y\|^k) \tag{6}$$

for all $x, y \in X$. Then there is a unique solution $G : X \to Y$ of Eq. (4) *with*

$$\|g(x) - G(x)\| \leq \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k} \qquad x \in X. \tag{7}$$

Theorem 2 *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a Banach space, $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^{2k} + |1-p|^{2k} < 1$, and $g : X \to Y$ satisfy*

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq A\|x\|^k\|y\|^k$$

for all $x, y \in X$. Then g is a solution to (4).

In this chapter, we complement these two theorems by considering the inequality

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq \delta \qquad x, y \in X \tag{8}$$

with a fixed positive real δ. In particular, we also obtain a description of solutions to (4).

Note that if we write $\overline{p} := 1 - p$, then Eq. (4) can be rewritten as follows:

$$f(px + \overline{p}y) + f(\overline{p}x + py) = f(x) + f(y). \tag{9}$$

We use this form of (4) in the sequel. Moreover, we consider a generalization of it with p and $\overline{p}$ being suitable functions, using the notions $px := p(x)$ and $\overline{p}x := px - x$ $(x \in X)$ for simplicity.

Actually, some results in such situation can be derived from [23]. Namely, from [23, Theorem 2] we can deduce the following.

Theorem 3 *Let $\delta \in (0, \infty)$, $(X, +)$ be a commutative group, $p : X \to \mathbb{X}$ be additive (i.e., $p(x + y) = p(x) + p(y)$ for $x, y \in X$), $\overline{p}(X) = p(X)$, and $g : X \to \mathbb{C}$ satisfy*

$$|g(px + (1 - p)y) + g((1 - p)x + py) - g(x) - g(y)| \leq \delta \qquad x, y \in X$$

for all $x, y \in X$. Then there is a solution $G : X \to \mathbb{C}$ of Eq. (4) *with*

$$\sup_{x \in X} |g(x) - G(x)| < \infty. \tag{10}$$

In this chapter, we provide a bit more precise estimations than (10), though we apply reasonings similar to those in [23].

2 Auxiliary Information and Lemmas

Let us start with a result that follows easily from [2, 24] (cf. [9]). We need for it the notion of the Fréchet difference operator. Let us recall that for a function f, mapping a semigroup $(S, +)$ into a group $(G, +)$,

$$\Delta_y f(x) = \Delta_y^1 f(x) := f(x + y) - f(x) \qquad x, y \in S,$$

$$\Delta_{t,z}^2 := \Delta_t \circ \Delta_z, \qquad \Delta_t^2 := \Delta_{t,t}^2 \qquad t, z \in S,$$

$$\Delta_{t,u,z}^3 := \Delta_t \circ \Delta_u \circ \Delta_z, \qquad \Delta_t^3 := \Delta_{t,t,t}^3 \qquad t, u, z \in S.$$

It is easy to check that

$$\Delta_{t,z}^2 f(x) = f(x + t + z) - f(x + t) - f(x + z) + f(x) \qquad x, t, z \in S,$$

$$\Delta_{t,z,u}^3 f(x) = f(x + t + z + u) - f(x + t + z) - f(x + t + u) - f(x + z + u) + f(x + t) + f(x + z) + f(x + u) - f(x) \qquad x, t, z, u \in S.$$

We refer to [12] for more information and further references concerning this subject. From [2, Theorem 4] (cf. [10, Theorem 7.6]) and [24, Theorem 9.1] we can easily derive the following proposition.

Proposition 1 *Let W be a normed space, $(V,+)$ be a commutative group, $\varepsilon \geq 0$, and $G : V \to W$ satisfy the inequality*

$$\|(\Delta_y^3 G)(x)\| \leq \varepsilon \qquad x, y \in V. \tag{11}$$

Assume that one of the following two hypotheses is valid.

(a) $\varepsilon = 0$.
(b) W is complete and V is divisible by 6 (i.e., for each $x \in V$, there is $y \in V$ with $x = 6y$).

Then there exist a constant $c \in W$, an additive mapping $a : V \to W$, and a symmetric biadditive mapping $b : V^2 \to W$ such that

$$\|G(x) - b(x,x) - a(x) - c\| \leq \frac{2\varepsilon}{3} \qquad x \in V.$$

Let us now recall two more stability results (see, e.g., [10, p. 13 and Theorem 3.1]).

Lemma 1 *Let $(V,+)$ be a commutative group, W be a Banach space, $\varepsilon \geq 0$, and $g : V \to W$ satisfy the inequality*

$$\|g(x+y) - g(x) - g(y)\| \leq \varepsilon \qquad x, y \in V.$$

Then there exists the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} g(2^n x) \qquad x \in V \tag{12}$$

and the function $A : V \to W$, defined in this way, is additive and

$$\|g(x) - A(x)\| \leq \varepsilon \qquad x \in V.$$

Lemma 2 *Let $(V,+)$ be a commutative group, W be a Banach space, $\varepsilon \geq 0$, and $g : V \to W$ satisfy the inequality*

$$\|g(x+y) + g(x-y) - 2\,g(x) - 2\,g(y)\| \leq \varepsilon \qquad x, y \in V.$$

Then there exists the limit

$$b(x) = \lim_{n\to\infty} 4^{-n} g(2^n x) \qquad x \in V \tag{13}$$

and the function $b : V \to W$, defined in this way, is quadratic and fulfills the inequality

$$\|g(x) - b(x)\| \leq \frac{\varepsilon}{2} \qquad x \in V.$$

In what follows, given a function p mapping a group $(V,+)$ into itself, for the sake of simplicity we write,

$$px := p(x), \qquad \overline{p}x := x - px \qquad x \in V.$$

The next proposition will be very useful in the proofs of our main results.

Lemma 3 *Let $(V, +)$ be a commutative group, $\varepsilon \geq 0$, $p : V \to V$ be a homomorphism with $p(V) = \overline{p}(V)$, and W be a normed space. Assume that $g : V \to W$ satisfies the inequality*

$$\|g(px + \overline{p}y) + g(\overline{p}x + py) - g(x) - g(y)\| \leq \epsilon \qquad x, y \in V. \tag{14}$$

Then the following two statements are valid.

(i) *If g is odd, then $\|\Delta^2_{z,u} g(x)\| \leq 4\varepsilon$ for $x, z, u \in V$.*

(ii) $\|\Delta^3_{t,u,z}\, g(x)\| \leq 8\varepsilon$ *for* $x, z, u, t \in V$.

Proof This proof is patterned on some reasonings from [23].

Take $z \in V$. There exists $w \in V$ with $pw = -\overline{p}z$, because $p(V) = \overline{p}(V)$ is a subgroup of V. Note that

$$\overline{p}(x + z) + p(y + w) = \overline{p}x + py \qquad x, y \in V,$$

whence replacing x by $x + z$ and y by $y + w$ in (14), we get

$$\begin{aligned} \|g(px &+ \overline{p}y + pz + \overline{p}w) + g(\overline{p}x + py) \\ &- g(x + z) - g(y + w)\| \leq \varepsilon \qquad x, y \in V. \end{aligned} \tag{15}$$

Now, (14) and (15) yield

$$\begin{aligned} \|g(x + z) &- g(x) - g(px + \overline{p}y + pz + \overline{p}w) \\ &+ g(px + \overline{p}y) + g(y + w) - g(y)\| \\ &\leq \|g(px + \overline{p}y + pz + \overline{p}w) + g(\overline{p}x + py) - g(x + z) - g(y + w)\| \\ &\quad + \|g(px + \overline{p}y) + g(\overline{p}x + py) - g(x) - g(y)\| \leq 2\varepsilon \qquad x, y \in V. \end{aligned} \tag{16}$$

Take $u \in V$. Analogously as before, we deduce that there is $v \in V$ with $\overline{p}v = -pu$. Clearly

$$p(x + u) + \overline{p}(y + v) = px + \overline{p}y \qquad x, y \in V.$$

Hence, replacing x by $x + u$ and y by $y + v$ in (16), we have

$$\begin{aligned} \|g(x + u + z) &- g(x + u) - g(px + \overline{p}y + pz + \overline{p}w) + g(px + \overline{p}y) \\ &+ g(y + w + v) - g(y + v)\| \leq 2\varepsilon \qquad x, y \in V. \end{aligned} \tag{17}$$

It is easily seen that (16) and (17) imply

$$\begin{aligned} \|g(x + u + z) &- g(x + u) - g(x + z) + g(x) \\ &+ g(y + w + v) - g(y + w) - g(y + v) + g(y)\| \\ &\leq \|g(px + \overline{p}y + pz + \overline{p}w) - g(px + \overline{p}y) \end{aligned} \tag{18}$$

$$\begin{aligned}
&- g(x+z) - g(y+w) + g(x) + g(y)\| \\
&+ \|g(px + \overline{p}y + pz + \overline{p}w) - g(px + \overline{p}y) \\
&- g(x+u+z) - g(y+w+v) \\
&+ g(x+u) + g(y+v)\| \le 4\varepsilon \qquad x, y \in V,
\end{aligned}$$

which with x replaced by $x+t$ yields

$$\begin{aligned}
\|g(x+t+u+z) &- g(x+t+u) - g(x+t+z) + g(x+t) + g(y+w+v) \\
&- g(y+w) - g(y+v) + g(y)\| \le 4\varepsilon \qquad t, x, y \in V.
\end{aligned}$$

Combining (18) and the latter inequality, we get statement (ii).

For the proof of (i), observe that (18) with x replaced by $-x-z-u$, under the assumption of the oddness of g, brings

$$\begin{aligned}
\| - g(x) &+ g(x+z) + g(x+u) - g(x+z+u) \\
&+ g(y+w+v) - g(y+w) - g(y+v) + g(y)\| \le 4\varepsilon \qquad x, y \in V,
\end{aligned} \tag{19}$$

whence and from (18) we have

$$\|2\, g(x) - 2\, g(x+z) - 2\, g(x+u) + 2\, g(x+z+u)\| \le 8\varepsilon \qquad x, y \in V. \tag{20}$$

This yields statement (i). □

The next corollary provides a description of solutions to (9), which will be useful in the sequel.

Corollary 1 *Let V and W be as in Proposition 1 and $p : V \to V$ be a homomorphism with $p(V) = \overline{p}(V)$. Then $f : V \to W$ satisfies* Eq. (9) *if and only if there exist $c \in W$, an additive $a : V \to W$ and a biadditive and symmetric $L : V^2 \to W$ such that*

$$f(x) = L(x, x) + a(x) + c \qquad x \in V, \tag{21}$$

$$L(px, \overline{p}x) = 0 \qquad x \in V. \tag{22}$$

Proof Let $f : V \to W$ be a solution of Eq. (9). Then (14) holds with $\varepsilon = 0$. Consequently, according to Lemma 3 (ii),

$$(\Delta_y^3 f)(x) = 0 \qquad x, y \in V.$$

Hence, on account of Proposition 1, there exist $c \in W$, an additive $a : V \to W$, and a quadratic $b : V \to W$ such that $f(x) = b(x) + a(x) + c$ for $x \in V$. Further, it is well known (see, e.g., [1]) that there exists a symmetric biadditive $L : V^2 \to W$ such that $b(x) = L(x, x)$ for $x \in V$, whence (21) holds. Now, it is easily seen that (9) (with $y = 0$) and (21) yield

$$L(px, px) + L(\overline{p}x, \overline{p}x) = L(x, x) \qquad x \in V$$

and consequently

$$-2\,L(px,\overline{p}x) = L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x) = 0 \qquad x \in V, \tag{23}$$

which gives (22).

The converse is a routine task. □

We need yet the following very simple lemma.

Lemma 4 *Let* $(V,+)$ *be a commutative group,* W *be a normed space,* $a, a_0 : V \to W$ *be additive,* $L, L_0 : V^2 \to W$ *be biadditive,* $c \in W$ *and*

$$M := \sup_{x\in V} \|a_0(x) - a(x) + L_0(x,x) - L(x,x) + c\| < \infty. \tag{24}$$

Then $a = a_0$ *and* $L = L_0$.

Proof That proof is actually a routine by now, but we present it here for the convenience of readers.

Note that

$$\|L_0(x,x) - L(x,x)\| \le \|a(x) - a_0(x)\| + \|c\| + M \qquad x \in V,$$

whence

$$\begin{aligned}\|L(x,x) - L_0(x,x)\| &= n^{-2}\|L(nx,nx) - L_0(nx,nx)\|\\ &\le n^{-2}(\|a(nx) - a_0(nx)\| + \|c\| + M)\\ &= n^{-1}\|a(x) - a_0(x)\| + n^{-2}(\|c\| + M) \qquad x \in V, n \in \mathbb{N},\end{aligned}$$

which yields $L = L_0$. Hence, by (24),

$$\begin{aligned}\|a(x) - a_0(x)\| &= n^{-1}\|a(nx) - a_0(nx)\|\\ &\le n^{-1}(\|c\| + M) \qquad x \in V, n \in \mathbb{N},\end{aligned}$$

and consequently $a = a_0$. □

3 The Main Stability Results

We start with two theorems describing odd and even solutions of functional inequality (14). They will help us to obtain the main result of the chapter (but they seem to be interesting, as well).

Theorem 4 *Let* $(V,+)$ *be a commutative group,* $\epsilon \ge 0$, $p : V \to V$ *be a homomorphism,* $p(V) = \overline{p}(V)$, *and* W *be a Banach space. Assume that* $g : V \to W$ *is odd and satisfies the inequality*

$$\|g(px + \overline{p}y) + g(\overline{p}x + py) - g(x) - g(y)\| \le \varepsilon \qquad x, y \in V. \tag{25}$$

Then there exists a unique additive function, $A : V \to W$*, such that*

$$\|g(x) - A(x)\| \le 4\varepsilon \qquad x \in V. \tag{26}$$

Moreover, (12) *holds and for every solution* $h : V \to W$ *of* (9) *such that*

$$\sup_{x \in V} \|g(x) - h(x)\| < \infty,$$

the function $A - h$ *is constant.*

Proof According to Lemma 3 (i),

$$\|g(x + z + u) - g(x + z) - g(x + u) + g(x)\| \le 4\varepsilon \qquad x, z, u \in V,$$

which with $x = 0$ yields

$$\|g(z + u) - g(z) - g(u)\| \le 4\varepsilon \qquad z, u \in V.$$

Hence Lemma 1 implies the existence and the form of A. It remains to show the statements on the uniqueness of A.

So, suppose that $A_0 : V \to W$ is additive and

$$\sup_{x \in V} \|g(x) - A_0(x)\| \le 4\varepsilon.$$

Then

$$\sup_{x \in V} \|A(x) - A_0(x)\| \le 8\varepsilon,$$

which implies that $A = A_0$.

Now, let $h : V \to W$ be a solution of (9) such that

$$\sup_{x \in V} \|g(x) - h(x)\| < \infty.$$

Then

$$M := \sup_{x \in V} \|A(x) - h(x)\| < \infty.$$

Further, by Corollary 1, $h(x) = a(x) + L(x, x) + c$ with some $c \in W$, an additive $a : V \to W$, and a biadditive and symmetric $L : V^2 \to W$. So, Lemma 4 implies that

$$L(x, x) = 0 \qquad x \in V$$

and $A = a$. □

Theorem 5 *Let* $(V, +)$ *be a commutative group,* $\varepsilon \ge 0$*,* $p : V \to V$ *be a homomorphism,* $p(V) = \overline{p}(V)$*, and* W *be a Banach space. Assume that* $g : V \to W$ *is even and satisfies inequality* (25)*. Then there exists a unique biadditive and symmetric mapping* $L : V^2 \to W$ *such that*

$$\|L(x, x) - g(x) + g(0)\| \le 4\varepsilon \qquad x \in V. \tag{27}$$

Moreover, (22) *holds,*

$$L(x,x) = \lim_{n\to\infty} 4^{-n} g(2^n x) \qquad x \in V \tag{28}$$

and, for every solution $h : V \to W$ *of* (9) *with*

$$\sup_{x\in V} \|g(x) - h(x)\| < \infty,$$

there is $c \in W$ *such that* $h(x) = L(x,x) + c$ *for* $x \in V$.

Proof Let $g_0 := g - g(0)$. Then g_0 fulfills (25) as well. According to Lemma 3 (ii),

$$\begin{aligned}\|g_0(x+t+z+u) &- g_0(x+t+z) - g_0(x+t+u) - g_0(x+z+u)\\ &+ g_0(x+t) + g_0(x+z) + g_0(x+u) - g_0(x)\| \le 8\varepsilon \qquad x,t,z,u \in S,\end{aligned}$$

whence (with $x = 0$ and $u = -t$) we obtain

$$\begin{aligned}\|g_0(z) - g_0(t+z) - g_0(0) - g_0(z-t) + g_0(t) + g_0(z) + g_0(-t) - g_0(0)\|\\ \le 8\varepsilon \qquad t,u,z \in V\end{aligned}$$

and consequently

$$\|2g_0(z) - g_0(t+z) - g_0(z-t) + 2g_0(t)\| \le 8\varepsilon \qquad t,z \in V.$$

Hence Lemma 2 implies the existence of L and (28).

Now we show that (22) holds. Clearly, (25) (with $y = 0$) yields

$$\|g(px) + g(\overline{p}x) - g(x) - g(0)\| \le \varepsilon \qquad x \in V.$$

So, (27) implies that

$$\begin{aligned}\|L(px,px) &+ L(\overline{p}x,\overline{p}x) - L(x,x)\|\\ &\le \|L(px,px) + g(0) - g(px)\|\\ &\quad + \|L(\overline{p}x,\overline{p}x) + g(0) - g(\overline{p}x)\|\\ &\quad + \|g(x) - L(x,x) - g(0)\|\\ &\quad + \|g(px) + g(\overline{p}x) - g(x) - g(0)\| \le 13\varepsilon \qquad x \in V.\end{aligned} \tag{29}$$

Since b is biadditive and it is very easy to check that

$$-2\,L(px,\overline{p}x) = L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x) \qquad x \in V,$$

from (29), we get

$$\begin{aligned}2k^2\|L(px,\overline{p}x)\| &= \|L(pkx,pkx) + L(\overline{p}kx,\overline{p}kx) - L(kx,kx)\|\\ &\le 13\varepsilon \qquad x \in V, k \in \mathbb{N},\end{aligned} \tag{30}$$

which means that (22) holds.

It remains to show the statements on the uniqueness of L. So, first suppose that $L_0 : V^2 \to W$ is symmetric, biadditive, and

$$\sup_{x\in V} \|L_0(x,x) - g(x) + g(0)\| \le 4\varepsilon.$$

Then

$$\sup_{x\in V} \|L_0(x,x) - L(x,x)\| \le 8\varepsilon,$$

whence from Lemma 4 we deduce that $L_0 = L$.

Now, assume that $h : V \to W$ is a solution of (9) with

$$\sup_{x\in V} \|g(x) - h(x)\| < \infty.$$

This implies that

$$M := \sup_{x\in V} \|L(x,x) - h(x)\| < \infty.$$

Further, according to Corollary 1,

$$h(x) = a(x) + S(x,x) + c \qquad x \in V$$

with some $c \in W$, an additive $a : V \to W$, and a biadditive and symmetric $S : V^2 \to W$. Clearly, by Lemma 4, $L = S$ and $a(x) = 0$ for every $x \in V$. Hence

$$h(x) = L(x,x)+c \qquad x \in V. \qquad \square$$

In what follows, given a function g mapping a group $(V,+)$ into a real linear space W, by g_o and g_e, we denote the odd and even parts of g, i.e.,

$$g_o(x) := \frac{g(x) - g(-x)}{2} \qquad x \in V,$$

$$g_e(x) := \frac{g(x) + g(-x)}{2} \qquad x \in V.$$

The next theorem is the main result in this chapter.

Theorem 6 *Let $(V,+)$ be a commutative group, $p : V \to V$ be a homomorphism such that $p(V) = \overline{p}(V)$, W be a Banach space, $\varepsilon \ge 0$ and $g : V \to W$ satisfy inequality (25). Then there exist a unique additive function $a : V \to W$ and a unique biadditive function $L : V^2 \to W$ such that*

$$\|g(x) - a(x) - L(x,x) - g(0)\| \le 8\varepsilon \qquad x \in V. \tag{31}$$

Moreover, (22) holds,

$$a(x) = \lim_{n\to\infty} 2^{-n} g_o(2^n x), \qquad L(x,x) = \lim_{n\to\infty} 4^{-n} g_e(2^n x) \qquad x \in V \tag{32}$$

and, for every solution $h : V \to W$ of (9) with

$$\sup_{x \in V} \|g(x) - h(x)\| < \infty, \tag{33}$$

there is $c \in W$ such that $h(x) = a(x) + L(x,x) + c$ for $x \in V$.

If V is divisible by 6, then there exists $c_0 \in W$ with

$$\|g(x) - a(x) - L(x,x) - c_0\| \le \frac{16\varepsilon}{3} \qquad x \in V. \tag{34}$$

Proof It is easily seen that g_o and g_e satisfy inequalities analogous to (25). So, by Theorems 4 and 5, there exist an additive function $a : V \to W$ and a symmetric biadditive function $L : V^2 \to W$ such that

$$\|g_o(x) - a(x)\| \le 4\varepsilon, \qquad \|g_e(x) - L(x,x) - g(0)\| \le 4\varepsilon \qquad x \in V. \tag{35}$$

Moreover, (32) holds and, clearly,

$$\begin{aligned} \|g(x) - a(x) - L(x,x) - g(0)\| &\le \|g_o(x) - a(x)\| \\ &+ \|g_e(x) - L(x,x) - g(0)\| \le 8\varepsilon \qquad x \in V. \end{aligned} \tag{36}$$

Further, (25) (with $y = 0$) yields

$$\|g_e(px) + g_e(\overline{p}x) - g_e(x) - g(0)\| \le \varepsilon \qquad x \in V.$$

Hence analogous to (29), from (35) we derive that

$$\|L(px,px) + L(\overline{p}x,\overline{p}x) - L(x,x)\| \le 13\varepsilon \qquad x \in V, \tag{37}$$

whence (30) holds, which implies (22).

For the proof of uniqueness of a and L, suppose that $a_0 : V \to W$ is additive, $L_0 : V^2 \to W$ is biadditive, and

$$\|g(x) - a_0(x) - L_0(x,x) - g(0)\| \le 8\varepsilon \qquad x \in V. \tag{38}$$

Then

$$\|a_0(x) - a(x) - L_0(x,x) - L(x,x)\| \le 16\varepsilon \qquad x \in V \tag{39}$$

and consequently, by Lemma 4, $L = L_0$ and $a = a_0$.

Now, let $h : V \to W$ be a solution of (9) fulfilling condition (33). Then, in view of (31),

$$M := \sup_{x \in V} \|a(x) + L(x,x) + g(0) - h(x)\| < \infty \tag{40}$$

and, according to Corollary 1, $h(x) = a_0(x) + L_0(x,x) + c$ with some $c \in W$, an additive $a_0 : V \to W$ and a biadditive and symmetric $L_0 : V^2 \to W$. Hence, again

Lemma 4 implies that $L = L_0$ and $a = a_0$. Consequently $h(x) = L(x, x) + a(x) + c$ for $x \in V$.

Finally assume that V is divisible by 6. Then, in view of Lemma 3 (ii), we have

$$\|(\Delta_y^3 g)(x)\| \leq 8\varepsilon \qquad x, y \in V.$$

Further, by Proposition 1, there are $c_0 \in W$, an additive $a_0 : V \to W$ and a biadditive and symmetric $b_0 : V^2 \to W$ such that

$$\|g(x) - b_0(x, x) - a_0(x) - c\| \leq \frac{16}{3}\varepsilon \qquad x \in V. \tag{41}$$

In view of (31) and Lemma 4, we must have $a_0 = a$ and $L_0 = L$. □

For some discussions on a special case of condition (22), we refer to [7] (see also [6, 8, 13]).

Remark 1 There arises natural questions whether (under reasonable suitable assumptions) we can get some better estimations than in (31) and (34) and whether the assumption of divisibility of V by 6 is necessary to get (34). Also, it would be interesting to know if we can have $c_0 = g(0)$ in (34).

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Multiplicative Ostrowski and Trapezoid Inequalities

Pietro Cerone, Sever S. Dragomir and Eder Kikianty

Abstract We introduce the multiplicative Ostrowski and trapezoid inequalities, that is, providing bounds for the comparison of a function f and its integral mean in the following sense:

$$f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \text{ and } f(b)^{\frac{b-x}{b-a}}f(a)^{\frac{x-a}{b-a}}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right].$$

We consider the cases of absolutely continuous and logarithmic convex functions. We apply these inequalities to provide approximations for the integral of f; and the first moment of f around zero, that is, $\int_a^b xf(x)dx$; for an absolutely continuous function f on $[a,b]$.

Keywords Ostrowski inequality · Trapezoid inequality · Logarithmic convex function

1 Introduction

Comparison between functions and integral means is incorporated in Ostrowski type inequalities. The first result in this direction is due to Ostrowski [27].

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Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with the property that $|f'(t)| \le M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M, \quad x \in [a,b]. \quad (1)$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

More inequalities of Ostrowski type have been generalised for functions which are not necessarily differentiable, namely, absolutely continuous, Hölder continuous, and convex functions. We refer to Sect. 2 for the details of these inequalities.

Inequalities providing upper bounds for the quantity

$$\left| \frac{(x-a) f(a) + (b-x) f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\,dt \right|, \qquad x \in [a,b] \quad (2)$$

are known in the literature as *generalized trapezoid inequalities*. It has been shown in Dragomir [7] (cf. [6]) that

$$\left| \frac{(x-a) f(a) + (b-x) f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \quad (3)$$

for any $x \in [a,b]$, provided that f is of bounded variation on $[a,b]$. In particular, we have the *trapezoid inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{2} \bigvee_a^b (f). \quad (4)$$

The constant $\frac{1}{2}$ is the best possible. The trapezoid inequalities have also been developed for other types of functions, such as absolutely continuous and convex functions. We refer to Sect. 2 for the details of these inequalities.

Motivated by the above results, we intend to develop the Ostrowski and trapezoid inequalities. In particular, we are interested in the multiplicative Ostrowski and trapezoid inequalities, that is, providing bounds for the comparison of a function f and its integral mean in the following sense:

$$f(x) \exp\left[-\frac{1}{b-a} \int_a^b \log f(t)\,dt \right] \text{ and } f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \exp\left[-\frac{1}{b-a} \int_a^b \log f(t)\,dt \right].$$

We summarise the results concerning absolutely continuous functions and logarithmic convex functions in Sect. 3. In Sect. 4, we apply these inequalities to provide approximations for the integral of f and the first moment of f around zero, that is,

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b x f(x)\,dx$$

for an absolutely continuous function f on $[a,b]$.

2 Results Concerning the Ostrowski and Trapezoid Inequalities

This section serves as a reference point for the developments of the Ostrowski and trapezoid inequalities. Readers who are familiar with these developments may skip this section.

We start with the Ostrowski type inequalities. The following results for absolutely continuous functions hold (cf. [19–21]).

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty, & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha+1} + \left(\frac{b-x}{b-a}\right)^{\alpha+1}\right]^{\frac{1}{\alpha}}(b-a)^{\frac{1}{\alpha}}\left\|f'\right\|_\beta, & \text{if } f' \in L_\beta[a,b] \\ & \frac{1}{\alpha}+\frac{1}{\beta}=1, \alpha>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_1; & \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, that is,

$$\|g\|_{[a,b],\infty} := ess\sup_{t\in[a,b]}|g(t)| \text{ and } \|g\|_{[a,b],r} := \left(\int_a^b |g(t)|^r dt\right)^{\frac{1}{r}},\ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from Fink's result [23]. If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (cf. Dragomir et al. [22] and the references therein for earlier contributions):

Theorem 3 *Let $f : [a,b] \to \mathbb{R}$ be of $r-H-$ Hölder type, that is,*

$$|f(x) - f(y)| \leq H|x-y|^r, \text{ for all } x, y \in [a,b], \tag{5}$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then, for all $x \in [a,b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{6}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, that is, f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitz continuous functions (with constant $L > 0$) (cf. [8]):

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) L, \tag{7}$$

where $x \in [a, b]$. Here the constant $\frac{1}{4}$ is also best possible.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (cf. [11]).

Theorem 4 *Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{8}$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we further assume that f is monotonically increasing, then the inequality (8) may be improved in the following manner [9] (cf. [18]).

Theorem 5 *Let $f : [a, b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:*

$$\begin{aligned}
&\left| f(x) - \tfrac{1}{b-a} \textstyle\int_a^b f(t) dt \right| \\
&\leq \tfrac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \textstyle\int_a^b \operatorname{sgn}(t-x) f(t) dt \right\} \\
&\leq \tfrac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \} \\
&\leq \left[\tfrac{1}{2} + \left| \tfrac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)].
\end{aligned} \tag{9}$$

All the inequalities in (9) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [13]:

Theorem 6 *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in (a, b)$ one has the inequality*

$$\begin{aligned}
&\frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \\
&\leq \int_a^b f(t)\, dt - (b-a) f(x) \\
&\leq \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].
\end{aligned} \tag{10}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x = a$ or $x = b$.

For other Ostrowski's type inequalities for the Lebesgue integral, we refer to Anastassiou [1], Cerone and Dragomir [2, 4], Cerone, Dragomir and Roumeliotis [5], and Dragomir [8, 9, 16]. Inequalities for the Riemann–Stieltjes integral may be found in Dragomir [10, 12]; while the generalization for isotonic functionals was provided in Dragomir [15]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph by Dragomir [17].

Now we recall the results concerning the trapezoid type inequalities. If f is absolutely continuous on $[a,b]$, then (see [3], p. 93)

$$\left|\frac{(x-a)\,f(a)+(b-x)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_{[a,b],\infty}, & \text{if} f'\in L_\infty[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p}, & \\ \qquad \text{if} f'\in L_p[a,b], \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_{[a,b],1}, & \end{cases} \tag{11}$$

for any $x\in[a,b]$. Here, $\|\cdot\|_{[a,b],p}$ are the usual Lebesgue norms.

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \frac{1}{4}(b-a)\left\|f'\right\|_\infty, & \text{if} f'\in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\left\|f'\right\|_p, & \text{if} f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\left\|f'\right\|_1. & \end{cases} \tag{12}$$

The constants $\frac{1}{4}$, $\frac{1}{2(q+1)^{1/q}}$ and $\frac{1}{2}$ are the best possible. Finally, for convex functions $f:[a,b]\to\mathbb{R}$, we have [14]

$$\begin{aligned} &\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\ &\leq (b-x)\,f(b)+(x-a)\,f(a)-\int_a^b f(t)\,dt \\ &\leq \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_-(a)\right] \end{aligned} \tag{13}$$

for any $x\in(a,b)$, provided that $f'_-(b)$ and $f'_+(a)$ are finite. As above, the second inequality also holds for $x=a$ and $x=b$ and the constant $\frac{1}{2}$ is the best possible in

both sides of (13). In particular, we have

$$\begin{aligned}&\frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\&\le\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\le\frac{1}{8}(b-a)\left[f'_-(b)-f'_-(a)\right].\end{aligned}\tag{14}$$

The constant $\frac{1}{8}$ is best possible in both inequalities. For other recent results on the trapezoid inequality, we refer to Dragomir [13], Kechriniotis and Assimakis [24], Liu [25], Mercer [26] and Ujevíc [28].

3 Results

We present our main results in this section. We start with the first of our main theorems.

Theorem 7 *Let $f:[a,b]\to(0,\infty)$ be an absolutely continuous function and $\gamma,\Gamma\in\mathbb{R}$ such that*

$$\gamma f(t)\le f'(t)\le\Gamma f(t),\quad \textit{for almost all } t\in[a,b].$$

Then, we have

$$\begin{aligned}\exp\left[\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}\right]&\le f(x)\exp\left[-\frac{1}{b-a}\int_a^b\log f(t)\,dt\right]\\&\le\exp\left[\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}\right],\end{aligned}\tag{15}$$

for any $x\in[a,b]$. In particular, we have

$$\begin{aligned}\exp\left[-\frac{1}{8}(\Gamma-\gamma)(b-a)\right]&\le f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b\log f(t)\,dt\right]\\&\le\exp\left[\frac{1}{8}(\Gamma-\gamma)(b-a)\right].\end{aligned}\tag{16}$$

The constant $\frac{1}{8}$ is best possible in (16).

Proof We use the Montgomery identity

$$g(x)-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\left[\int_a^x(t-a)g'(t)\,dt+\int_x^b(t-b)g'(t)\,dt\right]\tag{17}$$

where $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$. If we write (17) for the functions $g(t) = \log f(t)$, then we get

$$\begin{aligned} \log f(x) = &\ \frac{1}{b-a}\int_a^b \log f(t)\,dt \\ &+\frac{1}{b-a}\left[\int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt\right]. \end{aligned} \tag{18}$$

Taking the exponential of (18), and multiplying the result by $\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$, we have

$$\begin{aligned} & f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\ = &\ \exp\left\{\frac{1}{b-a}\left[\int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt\right]\right\} \end{aligned} \tag{19}$$

which can be considered as the *multiplicative Montgomery identity*. Now, since

$$\gamma \le \frac{f'(t)}{f(t)} \le \Gamma, \quad \text{for almost all } t \in [a,b],$$

it implies that

$$\gamma\int_a^x (t-a)\,dt \le \int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt \le \Gamma\int_a^x (t-a)\,dt,$$

which is equivalent to

$$\frac{1}{2}\gamma(x-a)^2 \le \int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt \le \frac{1}{2}\Gamma(x-a)^2. \tag{20}$$

Also,

$$\Gamma\int_x^b (t-b)\,dt \le \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt \le \gamma\int_x^b (t-b)\,dt,$$

which is equivalent to

$$-\frac{1}{2}\Gamma(b-x)^2 \le \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt \le -\frac{1}{2}\gamma(b-x)^2. \tag{21}$$

Adding inequalities (20) and (21) and dividing the resulted inequalities by $b-a > 0$ gives us

$$\begin{aligned} & \frac{1}{2(b-a)}\left[\gamma(x-a)^2 - \Gamma(b-x)^2\right] \\ & \le \frac{1}{b-a}\left[\int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt\right] \\ & \le \frac{1}{2(b-a)}\left[\Gamma(x-a)^2 - \gamma(b-x)^2\right], \end{aligned} \tag{22}$$

for $x \in [a, b]$. Utilising (19) and (22), we get (15); with (16) as its special case, that is, when $x = \frac{a+b}{2}$. The proof for the best possible constant is given in Remark 1 (via the sharpness of $\frac{1}{4}$ in (23)).

Remark 1 If $|f'(t)| \le Mf(t)$ for almost every $t \in [a, b]$, then by (15), for $\gamma = -M$ and $\Gamma = M$, we get

$$\begin{aligned}
&\exp\left[-\frac{M}{b-a}\left(\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right)\right]\\
&\le f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\\
&\le \exp\left[\frac{M}{b-a}\left(\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right)\right].
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\exp\left[-\frac{1}{4}M(b-a)\right] \le f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]&\\
\le \exp\left[\frac{1}{4}M(b-a)\right],& \qquad (23)
\end{aligned}$$

with $\frac{1}{4}$ as the best constant. To verify this, suppose that (23) holds for constants A, B instead of $-\frac{1}{4}$ and $\frac{1}{4}$, respectively, that is,

$$\exp\left[AM(b-a)\right] \le f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \qquad (24)$$

$$\exp\left[BM(b-a)\right] \ge f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]. \qquad (25)$$

Suppose in (24), $f(x) = \exp\left(|x - \frac{a+b}{2}|\right)$, thus $M = 1$, and we have

$$\exp\left[A(b-a)\right] \le \exp\left[-\frac{1}{4}(b-a)\right].$$

Since the exponential function is strictly increasing, we now have $A(b-a) \le -\frac{1}{4}(b-a)$; which asserts that $A \le -\frac{1}{4}$ since $a < b$. Now suppose in (25) that $f(x) = \exp\left(-|x - \frac{a+b}{2}|\right)$, again, $M = 1$ and we have

$$\exp\left[B(b-a)\right] \ge \exp\left[\frac{1}{4}(b-a)\right].$$

By similar arguments, we conclude that $B \ge \frac{1}{4}$.

We have the results for multiplicative trapezoid inequalities in the following.

Theorem 8 *Let $f : [a,b] \to (0,\infty)$ be an absolutely continuous function and $\gamma, \Gamma \in \mathbb{R}$ such that*

$$\gamma f(t) \le f'(t) \le \Gamma f(t), \quad \text{for almost all } t \in [a,b].$$

Then, we have

$$\begin{aligned}
&\exp\left[\frac{\gamma(b-x)^2 - \Gamma(x-a)^2}{2(b-a)}\right] \\
&\le f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
&\le \exp\left[\frac{\Gamma(b-x)^2 - \gamma(x-a)^2}{2(b-a)}\right],
\end{aligned} \tag{26}$$

for any $x \in [a,b]$. In particular, we have

$$\begin{aligned}
\exp\left[-\frac{1}{8}(\Gamma-\gamma)(b-a)\right] &\le \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
&\le \exp\left[\frac{1}{8}(\Gamma-\gamma)(b-a)\right].
\end{aligned} \tag{27}$$

The constant $\frac{1}{8}$ is best possible in (27).

Proof We use the generalised trapezoid identity

$$\frac{(b-x)g(b)+(x-a)g(a)}{b-a} - \frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\int_a^b (t-x)g'(t)\,dt, \tag{28}$$

that holds for any $x \in [a,b]$ and g an absolutely continuous function. If we write (28) for the function $g(t) = \log f(t)$, then we get

$$\begin{aligned}
&\frac{(b-x)\log f(b) + (x-a)\log f(a)}{b-a} - \frac{1}{b-a}\int_a^b \log f(t)\,dt \\
&= \frac{1}{b-a}\int_a^b (t-x)\frac{f'(t)}{f(t)}\,dt \\
&= \frac{1}{b-a}\left[\int_a^x (t-x)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-x)\frac{f'(t)}{f(t)}\,dt\right],
\end{aligned} \tag{29}$$

for $x \in [a,b]$. By taking the exponential of (29), we have

$$\begin{aligned}
&f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
&= \exp\left\{\frac{1}{b-a}\left[\int_a^x (t-x)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-x)\frac{f'(t)}{f(t)}\,dt\right]\right\},
\end{aligned} \tag{30}$$

for $x \in [a,b]$, which is the multiplicative generalised trapezoid identity. Similarly to the expositions in the proof of Theorem 7 and using the assumption that

$$\gamma \le \frac{f'(t)}{f(t)} \le \Gamma, \quad \text{for almost all } t \in [a,b],$$

we have

$$\begin{aligned}
&\frac{1}{2(b-a)}\left[\gamma(b-x)^2 - \Gamma(x-a)^2\right] \\
&\le \frac{1}{b-a}\left[\int_a^x (t-x)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-x)\frac{f'(t)}{f(t)}\,dt\right] \\
&\le \frac{1}{2(b-a)}\left[\Gamma(b-x)^2 - \gamma(x-a)^2\right].
\end{aligned} \tag{31}$$

Taking the exponential of (31) and utilising (30), we get the desired result (26); with (27) as a special case when $x = \frac{a+b}{2}$. The proof for the best possible constant is given in Remark 2 (by inequality (32)).

Remark 2 If $|f'(t)| \le Mf(t)$ for almost all $t \in [a,b]$, then by (26) we get (for $\gamma = -M$, $\Gamma = M$):

$$\begin{aligned}
&\exp\left\{-\frac{M}{(b-a)}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right\} \\
&\le f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
&\le \exp\left\{\frac{M}{(b-a)}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right\}
\end{aligned}$$

for any $x \in [a,b]$. In particular, we have

$$\begin{aligned}
\exp\left[-\frac{1}{4}M(b-a)\right] &\le \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
&\le \exp\left[\frac{1}{4}M(b-a)\right],
\end{aligned} \tag{32}$$

with $\frac{1}{4}$ as the best constant. To verify this, suppose that (32) holds for constants C, D instead of $-\frac{1}{4}$ and $\frac{1}{4}$, respectively, that is,

$$\exp\left[CM(b-a)\right] \le \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \tag{33}$$

$$\exp\left[DM(b-a)\right] \ge \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]. \tag{34}$$

Suppose in (33), $f(x) = \exp\big(-|x - \frac{a+b}{2}|\big)$, thus $M = 1$, and we have

$$\exp\left[C(b-a)\right] \leq \exp\left[-\frac{1}{4}(b-a)\right].$$

Since the exponential function is strictly increasing, we now have $C(b-a) \leq -\frac{1}{4}(b-a)$; which asserts that $C \leq -\frac{1}{4}$ since $a < b$. Now suppose in (34) that $f(x) = \exp\big(|x - \frac{a+b}{2}|\big)$, again, $M = 1$ and we have

$$\exp\left[D(b-a)\right] \geq \exp\left[\frac{1}{4}(b-a)\right].$$

By similar arguments, we conclude that $D \geq \frac{1}{4}$.

4 Applications

In this section, we apply the results from Sect. 3 to provide approximations for the integral of f and the first moment of f around zero. We start with the inequalities for logarithmic convex functions, as tools to help us in providing the above mentioned approximations.

If $f : [a,b] \to (0,\infty)$ is logarithmic convex, that is, $\log f$ is convex, then $\log f$ is differentiable almost everywhere and

$$\frac{f'_+(a)}{f(a)} \leq (\log f(t))' = \frac{f'(t)}{f(t)} \leq \frac{f'_-(b)}{f(b)}, \quad t \in (a,b).$$

Also, by Hermite–Hadamard's inequality we have the bounds

$$\begin{aligned} \log f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \log f(t)\,dt \\ &\leq \frac{\log f(b) + \log f(a)}{2} = \log\sqrt{f(b)f(a)}. \end{aligned} \tag{35}$$

From (15), we have

$$\begin{aligned} &\exp\left[\frac{\frac{f'_+(a)}{f(a)}(x-a)^2 - \frac{f'_-(b)}{f(b)}(b-x)^2}{2(b-a)}\right]\exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right) \\ &\leq f(x) \\ &\leq \exp\left[\frac{\frac{f'_-(b)}{f(b)}(x-a)^2 - \frac{f'_+(a)}{f(a)}(b-x)^2}{2(b-a)}\right]\exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right), \end{aligned} \tag{36}$$

for all $x \in [a,b]$. Utilising (35), we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\exp\left[\frac{\frac{f'_+(a)}{f(a)}(x-a)^2 - \frac{f'_-(b)}{f(b)}(b-x)^2}{2(b-a)}\right] \\ & \leq f(x) \\ & \leq \sqrt{f(a)f(b)}\exp\left[\frac{\frac{f'_-(b)}{f(b)}(x-a)^2 - \frac{f'_+(a)}{f(a)}(b-x)^2}{2(b-a)}\right], \end{aligned} \tag{37}$$

for all $x \in [a,b]$. From (26), we have

$$\begin{aligned} & \exp\left[\frac{\frac{f'_+(a)}{f(a)}(b-x)^2 - \frac{f'_-(b)}{f(b)}(x-a)^2}{2(b-a)}\right]\exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right) \\ & \leq f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \\ & \leq \exp\left[\frac{\frac{f'_-(b)}{f(b)}(b-x)^2 - \frac{f'_+(a)}{f(a)}(x-a)^2}{2(b-a)}\right]\exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right), \end{aligned} \tag{38}$$

for all $x \in [a,b]$. Utilising (35), we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\exp\left[\frac{\frac{f'_+(a)}{f(a)}(b-x)^2 - \frac{f'_-(b)}{f(b)}(x-a)^2}{2(b-a)}\right] \\ & \leq f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \\ & \leq \sqrt{f(a)f(b)}\exp\left[\frac{\frac{f'_-(b)}{f(b)}(b-x)^2 - \frac{f'_+(a)}{f(a)}(x-a)^2}{2(b-a)}\right], \end{aligned} \tag{39}$$

for all $x \in [a,b]$.

Recall the error functions:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \text{and} \quad \operatorname{erfi}(z) = -i\,\operatorname{erf}(iz).$$

Proposition 1 *Let $f : [a,b] \to (0,\infty)$ be an absolutely continuous function and $\gamma, \Gamma \in \mathbb{R}$ such that*

$$\gamma f(t) \leq f'(t) \leq \Gamma f(t), \quad \textit{for almost all } t \in [a,b].$$

Then we have the following estimates for the integral of f on $[a,b]$:

$$\sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt+\frac{\gamma\Gamma}{2\alpha}\right]$$
$$\leq \int_a^b f(x)\,dx$$
$$\leq \sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt-\frac{\gamma\Gamma}{2\alpha}\right];$$

where $\alpha=(\Gamma-\gamma)/(b-a)$. Furthermore, if f is log convex, then we have

$$\sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right] f\left(\frac{a+b}{2}\right)\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)$$
$$\leq \int_a^b f(x)\,dx$$
$$\leq \sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\sqrt{f(a)f(b)}\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right).$$

Proof First, we note some useful identities to help us in our calculations:

$$\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}=-\frac{\Gamma-\gamma}{2(b-a)}\left(x-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2+\frac{(b-a)\gamma\Gamma}{2(\Gamma-\gamma)}; \tag{40}$$

$$\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}=\frac{\Gamma-\gamma}{2(b-a)}\left(x+\frac{b\gamma-a\Gamma}{\Gamma-\gamma}\right)^2-\frac{(b-a)\gamma\Gamma}{2(\Gamma-\gamma)}. \tag{41}$$

To simplify our calculations, we let

$$\alpha=\frac{\Gamma-\gamma}{b-a},\quad \beta_1=\frac{b\Gamma-a\gamma}{\Gamma-\gamma},\quad \beta_2=\frac{a\Gamma-b\gamma}{\Gamma-\gamma}$$

so now (40) and (41) become

$$\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}=-\frac{\alpha}{2}(x-\beta_1)^2+\frac{\gamma\Gamma}{2\alpha}; \tag{42}$$

$$\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}=\frac{\alpha}{2}(x-\beta_2)^2-\frac{\gamma\Gamma}{2\alpha}. \tag{43}$$

We integrate (15) with respect to x over $[a,b]$. We observe the integral

$$\int_a^b \exp\left[\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}\right]dx$$
$$=\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b \exp\left[-\frac{\alpha}{2}(x-\beta_1)^2\right]dx$$

$$= \sqrt{\frac{2}{\alpha}} \exp\left(\frac{\gamma\Gamma}{2\alpha}\right) \int_{-\frac{\Gamma}{\sqrt{2\alpha}}}^{-\frac{\gamma}{\sqrt{2\alpha}}} \exp\left(-u^2\right) du$$

$$= \sqrt{\frac{\pi}{2\alpha}} \exp\left(\frac{\gamma\Gamma}{2\alpha}\right) \left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].$$

Performing similar calculations, we get that:

$$\int_a^b \exp\left[\frac{\Gamma(x-a)^2 - \gamma(b-x)^2}{2(b-a)}\right] dx$$

$$= \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right) \int_a^b \exp\left[\frac{\alpha}{2}(x-\beta_2)^2\right] dx$$

$$= -\sqrt{\frac{2}{\alpha}} \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right) \int_{i\frac{\gamma}{\sqrt{2\alpha}}}^{i\frac{\Gamma}{\sqrt{2\alpha}}} i \exp\left(-u^2\right) du$$

$$= \sqrt{\frac{\pi}{2\alpha}} \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right) \left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].$$

Thus (15) becomes:

$$\sqrt{\frac{\pi}{2\alpha}} \exp\left(\frac{\gamma\Gamma}{2\alpha}\right) \left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]$$
$$\le \int_a^b f(x)\,dx \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$$
$$\le \sqrt{\frac{\pi}{2\alpha}} \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right) \left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].$$

Multiplying the above by $\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$ gives us the desired result. The last set of inequalities follows from (37), coupled with the fact that both functions, erf and erfi are monotonically increasing.

Proposition 2 *Let* $0 < a < b$ *and* $f : [a,b] \to (0,\infty)$ *be an absolutely continuous function and* $\gamma, \Gamma \in \mathbb{R}$ *such that*

$$\gamma f(t) \le f'(t) \le \Gamma f(t), \quad \textit{for almost all } t \in [a,b].$$

Then we have the following estimates for $\int_a^b x f(x)\,dx$*:*

$$\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \left\{\frac{1}{\alpha}\left(\exp\left(-\frac{\Gamma(b-a)}{2}\right) - \exp\left(\frac{\gamma(b-a)}{2}\right)\right)\right.$$
$$\left. + \sqrt{\frac{\pi}{2\alpha}}\beta_1 \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\right\}$$

$$\leq \int_a^b xf(x)\,dx$$

$$\leq \exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\left\{\frac{1}{\alpha}\left(\exp\left(\frac{\Gamma(b-a)}{2}\right) - \exp\left(-\frac{\gamma(b-a)}{2}\right)\right)\right.$$

$$\left. +\sqrt{\frac{\pi}{2\alpha}}\beta_2 \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\right\},$$

where

$$\alpha = \frac{\Gamma-\gamma}{b-a}, \quad \beta_1 = \frac{b\Gamma - a\gamma}{\Gamma-\gamma}, \quad \beta_2 = \frac{a\Gamma - b\gamma}{\Gamma-\gamma}.$$

Proof We multiply (15) with $x \geq 0$ and integrate the resulting inequality with respect to x over $[a, b]$. We observe the integral

$$\int_a^b x\exp\left[\frac{\gamma(x-a)^2 - \Gamma(b-x)^2}{2(b-a)}\right]dx$$

$$= \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b x\exp\left[-\frac{\alpha}{2}(x-\beta_1)^2\right]dx$$

$$= \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\frac{2}{\alpha}\int_{-\frac{\Gamma}{\sqrt{2\alpha}}}^{-\frac{\gamma}{\sqrt{2\alpha}}} u\exp(-u^2)\,du + \sqrt{\frac{2}{\alpha}}\beta_1\int_{-\frac{\Gamma}{\sqrt{2\alpha}}}^{-\frac{\gamma}{\sqrt{2\alpha}}}\exp(-u^2)\,du\right]$$

$$= \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\frac{1}{\alpha}\left(-\exp\left(-\frac{\gamma^2}{2\alpha}\right) + \exp\left(-\frac{\Gamma^2}{2\alpha}\right)\right)\right.$$

$$\left. + \sqrt{\frac{\pi}{2\alpha}}\beta_1\left(\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right)\right]$$

$$= \frac{1}{\alpha}\left(-\exp\left(\frac{\gamma(b-a)}{2}\right) + \exp\left(-\frac{\Gamma(b-a)}{2}\right)\right)$$

$$+ \sqrt{\frac{\pi}{2\alpha}}\beta_1 \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]$$

Performing similar calculations, we get that

$$\int_a^b x\exp\left[\frac{\Gamma(x-a)^2 - \gamma(b-x)^2}{2(b-a)}\right]dx$$

$$= \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b x\exp\left[\frac{\alpha}{2}(x-\beta_2)^2\right]dx$$

$$= \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[-\frac{2}{\alpha}\int_{\frac{\gamma}{\sqrt{2\alpha}}i}^{\frac{\Gamma}{\sqrt{2\alpha}}i} u\exp(-u^2)\,du + \sqrt{\frac{2}{\alpha}}\beta_2\int_{\frac{\gamma}{\sqrt{2\alpha}}i}^{\frac{\Gamma}{\sqrt{2\alpha}}i}(-i)\exp(-u^2)\,du\right]$$

$$
\begin{aligned}
&= \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\frac{1}{\alpha}\left(\exp\left(\frac{\Gamma^2}{2\alpha}\right) - \exp\left(\frac{\gamma^2}{2\alpha}\right)\right)\right. \\
&\quad \left. + \sqrt{\frac{\pi}{2\alpha}}\beta_2\left(\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right)\right] \\
&= \frac{1}{\alpha}\left(\exp\left(\frac{\Gamma(b-a)}{2}\right) - \exp\left(-\frac{\gamma(b-a)}{2}\right)\right) \\
&\quad + \sqrt{\frac{\pi}{2\alpha}}\beta_2 \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]
\end{aligned}
$$

Thus (15) becomes:

$$
\begin{aligned}
&\frac{1}{\alpha}\left(\exp\left(-\frac{\Gamma(b-a)}{2}\right) - \exp\left(\frac{\gamma(b-a)}{2}\right)\right) \\
&\sqrt{\frac{\pi}{2\alpha}}\beta_1 \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right] \\
&\le \int_a^b x f(x)\,dx\ \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
&\le \frac{1}{\alpha}\left(\exp\left(\frac{\Gamma(b-a)}{2}\right) - \exp\left(-\frac{\gamma(b-a)}{2}\right)\right) \\
&+\sqrt{\frac{\pi}{2\alpha}}\beta_2 \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right) - \operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].
\end{aligned}
$$

Multiplying the above by $\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$ gives us the desired result.

Remark 3 The inequalities in Proposition 2 can be simplified in the similar manner to that of Proposition 1 by assuming that f is logarithmic convex and using the estimates for $\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$ in (37).

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A Survey on Ostrowski Type Inequalities for Riemann–Stieltjes Integral

W. S. Cheung and Sever S. Dragomir

Abstract Some Ostrowski type inequalities for the Riemann–Stieltjes integral for various classes of integrands and integrators are surveyed. Applications for the midpoint rule and a generalised trapezoidal type rule are also presented.

Keywords Ostrowski type inequalities · Riemann–Stieltjes integral · Absolutely continuous function · Trapezoidal rule

1 Introduction

The following result is known in the literature as Ostrowski's inequality [27]:

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \le M$ for all $t \in (a,b)$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M \tag{1}$$

for all $x \in (a,b)$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The above result has been naturally extended for absolutely continuous functions and Lebesgue $p-$norms of the derivative f' in [20–22] and can be stated as:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have:*

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$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right] (b-a)^{\frac{1}{q}} \|f'\|_q & \\ \qquad \text{if } f' \in L_p[a,b],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; & \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1, & \end{cases} \tag{2}$$

where $\|\cdot\|_r$ $(r \in [1, \infty])$ *are the usual Lebesgue norms on* $L_r[a,b]$, *that is, we recall that*

$$\|g\|_\infty := ess \sup_{t \in [a,b]} |g(t)| \quad and \quad \|g\|_r := \left(\int_a^b |g(t)|^r dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{1/p}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense mentioned above.*

They can also be obtained, in a slightly different form, as particular cases of some results established by A.M. Fink [23] for $n-$time differentiable functions.

For other Ostrowski-type inequalities concerning Lipschitzian and $r-H-$H ölder type functions, see [11] and [18].

The cases of bounded variation functions and monotonic functions were considered in [14] and [10] while the case of convex functions was studied in [16].

In order to approximate the Riemann–Stieltjes integral $\int_a^b p(x)dv(x)$, where $p, v : [a,b] \to \mathbb{R}$ are functions for which the above integral exists, S.S. Dragomir established in [12] the following integral identity:

$$[u(b) - u(a)] f(x) - \int_a^b f(t)du(t)$$
$$= \int_a^x [u(t) - u(a)] df(t) + \int_x^b [u(t) - u(b)] df(t), \quad x \in [a,b] \tag{3}$$

provided that the involved *Riemann–Stieltjes integrals* exist. In the case $u(t) = t$, $t \in [a,b]$, the above identity reduces to the celebrated *Montgomery identity* (see [26], p. 565) that has been extensively used by many authors in obtaining various inequalities of Ostrowski type. For a comprehensive recent collection of works, see the book [19] and the papers [1–5, 7, 24, 28, 29, 30].

In an effort to obtain an Ostrowski-type inequality for the Riemann–Stieltjes integral, which obviously contains the weighted integrals case, S.S. Dragomir established [12] the following result:

Theorem 2 *Let* $f : [a,b] \to \mathbb{R}$ *be a function of bounded variation and* $u : [a,b] \to \mathbb{R}$ *a function of* $r - H-$*Hölder type, i.e.,*

$$|u(x) - u(y)| \leq H |x-y|^r \quad for\ any\ x, y \in [a,b], \tag{4}$$

where $r \in (0,1]$ and $H > 0$ are given. Then, for any $x \in [a,b]$,

$$\left|[u(b) - u(x)] f(x) - \int_a^b f(t)\, du(t)\right|$$
$$\le H \left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f) \right]$$
$$\le H \times \begin{cases} \left[(x-a)^r + (b-x)^r\right] \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}} \left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p \right]^{\frac{1}{p}} \\ \qquad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases} \tag{5}$$

where $\bigvee_c^d (f)$ denotes the total variation of f on the interval $[c,d]$.

The dual case was considered in [13] and can be stated as follows:

Theorem 3 *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a function of $r - H-$Hölder type. Then*

$$\left|[u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t)\right| \le H \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) \tag{6}$$

for any $x \in [a,b]$.

For other results concerning inequalities for Riemann–Stieltjes integrals, see [3], [24] and [25].

The aim of the present survey paper is to present some results of Ostrowski-type inequalities for Riemann–Stieltjes integrals $\int_a^b f(t)\, du(t)$ discovered by the authors. Applications to the midpoint rule and for a generalised trapezoidal rule are also pointed out.

2 General Bounds for Absolutely Continuous Functions

The following representation result is of interest [8]:

Lemma 1 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ such that the Riemann–Stieltjes integrals*

$$\int_a^b f(t)\, du(t) \qquad \text{and} \qquad \int_a^b (x-t) \left(\int_0^1 f'[\lambda t + (1-\lambda)x]\, d\lambda \right) du(t)$$

exist for each $x \in [a,b]$. Then

$$f(x)[u(b) - u(a)] - \int_a^b f(t)du(t)$$

$$= \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right)du(t) \tag{7}$$

or, equivalently,

$$\int_a^b u(t)df(t) - u(b)[f(b) - f(x)] - u(a)[f(x) - f(a)]$$
$$= \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right)du(t) \tag{8}$$

for each $x \in [a,b]$.

Proof Since f is absolutely continuous on $[a,b]$, hence, for any $x,t \in [a,b]$ with $x \neq t$, one has

$$\frac{f(x) - f(t)}{x-t} = \frac{\int_t^x f'(u)\,du}{x-t} = \int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda$$

giving the equality (see also [15]):

$$f(x) = f(t) + (x-t)\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda \tag{9}$$

for any $x,t \in [a,b]$.

Integrating the identity (9) we deduce

$$f(x)\int_a^b du(t) = \int_a^b f(t)du(t) + \int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right)du(t),$$

which is exactly the desired inequality (7).

Now, on utilising the integration by parts formula for the Riemann–Stieltjes integral, we have

$$f(x)[u(b) - u(a)] - \int_a^b f(t)\,du(t)$$
$$= f(x)[u(b) - u(a)] - \left[f(b)u(b) - f(a)u(a) - \int_a^b u(t)\,df(t)\right]$$
$$= \int_a^b u(t)\,df(t) - u(b)[f(b) - f(x)] - u(a)[f(x) - f(a)]$$

and the representation (8) is also obtained. □

For an absolutely continuous function $f : [a,b] \to \mathbb{R}$, let us denote by $\mu(f;x,t) := \left|\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right|$, where $(t,x) \in [a,b]^2$. It is obvious that, by the Hölder inequality, we have

$$\mu(f;x,t) \le \begin{cases} \left\|f'\right\|_{[t,x],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \left\|f'\right\|_{[t,x],p} & \text{if } f' \in L_p[a,b],\ p \ge 1, \end{cases} \tag{10}$$

where

$$\left\|f'\right\|_{[t,x],\infty} := \sup_{\substack{u\in[t,x]\\(u\in[x,t])}} \left|f'(u)\right|,$$

$$\left\|f'\right\|_{[t,x],p} := \left|\int_t^x \left|f'(u)\right|^p du\right|^{\frac{1}{p}}, \quad p \geq 1$$

and $t, x \in [a,b]$.

We can also state the following result of Ostrowski type for the Riemann–Stieltjes integral [8]:

Theorem 4 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function and $u : [a,b] \to \mathbb{R}$ a function of bounded variation on $[a,b]$. Then*

$$\left|[u(b)-u(a)]\, f(x) - \int_a^b f(t)\, du(t)\right| \leq M(x), \tag{11}$$

and, equivalently

$$\left|\int_a^b u(t)\, df(t) - u(b)[f(b)-f(x)] - u(a)[f(x)-f(a)]\right| \leq M(x), \tag{12}$$

where $M(x) = M_1(x) + M_2(x)$ and

$$M_1(x) := \bigvee_a^x (u) \sup_{t\in[a,x]} \left[(x-t)\,\mu(f;x,t)\right],$$

$$M_2(x) := \bigvee_x^b (u) \sup_{t\in[x,b]} \left[(t-x)\,\mu(f;x,t)\right],$$

for $x \in [a,b]$.

Remark 1 Using the notations in Theorem 4, we have

$$\begin{aligned} M_1(x) &\leq (x-a) \bigvee_a^x (u) \sup_{t\in[a,x]} \mu(f;x,t) \\ &\leq (x-a) \bigvee_a^x (u) \cdot \begin{cases} \left\|f'\right\|_{[a,x],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \left\|f'\right\|_{[a,x],p} & \text{if } f' \in L_p[a,b],\ p \geq 1, \end{cases} \\ M_2(x) &\leq (b-x) \bigvee_x^b (u) \sup_{t\in[x,b]} \mu(f;x,t) \\ &\leq (b-x) \bigvee_x^b (u) \cdot \begin{cases} \left\|f'\right\|_{[x,b],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \left\|f'\right\|_{[x,b],p} & \text{if } f' \in L_p[a,b],\ p \geq 1 \end{cases} \end{aligned}$$

for any $x \in [a,b]$.

Proof We use the fact that, if $p, v : [c,d] \to \mathbb{R}$ are such that p is continuous and v is of bounded variation, then the Riemann–Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and

$$\left|\int_c^d p(x)\,dv(x)\right| \le \sup_{x\in[c,d]} |p(x)| \bigvee_c^d (v).$$

Utilising the representation (7) we have

$$\begin{aligned}
&\left| f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t)\right| \\
&= \left|\int_a^x (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t)\right. \\
&\quad \left. + \int_x^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t)\right| \\
&\le \left|\int_a^x (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t)\right| \\
&\quad + \left|\int_x^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t)\right| \\
&\le \bigvee_a^x (u) \sup_{t\in[a,x]} [(x-t)\,\mu(f;x,t)] + \bigvee_x^b (u) \sup_{t\in[x,b]} [(t-x)\,\mu(f;x,t)] \\
&\le M_1(x) + M_2(x) =: M(x).
\end{aligned}$$

The other inequalities for M_1 and M_2 are obvious from the inequality (10) and the details are omitted. □

Remark 2 Hence, if we denote by $\|f'\|_{[c,d],p}$ the p norm on the interval $[c,d]$, where $1 \le p \le \infty$, then for $f' \in L_p[a,b]$, we have

$$\begin{aligned}
&\left| f(x)[u(b)-u(a)] - \int_a^b f(t)du(t)\right| \\
&\le (x-a)\bigvee_a^x (u)\,\|f'\|_{[a,x],p} + (b-x)\bigvee_x^b (u)\,\|f'\|_{[x,b],p} =: N(x), \qquad (13)
\end{aligned}$$

where $p \in [1, \infty]$ and $x \in [a, b]$.

Obviously one can derive many upper bounds for the function $N(x)$ defined above. We intend to present in the following only a few that are simple and perhaps of interest for applications.

Estimate 1

$$
\begin{aligned}
N(x) &\leq \left[(x-a) \bigvee_a^x (u) + (b-x) \bigvee_x^b (u) \right] \|f'\|_{[a,b],p} \\
&\leq \|f'\|_{[a,b],p} \cdot \begin{cases} \max\{x-a, b-x\} \left[\bigvee_a^x (u) + \bigvee_x^b (u) \right]; \\ \left[(x-a)^\alpha + (b-x)^\alpha \right]^{\frac{1}{\alpha}} \left[\left(\bigvee_a^x (u) \right)^\beta + \left(\bigvee_x^b (u) \right)^\beta \right]^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (b-a) \max\left\{ \bigvee_a^x (u), \bigvee_x^b (u) \right\} \end{cases} \\
&= \|f'\|_{[a,b],p} \cdot \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (u); \\ \left[(x-a)^\alpha + (b-x)^\alpha \right]^{\frac{1}{\alpha}} \left[\left(\bigvee_a^x (u) \right)^\beta + \left(\bigvee_x^b (u) \right)^\beta \right]^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (b-a) \left[\frac{1}{2} \bigvee_a^b (u) + \frac{1}{2} \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right] \end{cases}
\end{aligned} \tag{14}
$$

for any $x \in [a, b]$.

Estimate 2

$$
\begin{aligned}
N(x) &\leq \max\{x-a, b-x\} \left[\bigvee_a^x (u) \|f'\|_{[a,x],p} + \bigvee_x^b (u) \|f'\|_{[x,b],p} \right] \\
&= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \left[\bigvee_a^x (u) \|f'\|_{[a,x],p} + \bigvee_x^b (u) \|f'\|_{[x,b],p} \right] \\
&\leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \\
&\quad \times \begin{cases} \max\left\{ \|f'\|_{[a,x],p}, \|f'\|_{[x,b],p} \right\} \bigvee_a^b (u); \\ \left[\|f'\|_{[a,x],p}^p + \|f'\|_{[x,b],p}^p \right]^{\frac{1}{p}} \left[\left(\bigvee_a^x (u) \right)^q + \left(\bigvee_x^b (u) \right)^q \right]^{\frac{1}{q}} \\ \qquad \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2} \bigvee_a^b (u) + \frac{1}{2} \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right] \left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p} \right] \end{cases}
\end{aligned}
$$

$$= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]$$

$$\times \begin{cases} \max\left\{\left\|f'\right\|_{[a,x],p}, \left\|f'\right\|_{[x,b],p}\right\} \bigvee_a^b(u); \\ \left\|f'\right\|_{[a,b],p}\left[\left(\bigvee_a^x(u)\right)^q + \left(\bigvee_x^b(u)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u) + \frac{1}{2}\left|\bigvee_a^x(u) - \bigvee_x^b(u)\right|\right]\left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p}\right] \end{cases}$$

for any $x \in [a,b]$.

Estimate 3

$$N(x) \le \max\left\{\bigvee_a^x(u), \bigvee_x^b(u)\right\}\left[(x-a)\left\|f'\right\|_{[a,x],p} + (b-x)\left\|f'\right\|_{[x,b],p}\right]$$

$$= \left[\frac{1}{2}\bigvee_a^b(u) + \frac{1}{2}\left|\bigvee_a^x(u) - \bigvee_x^b(u)\right|\right] x$$

$$\left[(x-a)\left\|f'\right\|_{[a,x],p} + (b-x)\left\|f'\right\|_{[x,b],p}\right]$$

$$\le \left[\frac{1}{2}\bigvee_a^b(u) + \frac{1}{2}\left|\bigvee_a^x(u) - \bigvee_x^b(u)\right|\right]$$

$$\times \begin{cases} \max\left\{\left\|f'\right\|_{[a,x],p}, \left\|f'\right\|_{[x,b],p}\right\}(b-a); \\ \left[(x-a)^q + (b-x)^q\right]^{\frac{1}{q}}\left\|f'\right\|_{[a,b],p} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p}\right] \end{cases}$$

for each $x \in [a,b]$.

In practical applications, the midpoint rule, that results for $x = \frac{a+b}{2}$, is of obvious interest due to its simpler form [8].

Corollary 1 *With the assumptions in Theorem 4, we have the inequalities:*

$$\left|[u(b) - u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t)\right|$$

$$\le \frac{1}{2}(b-a)\left[\bigvee_a^{\frac{a+b}{2}}(u)\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],p} + \bigvee_{\frac{a+b}{2}}^b(u)\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],p}\right]$$

$$\leq \frac{1}{2}(b-a)\begin{cases} \max\left\{\|f'\|_{\left[a,\frac{a+b}{2}\right],p}, \|f'\|_{\left[\frac{a+b}{2},b\right],p}\right\}\bigvee_a^b(u); \\ \left[\|f'\|^{\alpha}_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|^{\alpha}_{\left[\frac{a+b}{2},b\right],p}\right]^{\frac{1}{\alpha}} \\ \qquad \times\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{\beta} + \left(\bigvee_{\frac{a+b}{2}}^b(u)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u) + \frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u) - \bigvee_{\frac{a+b}{2}}^b(u)\right|\right] \\ \qquad \times\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2},b\right],p}\right], \end{cases} \tag{15}$$

where $p \in [1,\infty]$.

From the above, it is obvious that we can get some appealing inequalities as follows:

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t)\right|$$
$$\leq \frac{1}{2}(b-a)\begin{cases} \|f'\|_{[a,b],\infty}\bigvee_a^b(u), & \text{if } f' \in L_\infty[a,b]; \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^q + \left(\bigvee_{\frac{a+b}{2}}^b(u)\right)^q\right]^{\frac{1}{q}} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f' \in L_p[a,b]; \\ \left[\frac{1}{2}\bigvee_a^b(u) + \frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u) - \bigvee_{\frac{a+b}{2}}^b(u)\right|\right]\|f'\|_{[a,b],1}. \end{cases} \tag{16}$$

Remark 3 Similar inequalities can be obtained for the generalised trapezoidal rule. We only state here the following simple results:

$$\left|\int_a^b u(t)\,df(t) - u(b)\left[f(b) - f\left(\frac{a+b}{2}\right)\right] - u(a)\left[f\left(\frac{a+b}{2}\right) - f(a)\right]\right|$$
$$\leq \frac{1}{2}(b-a)\begin{cases} \|f'\|_{[a,b],\infty}\bigvee_a^b(u), & \text{if } f' \in L_\infty[a,b]; \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^q + \left(\bigvee_{\frac{a+b}{2}}^b(u)\right)^q\right]^{\frac{1}{q}} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f' \in L_p[a,b]; \\ \left[\frac{1}{2}\bigvee_a^b(u) + \frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u) - \bigvee_{\frac{a+b}{2}}^b(u)\right|\right]\|f'\|_{[a,b],1} \end{cases}$$

provided that u is of bounded variation and f is absolutely continuous on $[a,b]$.

3 Bounds in the Case of $|f'|$ a Convex Function

Some of the above results can be improved provided that a convexity assumption for $|f'|$ is in place [8]:

Theorem 5 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, $u : [a,b] \to \mathbb{R}$ a function of bounded variation on $[a,b]$ and $x \in [a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$ (and the intervals can be reduced at a single point), then*

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{17}$$
$$\le \frac{1}{2}\left[\bigvee_a^x(u)\sup_{t\in[a,x]}\left\{(x-t)\left|f'(t)\right|\right\}+\bigvee_x^b(u)\sup_{t\in[x,b]}\left\{(t-x)\left|f'(t)\right|\right\}\right]$$
$$+\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\le \frac{1}{2}\left[(x-a)\bigvee_a^x(u)\left\|f'\right\|_{[a,x],\infty}+(b-x)\bigvee_x^b(u)\left\|f'\right\|_{[x,b],\infty}\right]$$
$$+\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

for any $x \in [a,b]$.

Proof As in the proof of Theorem 4, we have

$$\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|$$
$$\le \sup_{t\in[a,x]}\left[(x-t)\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right|\right]\bigvee_a^x(u)$$
$$+\sup_{t\in[x,b]}\left[(t-x)\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right|\right]\bigvee_x^b(u)$$
$$\le \sup_{t\in[a,x]}\left[(x-t)\int_0^1\left|f'[\lambda t+(1-\lambda)x]\right|d\lambda\right]\bigvee_a^x(u)$$
$$+\sup_{t\in[x,b]}\left[(t-x)\int_0^1\left|f'[\lambda t+(1-\lambda)x]\right|d\lambda\right]\bigvee_x^b(u)$$
$$\le \sup_{t\in[a,x]}\left[(x-t)\frac{\left|f'(t)\right|+\left|f'(x)\right|}{2}\right]\bigvee_a^x(u)$$

$$
\begin{aligned}
&+ \sup_{t\in[x,b]}\left[(t-x)\frac{\left|f'(t)\right|+\left|f'(x)\right|}{2}\right]\bigvee_x^b(u) \\
&\le \frac{1}{2}\left[\sup_{t\in[a,x]}\left\{(x-t)\left|f'(t)\right|\right\}\cdot\bigvee_a^x(u)+\sup_{t\in[x,b]}\left\{(t-x)\left|f'(t)\right|\right\}\cdot\bigvee_x^b(u)\right] \\
&+ \frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]
\end{aligned}
$$

which proves the first inequality in (17).

The second inequality in (17) is obvious using properties of sup and the theorem is completely proved. □

The midpoint inequality is of interest in applications and provides a much simpler inequality [8]:

Corollary 2 *If f and u are as above and $\left|f'\right|$ is convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, then*

$$
\begin{aligned}
&\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \qquad (18)\\
&\le \frac{1}{4}(b-a)\left[\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right] \\
&\quad+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u) \\
&\le \frac{1}{4}(b-a)\bigvee_a^b(u)\left[\left\|f'\right\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right].
\end{aligned}
$$

Remark 4 If we denote, from the second inequality in (17),

$$
L_1(x):=\frac{1}{2}\left[(x-a)\left\|f'\right\|_{[a,x],\infty}\bigvee_a^x(u)+(b-x)\left\|f'\right\|_{[x,b],\infty}\bigvee_x^b(u)\right]
$$

and

$$
L_2(x):=\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]
$$

for $x\in[a,b]$, then we can point out various upper bounds for the functions L_1 and L_2 on $[a,b]$.

For instance, we have

$$
L_1(x)\le\frac{1}{2}\left\|f'\right\|_{[a,b],\infty}\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]
$$

and by (17) we can state the following inequality of interest:

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)du(t)\right| \tag{19}$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases}$$

for each $x\in[a,b]$.

Remark 5 A similar result to (19) can be stated for the generalised trapezoidal rule, out of which we would like to note the following one that is of particular interest:

$$\left|\int_a^b u(t)df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right|$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases} \tag{20}$$

for each $x\in[a,b]$.

As in Corollary 2, the case $x=\frac{a+b}{2}$ in (20) provides the simple result

$$\left|\int_a^b u(t)\,df(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right|$$
$$\leq \frac{1}{4}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right]$$
$$+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u)$$
$$\leq \frac{1}{4}(b-a)\bigvee_a^b(u)\left[\|f'\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{21}$$

Remark 6 Similar inequalities may be stated if one assumes either that $|f'|$ is quasi-convex or that $|f'|$ is log-convex on $[a,x]$ and $[x,b]$. The details are left to the interested readers.

4 The Case of Monotonic Integrators

The following result may be stated [9].

Theorem 6 *Let $f : [a,b] \to \mathbb{R}$ be a function of r-H-H ölder type with $r \in (0,1]$ and $H > 0$, and $u : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. Then*

$$\begin{aligned}
&\left|[u(b)-u(a)]\, f(x) - \int_a^b f(t)du(t)\right| \\
&\le H\left[(b-x)^r u(b) - (x-a)^r u(a) + r\left\{\int_a^x \frac{u(t)}{(x-t)^{1-r}}dt - \int_x^b \frac{u(t)}{(t-x)^{1-r}}dt\right\}\right] \\
&\le H\left\{(b-x)^r\left[(u(b)-u(x)\right] + (x-a)^r\left[u(x)-u(a)\right]\right\} \\
&\le H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r [u(b)-u(a)] \qquad (22)
\end{aligned}$$

for any $x \in [a,b]$.

Proof First of all we remark that if $p : [a,b] \to \mathbb{R}$ is continuous and $v : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann–Stieltjes integral $\int_a^b p(t)dv(t)$ exists and:

$$\left|\int_a^b p(t)dv(t)\right| \le \int_a^b |p(t)|\, dv(t). \qquad (23)$$

Making use of this property and the fact that f is of r-H-Hölder type, we can state that

$$\begin{aligned}
\left|[u(b)-u(a)]\, f(x) - \int_a^b f(t)du(t)\right| &= \left|\int_a^b [f(x)-f(t)]du(t)\right| \\
&\le \int_a^b |f(x)-f(t)|\, du(t) \\
&\le H\int_a^b |x-t|^r du(t). \qquad (24)
\end{aligned}$$

By the integration by parts formula for the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_a^b |x-t|^r du(t) = \int_a^x (x-t)^r du(t) + \int_x^b (t-x)^r du(t) \\
&= (x-t)^r u(t)\Big|_a^x + r\int_a^x \frac{u(t)}{(x-t)^{1-r}}dt + (t-x)^r u(t)\Big|_x^b - r\int_x^b \frac{u(t)}{(t-x)^{1-r}}dt \\
&= (b-x)^r u(b) - (x-a)^r u(a) + r\left[\int_a^x \frac{u(t)}{(x-t)^{1-r}}dt - \int_x^b \frac{u(t)}{(t-x)^{1-r}}dt\right], \qquad (25)
\end{aligned}$$

which together with (24) proves the first inequality in (22).

Now, by the monotonicity property of u we have

$$\int_a^x \frac{u(t)dt}{(x-t)^{1-r}} \le u(x)\int_a^x \frac{dt}{(x-t)^{1-r}} = \frac{(x-a)^r u(x)}{r}$$

and

$$\int_x^b \frac{u(t)dt}{(t-x)^{1-r}} \ge u(x)\int_x^b \frac{dt}{(t-x)^{1-r}} = \frac{(b-x)^r u(x)}{r},$$

giving that

$$\int_a^x \frac{u(t)dt}{(x-a)^{1-r}} - \int_x^b \frac{u(t)dt}{(t-x)^{1-r}} \le \frac{1}{r}\left[(x-a)^r u(x) - (b-x)^r u(x)\right]. \tag{26}$$

This inequality implies that

$$\begin{aligned}
&(b-x)^r u(b) - (x-a)^r u(a) + r\left[\int_a^x \frac{u(t)}{(x-t)^{1-r}}dt - \int_x^b \frac{u(t)}{(t-x)^{1-r}}dt\right] \\
&\le (b-x)^r u(b) - (x-a)^r u(a) + (x-a)^r u(x) - (b-x)^r u(x) \\
&= (b-x)^r\,[u(b)-u(x)] + (x-a)^r\,[u(x)-u(a)]
\end{aligned}$$

and the second part of inequality (22) is also proved.

The last part is obvious by the property of max function and we omit the details here. □

Remark 7 If f is assumed to be L-Lipschitzian, i.e.,

$$|f(x)-f(y)| \le L|x-y| \quad \text{for any} \quad x,y\in[a,b], \tag{27}$$

where $L>0$ is given, then for $u:[a,b]\to\mathbb{R}$ being monotonic nondecreasing on $[a,b]$ the inequality (7) will produce the simple result:

$$\begin{aligned}
&\left|[u(b)-u(a)]\,f(x) - \int_a^b f(t)du(t)\right| \\
&\le L\left[bu(b)+au(a)-x\,[u(a)+u(b)] + \int_a^b \operatorname{sgn}(x-t)u(t)dt\right] \\
&\le L\,[(b-x)\,[u(b)-u(x)] + (x-a)\,[u(x)-u(a)]] \\
&\le L\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][u(b)-u(a)]
\end{aligned} \tag{28}$$

for any $x\in[a,b]$.

A particular case that may be useful in applications is the following midpoint-type inequality [9].

Corollary 3 *With the assumptions in Theorem 6, we have:*

$$\begin{aligned}
&\left|[u(b)-u(a)]\, f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right| \\
&\le H\left[\frac{(b-a)^r}{2^r}\,[u(b)-u(a)]+r\left\{\int_a^{\frac{a+b}{2}}\frac{u(t)dt}{(\frac{a+b}{2}-t)^{1-r}}-\int_{\frac{a+b}{2}}^b\frac{u(t)dt}{(t-\frac{a+b}{2})^{1-r}}\right\}\right] \\
&\le \frac{H(b-a)^r}{2^r}\,[u(b)-u(a)]\,. \qquad (29)
\end{aligned}$$

In particular, if f is a L-Lipschitzian function, we have

$$\begin{aligned}
&\left|[u(b)-u(a)]\, f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right| \\
&\le L\left[\frac{(b-a)\,[u(b)-u(a)]}{2}+\int_a^b sgn\left(\frac{a+b}{2}-t\right)u(t)dt\right] \\
&\le \frac{L\cdot(b-a)}{2}\,[u(b)-u(a)]\,. \qquad (30)
\end{aligned}$$

Remark 8 We observe that the first inequality in (30) is sharp. Indeed, if we choose $f,u:[a,b]\to\mathbb{R}$, $f(t)=\left|t-\frac{a+b}{2}\right|$, $u(t)=t-\frac{a+b}{2}$, we notice that f is L-Lipschitzian with the constant $L=1$ and u is monotonic nondecreasing on $[a,b]$. Also:

$$[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)dt=-\int_a^b\left|t-\frac{a+b}{2}\right|dt=-\frac{(b-a)^2}{4},$$

$$\begin{aligned}
&\frac{(b-a)\,[u(b)-u(a)]}{2}+\int_a^b \mathrm{sgn}\left(\frac{a+b}{2}-t\right)u(t)dt \\
&=\frac{(b-a)^2}{2}-\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{(b-a)^2}{4}
\end{aligned}$$

which shows that in both sides of (30) we have the same quantity $\frac{(b-a)^2}{4}$.

Remark 9 In terms of probability density functions, if $w:[a,b]\to[0,\infty)$ is such that $\int_a^b w(s)ds=1$, then writing out the inequality (22) for $u(t):=\int_a^t w(s)ds$, we obtain:

$$\begin{aligned}
&\left|f(x)-\int_a^b w(s)f(s)ds\right| \\
&\le H\left[(b-x)^r+r\left\{\int_a^x\frac{W(t)}{(x-t)^{1-r}}dt-\int_x^b\frac{W(t)}{(t-x)^{1-r}}dt\right\}\right] \\
&\le H\left[(b-x)^r\int_x^b w(s)ds+(x-a)^r\int_a^x w(s)ds\right] \\
&\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \qquad (31)
\end{aligned}$$

for any $x \in [a,b]$, where, as above, f is of r-H-Hölder type.

The Lipschitzian case provides the simpler inequality:

$$\begin{aligned}\left| f(x) - \int_a^b w(s)f(s)ds \right| &\leq L\ \left[b - x + \int_a^b \operatorname{sgn}(x-t)W(t)dt \right] \\ &\leq L\ \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \end{aligned} \tag{32}$$

for any $x \in [a,b]$.

Finally, the weighted trapezoidal inequality for Hölder continuous functions reads as

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \int_a^b w(s)f(s)ds \right| \\ &\leq H\left[\frac{(b-a)^r}{2^r} + r\left\{ \int_a^{\frac{a+b}{2}} \frac{W(t)}{(\frac{a+b}{2}-t)^{1-r}}dt - \int_{\frac{a+b}{2}}^b \frac{W(t)dt}{(t-\frac{a+b}{2})^{1-r}} \right\} \right] \\ &\leq \frac{H(b-a)^r}{2^r}, \end{aligned} \tag{33}$$

while for Lipschitzian functions it will have the form

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \int_a^b w(s)f(s)ds \right| \\ &\leq L \cdot \left[\frac{b-a}{2} + \int_a^b \operatorname{sgn}\left(\frac{a+b}{2} - t\right) W(t)dt \right] \leq \frac{1}{2}L(b-a). \end{aligned} \tag{34}$$

The uniform distribution $w(s) = \frac{1}{b-a}$, $s \in [a,b]$, will then provide the following inequality:

$$\begin{aligned}&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq H\left[(b-x)^r + \frac{r}{b-a}\left\{ \int_a^x \frac{(t-a)}{(x-t)^{1-r}}dt - \int_x^b \frac{(t-a)}{(t-x)^{1-r}}dt \right\} \right] \ (=: HT,\ \text{say}) \\ &\leq \frac{H}{b-a}\left[(b-x)^{r+1} + (x-a)^{r+1} \right] \leq H\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r. \end{aligned} \tag{35}$$

Since

$$\begin{aligned}\int_a^x \frac{(t-a)}{(x-t)^{1-r}}dt &= \int_a^x (t-a)(x-t)^{r-1}dt \\ &= (x-a)^{r+1}\int_0^1 s(1-s)^{r-1}ds \\ &= (x-a)^{r+1}B(2,r) = \frac{(x-a)^{r+1}}{r(r+1)},\end{aligned}$$

where $B(p,q) := \int_0^1 s^{p-1}(1-s)^{q-1}ds$, $p,q > 0$, is the *Euler's Beta function*, and

$$\begin{aligned}\int_x^b \frac{(t-a)}{(t-x)^{1-r}}dt &= \int_x^b \frac{t-b+b-a}{(t-x)^{1-r}}dt \\ &= (b-a)\int_x^b \frac{dt}{(t-x)^{1-r}} - \int_x^b \frac{(b-t)dt}{(t-x)^{1-r}} \\ &= (b-a)\cdot\frac{(b-x)^r}{r} - \int_x^b (b-t)(t-x)^{r-1}dt \\ &= (b-a)\cdot\frac{(b-x)^r}{r} - (b-x)^{r+1}\int_0^1 s(1-s)^{r-1}ds \\ &= (b-a)\cdot\frac{(b-x)^r}{r} - (b-x)^{r+1}B(2,r) \\ &= \frac{(b-a)(b-x)^r}{r} - \frac{(b-x)^{r+1}}{r(r+1)},\end{aligned}$$

hence T, defined above, has the form

$$\begin{aligned}T &= (b-x)^r + \frac{r}{b-a}\left\{\frac{(x-a)^{r+1}}{r(r+1)} - \frac{(b-a)(b-x)^r}{r} + \frac{(b-x)^{r+1}}{r(r+1)}\right\} \\ &= \frac{(x-a)^{r+1}+(b-\alpha)^{r+1}}{r+1}.\end{aligned}$$

Therefore, from the first inequality in (35) we deduce

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{H}{r+1}\left[\left(\frac{x-a}{b-a}\right)^{r+1} + \left(\frac{b-x}{b-x}\right)^{r+1}\right](b-a)^r \tag{36}$$

for any $x \in [a,b]$, which has been obtained before (see for instance [10] and [19]).

5 The Case of Monotonic Integrands

It is natural now to investigate the dual case, that is, where the integrand f is assumed to be monotonic nondecreasing while the integrator u is Hölder continuous [9].

Theorem 7 *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ of r-H-Hölder type. Then*

$$\begin{aligned}&\left|[u(b)-u(a)]f(x) - \int_a^b f(t)du(t)\right| \\ &\le H\left[\left[(x-a)^r-(b-a)^r\right]f(x) + r\left\{\int_x^b \frac{f(t)dt}{(b-t)^{1-r}} - \int_a^x \frac{f(t)dt}{(t-a)^{1-r}}\right\}\right]\end{aligned}$$

$$\leq H\left\{(b-x)^r\left[f(b)-f(x)\right]+(x-a)^r\left[f(x)-f(a)\right]\right\}$$
$$\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\left[f(b)-f(a)\right]. \tag{37}$$

Proof Utilising the integral identity (3) and the hypothesis, we have successively

$$\begin{aligned}
&\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)du(t)\right| \\
&\leq \int_a^x |u(t)-u(a)|\,df(t)+\int_x^b |u(s)-u(t)|\,df(t) \\
&\leq H\left[\int_a^x (t-a)^r df(t)+\int_x^b (b-t)^r df(t)\right] \\
&= H\left[(t-a)^r f(t)\Big|_a^x - r\int_a^x \frac{f(t)dt}{(t-a)^{1-r}}+(b-t)^r f(t)\Big|_x^b + r\int_x^b \frac{f(t)dt}{(b-t)^{1-r}}\right] \\
&= H\left[(x-a)^r f(x)-(b-x)^r f(x)+r\left\{\int_x^b \frac{f(t)dt}{(b-t)^{1-r}}-\int_a^x \frac{f(t)dt}{(t-a)^{1-r}}\right\}\right]
\end{aligned} \tag{38}$$

proving the first inequality in (37).

Since f is monotonic nondecreasing on $[a,b]$, hence

$$\int_x^b \frac{f(t)dt}{(b-t)^{1-r}} \leq f(b)\int_x^b \frac{dt}{(b-t)^{1-r}} = \frac{f(b)(b-x)^r}{r}$$

and

$$\int_a^x \frac{f(t)dt}{(t-a)^{1-r}} \geq f(a)\int_a^x \frac{dt}{(t-a)^{1-r}} = \frac{f(a)(x-a)^r}{r},$$

giving that

$$\int_x^b \frac{f(t)dt}{(b-t)^{1-r}} - \int_a^x \frac{f(t)dt}{(t-a)^{1-r}} \leq \frac{1}{r}\left[f(b)(b-x)^r - f(a)(x-a)^r\right]$$

which obviously implies that

$$\begin{aligned}
(x-a)^r f(x)-(b-x)^r f(x)+r&\left[\int_x^b \frac{f(t)dt}{(b-t)^{1-r}}-\int_a^x \frac{f(t)dt}{(t-a)^{1-r}}\right] \\
&\leq (x-a)^r f(x)-(b-x)^r f(x)+f(b)(b-x)^r-f(a)(x-a)^r \\
&= (b-x)^r\left[f(b)-f(x)\right]+(x-a)^r\left[f(x)-f(a)\right],
\end{aligned}$$

which together with (38) provides the second inequality in (37).

The last inequality is obvious, since

$$
\begin{aligned}
(b-x)^r\,[f(b)-f(x)]+(x-a)^r\,[f(x)-f(a)] &\\
&\leq \max\left\{(b-x)^r,(x-a)^r\right\}[f(b)-f(a)]\\
&=[\max\{b-x,x-a\}]^r\,[f(b)-f(a)]\\
&=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r[f(b)-f(a)]
\end{aligned}
$$

for any $x\in[a,b]$. □

Remark 10 The particular case of L-Lipschitzian functions provides a much simpler result:

$$
\begin{aligned}
&\left|[u(b)-u(a)]f(x)-\int_a^b f(t)du(t)\right|\\
&\leq L\left[(2x-a-b)f(x)+\int_a^b \operatorname{sgn}(t-x)f(t)dt\right]\\
&\leq L\{(b-x)[f(b)-f(x)]+(x-a)[f(x)-f(a)]\}\\
&\leq L\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)] \qquad (39)
\end{aligned}
$$

for any $x\in[a,b]$.

A particular case that can be useful in applications is the following one [9].

Corollary 4 *With the assumptions in Theorem 5 we have:*

$$
\begin{aligned}
&\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right|\\
&\leq rH\left\{\int_{\frac{a+b}{2}}^b \frac{f(t)dt}{(b-t)^{1-r}}-\int_a^{\frac{a+b}{2}}\frac{f(t)dt}{(t-a)^{1-r}}\right\}\\
&\leq \frac{H(b-a)^r}{2^r}[f(b)-f(a)]. \qquad (40)
\end{aligned}
$$

In particular, for u a L-Lipschitizian function, we have

$$
\begin{aligned}
&\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right|\\
&\leq L\int_a^b sgn\left(t-\frac{a+b}{2}\right)\cdot f(t)dt\leq\frac{1}{2}L\,[f(b)-f(a)]\,. \qquad (41)
\end{aligned}
$$

Remark 11 The inequalities (41) are sharp. Indeed, if we take $u,f:[a,b]\to\mathbb{R}$, $u(t)=\left|t-\frac{a+b}{2}\right|$ and $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, then u is L-Lipschitzian with $L=1$ and f is monotonic nondecreasing on $[a,b]$. Also,

$$
\begin{aligned}
&[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\\
&\qquad= -\left[\int_a^{\frac{a+b}{2}}(-1)\cdot d\left(\frac{a+b}{2}-t\right)+\int_{\frac{a+b}{2}}^b(+1)\cdot dt\left(t-\frac{a+b}{2}\right)\right]\\
&\qquad= -(b-a),
\end{aligned}
$$

and

$$\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)f(t)dt = b-a,$$

and then we get in all sides of the inequality (41) the same quantity $(b-a)$.

Remark 12 In the case when $u(t) = t$, $t \in [a,b]$, out of (39) we deduce the Ostrowski-type inequality:

$$
\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| &\le \frac{1}{b-a}\left[[2x-(a+b)]+\int_a^b \operatorname{sgn}(t-x)f(t)dt\right]\\
&\le \frac{1}{b-a}\{(b-x)[f(b)-f(x)]+(x-a)[f(x)-f(a)]\}\\
&\le \left[\frac{1}{2}+\frac{|x-\frac{a+b}{2}|}{b-a}\right][f(b)-f(a)]
\end{aligned}
$$

that has been obtained in [10] (see also [19]).

6 Some Results for a Generalised Trapezoidal Rule

In [17], the authors have considered the following *generalised trapezoidal formula*:

$$[u(b)-u(x)]\,f(b)+[u(x)-u(a)]\,f(a), \quad x\in[a,b]$$

to approximate the Riemann–Stieltjes integral $\int_a^b f(t)du(t)$. They proved the inequality

$$
\begin{aligned}
&\left|\int_a^b f(t)du(t)-[u(b)-u(x)]\,f(a)-[u(x)-u(a)]f(a)\right|\\
&\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f) \qquad (42)
\end{aligned}
$$

for any $x\in[a,b]$, provided that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and u is of r-H-Hölder type.

The best inequality one can obtain from (42) is the following:

$$\left|\int_a^b f(t)du(t) - \left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(a) - \left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a)\right|$$
$$\leq \frac{H(b-a)}{2^r} \bigvee_a^b (f). \tag{43}$$

We observe that if $p, v : [a,b] \to \mathbb{R}$ are a pair of functions for which the Riemann–Stieltjes integral $\int_a^b p(t)dv(t)$ exists, then, on application of the integration by parts formula, we have

$$\begin{aligned}
&[v(b) - v(a)]\, p(x) - \int_a^b p(t)dv(t) \\
&= [v(b) - v(a)]\, p(x) - \left[p(b)v(b) - p(a)v(a) - \int_a^b v(t)dp(t)\right] \\
&= \int_a^b v(t)dp(t) - v(a)[p(x) - p(a)] - v(b)[p(b) - p(x)].
\end{aligned} \tag{44}$$

Therefore, any inequality of Ostrowski type for the difference

$$[v(b) - v(a)]\, p(x) - \int_a^b p(t)dv(t)$$

would give a corresponding inequality for the generalised trapezoidal approximation of the dual Riemann–Stieltjes integral:

$$\int_a^b v(t)dp(t) - v(a)\,[p(x) - p(a)] - v(b)\,[p(b) - p(x)]\,.$$

If v is of r-H-Hölder type and p is of bounded variation, then by (5) and (44) we recapture the result from [6]:

$$\left|\int_a^b v(t)dp(t) - v(a)\,[p(x) - p(a)] - v(b)\,[p(b) - p(x)]\right|$$
$$\leq H\left[(x-a)^r \bigvee_a^x (p) + (b-x)^r \bigvee_x^b (p)\right]$$
$$\leq \begin{cases} H\,[(x-a)^r + (b-x)^r]\left[\frac{1}{2}\bigvee_a^b (p) + \frac{1}{2}\left|\bigvee_a^x (p) - \bigvee_x^b (p)\right|\right]; \\ H\,[(x-a)^{qr} + (b-x)^{qr}]^{1/q}\left[\left[\bigvee_a^x (p)\right]^p + \left[\bigvee_x^b (p)\right]^p\right]^{\frac{1}{p}} \\ \qquad \text{if} \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (p) \end{cases} \tag{45}$$

for $x \in [a,b]$.

If we use (6) and the identity (44) above, then we can get the result in (42).

Now, if p is of r-H-Hölder type and v is monotonic nondecreasing, then by Theorem 6 and (44) we have the inequality

$$\begin{aligned}
&\left|\int_a^b v(t)dp(t) - v(a)\left[p(x)-p(a)\right] - v(b)\left[p(b)-p(x)\right]\right| \\
&\leq H\left\{(b-x)^r v(b) - (x-a)^r v(a) + r\left\{\int_a^x \frac{v(t)}{(x-t)^{1-r}}dt - \int_x^b \frac{v(t)}{(t-x)^{1-r}}dt\right\}\right\} \\
&\leq H\left\{(b-x)^r\left[v(b)-v(x)\right] + (x-a)^r\left[v(x)-v(a)\right]\right\} \\
&\leq H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \left[v(b)-v(a)\right] \qquad (46)
\end{aligned}$$

for any $x \in [a,b]$.

Finally, by employing Theorem 7 and the identity (44), we can state that for p monotonic nondecreasing and v of r-H-Hö lder type, we have:

$$\begin{aligned}
&\left|\int_a^b v(t)dp(t) - v(a)[p(x)-p(a)] - v(b)[p(b)-p(x)]\right| \\
&\leq H\left[\left[(x-a)^r - (b-x)^r\right]p(x) + r\left\{\int_x^b \frac{p(t)dt}{(b-t)^{1-r}} - \int_a^x \frac{p(t)dt}{(t-a)^{1-r}}\right\}\right] \\
&\leq H\left[(b-x)^r[p(b)-p(x)] + (x-a)^r[p(x)-p(a)]\right] \\
&\leq H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \left[p(b)-p(a)\right] \qquad (47)
\end{aligned}$$

for each $x \in [a,b]$.

7 The Case of Hölder Continuous and Lipschitzian Functions

The following result may be stated [4]:

Theorem 8 *Let $f : [a,b] \to \mathbb{R}$ be a $r-H-H$ölder continuous function on $[a,b]$, i.e.,*

$$|f(x) - f(y)| \leq H|x-y|^r \qquad \textit{for any } x, y \in [a,b], \tag{48}$$

where $r \in (0,1]$ and $H > 0$ are given, and $u : [a,b] \to \mathbb{R}$ is an $L-$Lipschitzian function on $[a,b]$, that is,

$$|u(x) - u(y)| \leq L|x-y|^r \qquad \textit{for any } x, y \in [a,b], \tag{49}$$

then for any $x \in [a,b]$,

$$\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t)\right| \le \frac{LH}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right], \tag{50}$$

or, equivalently,

$$\left|\int_a^b u(t)\,df(t)-\{u(b)[f(b)-f(x)]+u(a)[f(x)-f(a)]\}\right| \le \frac{LH}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]. \tag{51}$$

Proof Note that if $p:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian, then the Riemann–Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt. \tag{52}$$

Utilising this property,

$$\begin{aligned}\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t)\right| &= \left|\int_a^b [f(x)-f(t)]\,du(t)\right| \\ &\le L\int_a^b |f(x)-f(t)|\,dt \\ &\le LH\int_a^b |x-t|^r\,dt \\ &= \frac{LH}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right],\end{aligned}$$

and the inequality (50) is proved.

Since, by the integration by parts formula for Riemann–Stieltjes integrals we have,

$$\begin{aligned}&[u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t) \\ &= \int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)],\end{aligned}$$

hence (51) is a direct consequence of (50). □

Remark 13 If f is assumed to be $K-$Lipschitzian, then from (50) and (51) we get the equivalent inequalities:

$$\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t)\right| \le HL\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 \tag{53}$$

and

$$\left| \int_a^b u(t)\, df(t) - u(b)[f(b) - f(x)] - u(a)[f(x) - f(a)] \right|$$
$$\leq HL \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2$$

for each $x \in [a, b]$.

The midpoint inequality is useful for numerical implementation and is incorporated in the following corollary [4].

Corollary 5 *With the assumptions of Theorem 8,*

$$\left| [u(b) - u(a)]\, f\left(\frac{a+b}{2} \right) - \int_a^b f(t)\, du(t) \right| \leq \frac{1}{2^r (r+1)} LH (b-a)^{r+1}, \tag{54}$$

and

$$\left| \int_a^b u(t) df(t) - u(b) \left[f(b) - f\left(\frac{a+b}{2} \right) \right] - u(a) \left[f\left(\frac{a+b}{2} \right) - f(a) \right] \right|$$
$$\leq \frac{1}{2^r (r+1)} LH (b-a)^{r+1}, \tag{55}$$

respectively.

Remark 14 If $u(t) = t$ in the above, then the results for the Riemann integral obtained in [18] are recaptured.

Remark 15 In terms of probability density functions, if $w : [a, b] \to [0, \infty)$ is such that $w \in L_\infty [a, b]$, i.e., $\|w\|_{[a,b],\infty} := ess\sup_{t \in [a,b]} |w(t)| < \infty$, and $\int_a^b w(s)\, ds = 1$, then the function $u(t) = \int_a^t w(s)\, ds$ is $L-$Lipschitzian with the constant $L = \|w\|_{[a,b],\infty}$ and the inequalities (50) and (51) can be written as:

$$\left| f(x) - \int_a^b w(t) f(t) dt \right| \leq \frac{H \|w\|_{[a,b],\infty}}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right] \tag{56}$$

and

$$\left| \int_a^b \left(\int_a^t w(s) ds \right) df(t) - f(b) - f(x) \right|$$
$$\leq \frac{H \|w\|_{[a,b],\infty}}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right] \tag{57}$$

for any $x \in [a, b]$.

The dual case, that is, when f is Lipschitzian and u is Hölder continuous admits some slight variations as follows [4].

Theorem 9 *Let $x \in [a,b]$ and assume that f is L_1- Lipschitzian on the interval $[a,x]$ and L_2-Lipschitzian on the interval $[a,b]$ $(L_1, L_2 > 0)$ while the function $u : [a,b] \to \mathbb{R}$ satisfies some local Hölder conditions (properties), namely,*

$$|u(t) - u(a)| \leq H_1 |t-a|^{\alpha_1} \qquad \text{for any } t \in [a,x] \tag{58}$$

and

$$|u(b) - u(t)| \leq H_2 |t-b|^{\alpha_2} \qquad \text{for any } t \in [x,b], \tag{59}$$

where $H_1, H_2 > 0$, $\alpha_1, \alpha_2 \in (-1, \infty)$ (notice the difference for α_1, α_2), then,

$$\begin{aligned} &\left|[u(b)-u(a)]f(x) - \int_a^b f(t)du(t)\right| \\ &\leq \frac{L_1 H_1 (x-a)^{\alpha_1+1}}{\alpha_1+1} + \frac{L_2 H_2 (b-x)^{\alpha_2+1}}{\alpha_2+1} \end{aligned} \tag{60}$$

or, equivalently,

$$\begin{aligned} &\left|\int_a^b u(t)df(t) - u(b)[f(b)-f(x)] - u(a)[f(x)-f(a)]\right| \\ &\leq \frac{L_1 H_1 (x-a)^{\alpha_1+1}}{\alpha_1+1} + \frac{L_2 H_2 (b-x)^{\alpha_2+1}}{\alpha_2+1}. \end{aligned} \tag{61}$$

Proof We use the following generalisation of the *Montgomery identity* for the Riemann–Stieltjes integral established by S.S. Dragomir [12]:

$$\begin{aligned} &[u(b)-u(a)]f(x) - \int_a^b f(t)du(t) \\ &= \int_a^x [u(t)-u(a)]df(t) + \int_x^b [u(t)-u(b)]df(t) \end{aligned} \tag{62}$$

for any $x \in [a,b]$.

Taking the modulus we have

$$\begin{aligned} &\left|[u(b)-u(a)]f(x) - \int_a^b f(t)du(t)\right| \\ &\leq \left|\int_a^x [u(t)-u(a)]df(t)\right| + \left|\int_x^b [u(t)-u(b)]df(t)\right| \\ &\leq L_1 \int_a^x |u(t)-u(a)|dt + L_2 \int_x^b |u(t)-u(b)|dt \\ &\leq H_1 L_1 \int_a^x (t-a)^{\alpha_1} dt + H_2 L_2 \int_x^b (b-x)^{\alpha_2} dt \\ &= \frac{H_1 L_1 (x-a)^{\alpha_1+1}}{\alpha_1+1} + \frac{H_2 L_2 (b-x)^{\alpha_2+1}}{\alpha_2+1}, \end{aligned}$$

and the inequality (60) is obtained. □

Remark 16 It is obvious that, if we assume that f is $K-$Lipschitzian on the whole interval $[a,b]$ while u is of the $q-$Hö lder type with $q \in (0,1]$, then from Theorem 9 we can obtain the following inequality which is the dual of (50):

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \le \frac{KH}{q+1}\left[(x-a)^{q+1}+(b-x)^{q+1}\right] \tag{63}$$

for any $x \in [a,b]$. □

Remark 17 From the tools utilised in the proofs of Theorem 8 and 9, one can easily realise that if in the first result it is natural to assume the global property of $r-H-$Hölder continuity for the integrand and $L-$Lipschitzian property for the integrator, then in the second theorem the local properties around the end-points a and b qualify as natural as well. Moreover, we observe that in (51) the order of approximation is $\min(\alpha_1,\alpha_2)+1$ which can be higher than the order of approximation in (50) which is $r+1$ (maximum 2 for $r=1$). However, this can be improved if some local conditions around $x \in [a,b]$ are assumed.

If u is T_1-Lipschitzian on $[a,x]$ and T_2- Lipschitzian on $[x,b]$ and the function f satisfies around x the following conditions

$$|f(t)-f(x)| \le V_1\,|t-x|^{\beta_1}, \qquad t \in [a,x],$$

and

$$|f(t)-f(x)| \le V_2\,|t-x|^{\beta_2}, \qquad t \in [x,b],$$

where $V_1, V_2 > 0$, $\beta_1, \beta_2 \in (-1,\infty)$ are given, then, following the proof of Theorem 8, we have,

$$\begin{aligned}
&\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \\
&= \left|\int_a^x (f(x)-f(t))\,du(t)+\int_x^b (f(x)-f(t))\,du(t)\right| \\
&\le \left|\int_a^x (f(x)-f(t))\,du(t)\right|+\left|\int_x^b (f(x)-f(t))\,du(t)\right| \\
&\le T_1\int_a^x |f(x)-f(t)|\,dt+T_2\int_x^b |f(x)-f(t)|\,dt \\
&\le \frac{T_1V_1(x-a)^{\beta_1+1}}{\beta_1+1}+\frac{T_2V_2(b-x)^{\beta_2+1}}{\beta_2+1},
\end{aligned}$$

giving a similar result to the one in Theorem 9.

8 The Case of Monotonic and Lipschitzian Functions

The case where the integrator is monotonic nondecreasing is incorporated in the following result [4]:

Theorem 10 *Let $x \in [a,b]$ and assume that $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,x]$ and $[x,b]$ (it may not be monotonic nondecreasing on the whole of $[a,b]$). If u is L_1-Lipschitzian on $[a,x]$ and L_2-Lipschitzian on $[x,b]$, then,*

$$
\begin{aligned}
&\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \\
&\leq L_2 \int_x^b f(t)\, dt - L_1 \int_a^x f(t)\, dt - [L_2 (b-x) - L_1 (x-a)] f(x) \\
&\leq L_2 (b-x) [f(b) - f(x)] + L_1 (x-a) [f(x) - f(a)] \\
&\leq \max\{L_1, L_2\} ((b-x)[f(b) - f(x)] + (x-a)[f(x) - f(a)]) \\
&\leq \max\{L_1, L_2\} \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] [f(b) - f(a)]; \\ \left[\frac{1}{2}[f(b) - f(a)] + \frac{1}{2}\left|f(x) - \frac{f(a)+f(b)}{2}\right|\right] (b-a), \end{cases}
\end{aligned} \tag{64}
$$

and a similar inequality holds for the generalised trapezoidal rule.

Proof As in the proof of Theorem 8 above, we have,

$$
\begin{aligned}
&\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \\
&\leq L_1 \int_a^x |f(x) - f(t)|\, dt + L_2 \int_x^b |f(x) - f(t)|\, dt \\
&= L_1 (x-a) f(x) - L_1 \int_a^x f(t)\, dt + L_2 \int_x^b f(t)\, dt - L_2 (b-x) f(x) \\
&= L_2 \int_x^b f(t)\, dt - L_1 \int_a^x f(t)\, dt - [L_2 (b-x) - L_1 (x-a)] f(x),
\end{aligned}
$$

proving the first inequality in (64).

Now, on utilising the monotonicity property of f on both intervals, we have

$$
\int_x^b f(t)\, dt \leq (b-x) f(b) \quad \text{and} \quad \int_a^x f(t)\, dt \geq (x-a) f(a),
$$

which implies that,

$$
\begin{aligned}
&L_2 \int_x^b f(t)\, dt - L_1 \int_a^x f(t)\, dt - [L_2 (b-x) - L_1 (x-a)] f(x) \\
&\leq L_2 (b-x) f(b) - L_1 (x-a) f(a) - [L_2 (b-x) - L_1 (x-a)] f(x) \\
&= L_2 (b-x) [f(b) - f(x)] + L_1 (x-a) [f(x) - f(a)],
\end{aligned}
$$

that is, the second inequality in (64).

The last part is obvious by the property of the max function and we omit the details. □

Corollary 6 *If $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ and u is L_1-Lipschitzian on the first interval and L_2- Lipschitzian on the second, then*

$$
\begin{aligned}
&\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right| \\
&\le L_2\int_{\frac{a+b}{2}}^b f(t)dt - L_1\int_a^{\frac{a+b}{2}} f(t)dt - \frac{b-a}{2}(L_2-L_1)f\left(\frac{a+b}{2}\right) \\
&\le \frac{b-a}{2}\left[L_2[f(b)-f(x)]+L_1[f(x)-f(a)]\right] \\
&\le \frac{b-a}{2}\max\{L_1,L_2\}[f(b)-f(a)].
\end{aligned}
\tag{65}
$$

Remark 18 The case $u(t) = t$ (therefore $L_1 = L_2 = 1$) retrieves the results obtained earlier for the Riemann integral in [10].

The dual case is incorporated in the following result [4]:

Theorem 11 *Let $x \in [a,b]$ and assume that u is monotonic nondecreasing on both $[a,x]$ and $[x,b]$, then,*

$$
\begin{aligned}
&\left|[u(b)-u(x)]f(x)-\int_a^b f(t)du(t)\right| \\
&\le L_2(b-x)u(b)+L_1(x-a)u(a)+L_1\int_a^x u(t)dt - L_2\int_x^b u(t)dt \\
&\le L_1(x-a)(u(x)-u(a))+L_2(b-x)(u(b)-u(x)) \\
&\le \max\{L_1,L_2\}[(x-a)(u(x)-u(a))+(b-x)(u(b)-u(x))] \\
&\le \max\{L_1,L_2\}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][u(b)-u(a)]; \\ \left[\frac{1}{2}[u(b)-u(a)]+\frac{1}{2}\left|u(x)-\frac{u(a)+u(b)}{2}\right|\right](b-a),\end{cases}
\end{aligned}
\tag{66}
$$

and a similar inequality holds for the generalised trapezoidal rule.

Proof As in the proof of Theorem 9 above, we have,

$$
\begin{aligned}
&\left|[u(b)-u(x)]f(x)-\int_a^b f(t)\,du(t)\right| \\
&\le L_1\int_a^x |u(t)-u(a)|\,dt + L_2\int_x^b |u(t)-u(b)|\,dt \\
&= L_1\int_a^x u(t)\,dt - L_1(x-a)u(a)+L_2(b-x)u(b)-L_2\int_x^b u(t)\,dt
\end{aligned}
$$

and the first inequality in (66) is proved.

By the monotonicity of u in both intervals $[a, x]$ and $[x, b]$ we have,

$$\int_a^x u(t)\,dt \le (x-a)\,u(x) \quad \text{and} \quad \int_x^b u(t)\,dt \ge (b-x)\,u(x),$$

which gives

$$\begin{aligned}
&L_1 \int_a^x u(t)\,dt - L_1(x-a)\,u(a) + L_2(b-x)\,u(b) - L_2 \int_x^b u(t)\,dt \\
&\le L_1(x-a)\,u(x) - L_1(x-a)\,u(a) + L_2(b-x)\,u(b) - L_2(b-x)\,u(x) \\
&= L_1(x-a)\,[u(x) - u(a)] + L_2(b-x)\,[u(b) - u(x)]
\end{aligned}$$

and the second part of (66) also holds.

The last part is obvious and the details are omitted. □

Corollary 7 *If u is monotonic on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ while f is L_1-Lipschitzian on the first interval and L_2-Lipschitzian on the second, then*

$$\begin{aligned}
&\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right) - \int_a^b f(t)du(t)\right| \\
&\le \frac{b-a}{2}\,[L_2 u(b) - L_1 u(a)] + L_1 \int_a^{\frac{a+b}{2}} u(t)dt - L_2 \int_{\frac{a+b}{2}}^b u(t)dt \\
&\le \frac{b-a}{2}\left\{L_1\left[u\left(\frac{a+b}{2}\right) - u(a)\right] + L_2\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right\} \\
&\le \frac{b-a}{2}\max\{L_1, L_2\}\,[u(b)-u(a)]. \qquad (67)
\end{aligned}$$

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Invariance in the Family of Weighted Gini Means

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Abstract Given two means M and N, the mean P is called (M, N)-invariant if $P(M, N) = P$. At the same time the mean N is called complementary to M with respect to P. We use the method of series expansion of means to determine the complementary with respect to a weighted Gini mean. The invariance in the family of weighted Gini means is also studied. The computer algebra Maple was used for solving some complicated systems of equations.

Keywords Weighted Gini mean · Complementary mean · Invariance in a class of means.

1 Means

A **mean** is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a, b) \leq M(a, b) \leq \max(a, b), \ \forall a, b > 0.$$

Each mean is **reflexive**, that is

$$M(a, a) = a, \ \forall a > 0.$$

A mean is **symmetric** iff

$$M(b, a) = M(a, b), \ \forall a, b > 0,$$

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homogeneous iff

$$M(ta, tb) = tM(a, b),\ \forall a, b, t > 0,$$

it is **strict** iff

$$[M(a, b) - a]\,[M(a, b) - b] \neq 0, \text{for } a \neq b,$$

and **strictly isotone** iff for each $a, b > 0$, the functions $M(a, .)$ and $M(., b)$ are strictly increasing.

A reflexive function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called also a **pre-mean**.

We shall refer here to the following families of means:

- the **weighted Gini means**(or **sum means**) $\mathcal{S}_{p,q;\lambda}$, defined for $p \neq q$ by

$$\mathcal{S}_{p,q;\lambda}(a, b) = \left(\frac{\lambda a^p + (1-\lambda)b^p}{\lambda a^q + (1-\lambda)b^q}\right)^{\frac{1}{p-q}}, \lambda \in (0, 1)\,;$$

- the **weighted Lehmer means** (or **generalized counter-harmonic means**) $\mathcal{C}_{p;\lambda} = \mathcal{S}_{p,p-1;\lambda}$;
- the **weighted power means** $\mathcal{P}_{q;\lambda} = \mathcal{S}_{q,0;\lambda}$;
- the **weighted arithmetic means** $\mathcal{A}_\lambda = \mathcal{P}_{1;\lambda}$;
- the **weighted harmonic means** $\mathcal{H}_\lambda = \mathcal{P}_{-1;\lambda}$;
- the **weighted geometric means** $\mathcal{G}_\lambda$, defined by

$$\mathcal{G}_\lambda(a, b) = a^\lambda b^{1-\lambda}.$$

The symmetric means $\mathcal{S}_{p,q;1/2}, \mathcal{C}_{p;1/2}, \mathcal{P}_{q;1/2}, \mathcal{A}_{1/2}, \mathcal{H}_{1/2}$ and $\mathcal{G}_{1/2}$ are written simply as $\mathcal{S}_{p,q}, \mathcal{C}_p, \mathcal{P}_q, \mathcal{A}, \mathcal{H}$ respectively $\mathcal{G}$. For $\lambda = 0$ or $\lambda = 1$, we have

$$\mathcal{S}_{p,q;0} = \Pi_2 \text{ and } \mathcal{S}_{p,q;1} = \Pi_1, \forall p, q \in \mathbb{R}\,,$$

where Π_1 and Π_2 are the **projections** defined by

$$\Pi_1(a, b) = a,\ \Pi_2(a, b) = b,\ \forall a, b > 0,$$

respectively. For $\lambda \notin [0, 1]$, $\mathcal{S}_{p,q;\lambda}$ are only pre-means for all $p, q \in \mathbb{R}$.

Some families of means are defined with respect to arbitrary functions. For instance, given a fixed mean M and a bijection $f : \mathbb{R}_+ \to \mathbb{R}_+$, we can construct a mean $M(f)$ defined by

$$M(f)(a, b) = f^{-1}(M(f(a), f(b))), \forall a, b > 0.$$

If we take $M = \mathcal{A}_\lambda$, we get the family of **weighted quasi-arithmetic means**. In a similar way, the **Beckenbach-Gini means** are defined by

$$C_f(a, b) = \frac{af(a) + bf(b)}{f(a) + f(b)}, \forall a, b > 0,$$

where f is a positive function. A generalized quasi-arithmetic mean $\mathcal{A}^{[f,g]}$, is defined by

$$\mathcal{A}^{[f,g]}(a,b) = (f+g)^{-1}\left(f(a)+g(b)\right),$$

where $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ and $f+g$ is a bijection. A Lagrangian quasi-arithmetic mean $\mathcal{A}_{[\mu]}^{[f]}$ is defined by

$$\mathcal{A}_{[\mu]}^{[f]}(a,b) = f^{-1}\left(\int_0^1 f\left(tx+(1-t)\,y\right)d\mu(t)\right),$$

where $f : [0,1] \to [0,1]$ is a bijection.

More details on means can be found in [5]. Let us underline that the notations can be different from one paper to another.

2 Invariance of Means

Given three functions $M, N, P : \mathbb{R}_+^2 \to \mathbb{R}_+$, we can compose them, obtaining a new function $P(M,N) : \mathbb{R}_+^2 \to \mathbb{R}_+$, defined by

$$P(M,N)(a,b) = P(M(a,b), N(a,b)), \forall a,b > 0.$$

If M, N, P are means (pre-means) then $P(M,N)$ is also a mean (respectively a pre-mean).

Definition 1 The function P is called $(M,N)-$ **invariant** if it verifies

$$P(M,N) = P.$$

Obviously we have the following **duality property**:

Lemma 1 *If the symmetric mean P is $(M,N)-$ invariant, then it is also $(N,M)-$ invariant.*

The following property was proved in [36].

Lemma 2 *If the means M and N are symmetric and P is $(M,N)-$ invariant, then P is also symmetric.*

A similar result can be also proved.

Lemma 3 *If the means P and M are symmetric, P is strictly isotonic and $(M,N)-$ invariant, then N is also symmetric.*

Proof We have

$$P(M(a,b), N(a,b)) = P(a,b), P(M(b,a), N(b,a)) = P(b,a), \forall a,b \in J.$$

As P and M are symmetric, the second equality gives

$$P(M(a,b), N(b,a)) = P(a,b), \forall a,b \in J,$$

thus

$$P(M(a,b), N(a,b)) = P(M(a,b), N(b,a)), \forall a, b \in J.$$

The strict isotony of P implies the symmetry of N. □

These properties are related to the following problem. Given two functions $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ and two numbers $a_0, b_0 \in \mathbb{R}_+$, we can define a (Gaussian) **double sequence** by:

$$a_{n+1} = M(a_n, b_n), b_{n+1} = N(a_n, b_n), \forall n > 0.$$

If M, N are means which have some properties (for instance, one of them is continuous and strict (see [34])), the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are convergent to a common limit $P(a_0, b_0)$. Moreover P also defines a mean. C. F. Gauss was the first author who related the problem of determining the common limit of the double sequences, to the invariance of the mean P with respect to (M, N), in the special case in which M is the arithmetic mean while N is the geometric mean. A general **invariance principle** was proved in [3]. It was generalized for pre-means in [37]:

Theorem 1 *Let P be a continuous pre-mean and M and N be two functions such that P is $(M, N)-$ invariant. If the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are convergent to a common limit l, then $l = P(a_0, b_0)$.*

3 Invariance in a Family of Means

Given a family $\mathcal{Z}$ of means, we can consider three problems of invariance:

- A first problem is that of the study of the **invariance** of a given mean P **with respect to the family** $\mathcal{Z}$. This means the determining of all the pairs of means (M, N) from $\mathcal{Z}$ such that P is $(M, N)-$ invariant.
- A second problem is named **invariance in the family** $\mathcal{Z}$. It consists of determining all the triples of means (P, M, N) from $\mathcal{Z}$ such that P is $(M, N)-$ invariant.
- A third type of problem was called **reproducing identities** and assumes determining quadruples of means (P, M, N, Q) from $\mathcal{Z}$ such that

$$P\,(M, N) = Q. \tag{1}$$

This problem has the trivial solution

$$P\,(M, M) = M, \tag{2}$$

the solutions of the invariance problem

$$P\,(M, N) = P, \tag{3}$$

but it can have other solutions also.

Many problems of the first type were formulated as functional equations. The first one was related to the invariance of the arithmetic mean $\mathcal{A}$ with respect to the family of quasi-arithmetic means $\mathcal{A}(f)$. It was solved in [33] for analytical functions f and in [30] for the second order continuously differentiable functions f. It was called the Matkowski-Sutô problem (see [16]). The regularity assumptions were weakened step-by step in [19, 16, 17], arriving at simple continuity hypothesis on the functions f.

The problem of invariance of the arithmetic mean $\mathcal{A}$ was studied later:

- with respect to the family of Lagrangian means, in [32];
- with respect to the family of Beckenbach-Gini means, in [20];
- with respect to the family of weighted quasi-arithmetic means $\mathcal{A}_\lambda(f)$, in [1] and [18];
- with respect to the family of generalized quasi-arithmetic means $\mathcal{A}^{[f,g]}$, in [29];
- with respect to the family of Lagrangian quasi-arithmetic means $\mathcal{A}^{[f]}_{[\mu]}$, in [32].

The problem of invariance of the geometric mean $\mathcal{G}$ with respect to the family of Lagrangian means was studied in [22].

The problem of invariance was studied in the family of Beckenbach-Gini means in [31], in the family of Greek means in [35], and in the family of weighted quasi-arithmetic means in [27] and [26].

The first reproducing identities problem was studied in [4] for the families of Lehmer means and for that of power means.

4 Complementary of a Mean with Respect to Another Mean

Given two means M and N, it is very difficult to find a mean P which is (M, N) -invariant, as can be seen in the case considered by C. F. Gauss: $M = \mathcal{A}$ and $N = \mathcal{G}$ (see [3]). Another method was considered to overcome this situation. The idea was taken from [21] where two means M and N are called **complementary** (with respect to $\mathcal{A}$) if $M + N = 2\mathcal{A}$. We remark that for every mean M, the function $2 \cdot \mathcal{A} - M$ is again a mean. Thus the complementary of every mean M exists and it is denoted by ${}^cM$. The most interesting example of a mean defined on this way is the contraharmonic mean given by $\mathcal{C} = {}^c\mathcal{H}$. A second notion of this type also considered in [21] is the following: two means M and N are called **inverses** (with respect to $\mathcal{G}$) if $M \cdot N = \mathcal{G}^2$. Again, for every (nonvanishing) mean M, the expression $\mathcal{G}^2/M$ gives a mean, the inverse of M, which we denote by ${}^iM$. For example we have

$${}^i\mathcal{A} = \mathcal{H}.$$

In [34] and then in [30] it was proposed a generalization of complementariness and of inversion.

Definition 2 A mean N is called **complementary to** M **with respect to** P (or $P-$ **complementary** to M) if it verifies

$$P(M, N) = P.$$

Remark 1 Of course this is equivalent with the property that P is (M, N)—invariant, but sometimes we can easier determine the mean N which is the P—complementary of M, than to determine the mean P which is (M, N)—invariant.

Remark 2 The P—complementary of a given mean does not necessarily exist nor is unique. For example the Π_1—complementary of Π_1 is any mean M, but no mean $M \neq \Pi_1$ has a Π_1—complementary. If a given mean M has a unique P—complementary mean N, we denote it by ${}^P M$.

Proposition 1 *For every mean M we have*

$$ {}^M M = M, \tag{4} $$

$$ {}^M \Pi_1 = \Pi_2, \tag{5} $$

$$ {}^{\Pi_2} M = \Pi_2 \tag{6} $$

and if P is a symmetric mean then

$$ {}^P \Pi_2 = \Pi_1. \tag{7} $$

Remark 3 In what follows, we shall call these results as **trivial cases** of complementariness. We shall denote also

$$ {}^{\Pi_1} \Pi_1 = M, \tag{8} $$

meaning that $\Pi_1 (\Pi_1, M) = \Pi_1$.

Remark 4 Of course, we are interested in determining non trivial cases. The complementariness with respect to $\mathcal{P}_{m;\lambda}$ for $\lambda \neq 1$ was considered in [8]. If we denote it by ${}^{\mathcal{P}(m;\lambda)}M$, we find the expression

$$ {}^{\mathcal{P}(m;\lambda)}M = \left[\frac{\left(\mathcal{P}_{m;\lambda}\right)^m - \lambda \cdot M^m}{1-\lambda} \right]^{\frac{1}{m}}, \ m \neq 0, $$

while, for $m = 0$ we have

$$ {}^{\mathcal{G}(\lambda)}M = \left(\frac{\mathcal{G}_\lambda}{M^\lambda} \right)^{\frac{1}{1-\lambda}}. $$

Lemma 4 *The pre-mean ${}^{\mathcal{P}(m;\lambda)}M$ is a mean for every mean M and each $m \in \mathbb{R}$ if and only if* $0 \leq \lambda \leq \frac{1}{2}$.

Remark 5 This complementary can exist also for $1/2 < p < 1$, but only for some means. For example, we have

$$ {}^{\mathcal{G}(\lambda)}\mathcal{G}_\mu = \mathcal{G}_{\frac{\lambda(1-\mu)}{1-\lambda}}, \tag{9} $$

and the result is a mean for $0 < \lambda \leq \frac{1}{2-\mu}$. We shall refer also to the following special cases

$$^{\mathcal{G}(\lambda)}\mathcal{G}_{\frac{3\lambda-1}{2\lambda}} = \mathcal{G}; \tag{10}$$

$$^{\mathcal{G}(\lambda)}\mathcal{G} = \mathcal{G}_{\frac{\lambda}{2(1-\lambda)}}; \tag{11}$$

$$^{\mathcal{G}(\lambda)}\mathcal{G}_{\frac{2\lambda-1}{\lambda}} = \Pi_1; \tag{12}$$

$$^{\mathcal{G}(1/3)}\Pi_2 = \mathcal{G}; \tag{13}$$

$$^{\mathcal{G}(2/3)}\mathcal{G} = \Pi_1; \tag{14}$$

and

$$^{\mathcal{G}}\mathcal{G}_\mu = \mathcal{G}_{1-\mu}. \tag{15}$$

5 Series Expansion of Means

For the study of some problems related to means, the power series expansions was used in [28]. Let M be a symmetric and homogeneous mean. Without loss of generality we may assume that M acts on the positive numbers $a \geq b$ and

$$M(a,b) = aM(1,b/a) = aM(1,1-t),$$

where

$$0 \leq t = 1 - b/a < 1.$$

For many problems it suffices to consider only the normalized function $M(1,1-t)$ even if the mean M is not symmetric nor homogeneous. We shall give explicit Taylor series coefficients of the normalized function for some means. In order to avoid complicating the presentation, we shall call them series expansions of the corresponding means. For some means, determining all the coefficients is impossible. In these cases, a recurrence relation for the coefficients will be very useful. It gives a way to calculate as many coefficients as desired. Such a formula was given by Euler (see [23]) in the following:

Theorem 2 *If the function f has the Taylor series*

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n,$$

p is a real number and

$$[f(x)]^p = \sum_{n=0}^{\infty} b_n \cdot x^n,$$

then we have the recurrence relation

$$\sum_{k=0}^{n} [k(p+1) - n] \cdot a_k \cdot b_{n-k} = 0,\ n \geq 0. \tag{16}$$

In [11] it was proved the following

Theorem 3 *The first terms of the power series expansion of the weighted Gini mean $\mathcal{S}_{p,p-r;t}$, with $r \neq 0$, $t \in (0,1)$ are*

$$\mathcal{S}_{p,p-r;t}(1, 1-x) = 1 - (1-t) \cdot x + t\,(1-t)\,(2p-r-1) \cdot \frac{x^2}{2!} - t$$
$$\cdot\,(1-t)\left[t\left(6p^2 - 6p\,(r+1) + (r+1)\,(2r+1)\right) - 3p\,(p-r) - (r-1)\,(r+1)\right] \cdot \frac{x^3}{3!}$$
$$-t\,(1-t) \cdot \left[t^2(-24p^3 + 36p^2\,(r+1) - 12p\,(r+1)\,(2r+1) + (r+1)\,(2r+1)\right.$$
$$\cdot\,(3r+1)\,) + t(24p^3 - 12p^2\,(3r+1) + 12p\,(r+1)\,(2r-1) - 3\,(r+1)\,(2r+1)$$
$$\left.\cdot\,(r-1)\,) - 4p^3 + 6p^2\,(r-1) - 2p\left(2r^2 - 3r - 1\right) + (r-2)\,(r-1)\,(r+1)\right] \cdot \frac{x^4}{4!} -$$
$$-t\,(1-t) \cdot \left[t^3\left(120p^4 - 240p^3\,(r+1) + 120p^2(r+1)(2r+1) -\right.\right.$$
$$-20p(r+1)(2r+1)(3r+1) + (r+1)(2r+1)(3r+1)(4r+1)) +$$
$$+t^2\left(-180p^4 + 180p^3(2r+1) - 90p^2(r+1)(4r-1) + 30p(r+1)(2r+1)\right.$$
$$\cdot(3r-2) - 6(r-1)(r+1)(2r+1)(3r+1)) + t\left(70p^4 - 20p^3(7r-2) + 10p^2\right.$$
$$\left.\cdot(14r^2 - 6r - 9) - 10p(r+1)(7r^2 - 12r + 3) + (r-1)(2r+1)(7r-11)(r+1)\right)$$
$$-5p^4 + 10p^3(r-2) - 5p^2(2r^2 - 6r + 3) + 5p(r-2)(r^2 - 2r - 1)$$
$$\left.-(r+1)(r-1)(r-2)(r-3)\right] \cdot \frac{x^5}{5!} + \cdots$$

Taking $r = 1$ we get the first terms of the weighted Lehmer mean $\mathcal{C}_{p;t}$. The first terms of $\mathcal{C}_p$ were given in [24]. Also, for $r = p$ we get the first terms of the weighted power mean $\mathcal{P}_{p;t}$ which were determined in [6]. Its first part was given for $\mathcal{P}_p$ in [28].

Using series expansion of means, in [28] it was proved that the families of means $\mathcal{P}_q$ and $\mathcal{C}_p$ have in common only the arithmetic mean, geometric mean, and harmonic mean. More generally in [11] is proved the following result:

Theorem 4 *The families of weighted means $\mathcal{P}_{q;t}$ and $\mathcal{C}_{p;s}$ have in common only the weighted arithmetic mean $\mathcal{A}_t$, the geometric mean $\mathcal{G}$, the weighted harmonic mean $\mathcal{H}_t$, and the first and the second projection Π_1 and Π_2.*

6 Generalized Inverses of Means

The basic results related to **generalized inverses** of means, that is to complementary with respect to $\mathcal{G}_\lambda$, were given in [6]. We denote the $\mathcal{G}_\lambda$−complementary of M by ${}^{\mathcal{G}(\lambda)}M$. For $\lambda = 1/2$ we use the simpler notation ${}^{\mathcal{G}}M$.

Theorem 5 *If the mean M has the series expansion*

$$M(1, 1-x) = 1 + \sum_{n=1}^{\infty} a_n x^n,$$

then the first terms of the series expansion of its generalized inverse ${}^{\mathcal{G}(\lambda)}M$ are

$$\begin{aligned}
{}^{\mathcal{G}(\lambda)}M(1, 1-x) &= 1 - (1 + \alpha \cdot a_1) \cdot x + \frac{\alpha}{2}\left[(\alpha+1) \cdot a_1^2 + 2\,(a_1 - a_2)\right] \cdot x^2 \\
&- \frac{\alpha}{6}\left[(\alpha+1)(\alpha+2) \cdot a_1^3 + 3\,(\alpha+1) \cdot a_1\,(a_1 - 2a_2) + 6\,(a_3 - a_2)\right] \cdot x^3 \\
&+ \frac{\alpha}{24}\left[(\alpha+1)(\alpha+2)(\alpha+3) \cdot a_1^4 + 4a_1^2\,(\alpha+1)(\alpha+2)(a_1 - 3a_2)\right. \\
&\left. + 12\,(\alpha+1)\left(a_2^2 - 2a_1\,(a_2 - a_3)\right) + 24\,(a_3 - a_4)\right] \cdot x^4 - \frac{\alpha}{5!}\left[(\alpha+1)(\alpha+2) \cdot\right. \\
&\cdot (\alpha+3)(\alpha+4) \cdot a_1^5 + 5a_1^3\,(\alpha+1)(\alpha+2)(\alpha+3)(a_1 - 4a_2) - 60a_1^2 \cdot \\
&\cdot (\alpha+1)(\alpha+2)(a_2 - a_3) + 60a_1\,(\alpha+1)\left((\alpha+2)\,a_2^2 + 2\,(a_3 - a_4)\right) \\
&\left. + 60a_2\,(\alpha+1)\,(a_2 - 2a_3) - 120\,(a_4 - a_5)\right] \cdot x^5 + \cdots,
\end{aligned}$$

where

$$\alpha = \frac{\lambda}{1-\lambda}.$$

The series expansion of the generalized inverse of $S_{p,p-r;\mu}$ was given in [7].

Corollary 1 *The first terms of the series expansion of the generalized inverse of $\mathcal{S}_{p,p-q;\mu}$ are*

$$\begin{aligned}
{}^{\mathcal{G}(\lambda)}\mathcal{S}_{p,p-q;\mu}\,(1, 1-x) &= 1 - (\alpha\mu - \alpha + 1) \cdot x - \alpha\,(1-\mu)\left[(\alpha + 2p - q)\,\mu\right. \\
&\left. - (\alpha - 1)\right] \cdot \frac{x^2}{2!} + \alpha\,(1-\mu)\left\{\left[6p^2 + 6\,(\alpha - q)\,p + (\alpha - q)(\alpha - 2q)\right]\mu^2 - \left[3p^2\right.\right. \\
&\left.\left. -3\,(q - 2\alpha)\,p + (2\alpha - q)\,(\alpha - q)\right]\mu + (\alpha - 1)(\alpha + 1)\right\} \cdot \frac{x^3}{3!} - \alpha\,(1-\mu)\left\{\left[24p^3\right.\right. \\
&+ 36\,(\alpha - q)\,p^2 + 12\,(\alpha - q)(\alpha - 2q)\,p + (\alpha - q)(\alpha - 2q)\,(\alpha - 3q)\right]\mu^3 + \left[-24p^3\right. \\
&\cdot p^2 - 12\,(2\alpha - 2q + 1)(\alpha - q)\,p - (\alpha - 2q)(\alpha - q)\,(3\alpha + 2 - 3q)\right]\mu^2 + \left[4p^3\right. \\
&+ 12\,(3q - 4\alpha - 1) + 6\,(2\alpha - q + 1)\,p^2 + 2\left(6\alpha\,(2\alpha - 2q + 1) - 3q + 2q^2 - 1\right)p \\
&\left.\left. + (\alpha - q)(3\alpha^2 + 4\alpha - 3q\alpha - 2q + q^2 - 1)\right]\mu - (\alpha - 1)(\alpha + 1)(\alpha + 2)\right\} \cdot \frac{x^4}{4!}
\end{aligned}$$

$$+\alpha(1-\mu)\left\{\left[120p^4+240(\alpha-q)p^3+120(\alpha-q)(\alpha-2q)p^2\right.\right.$$
$$+20(\alpha-q)(\alpha-2q)(\alpha-3q)p+(\alpha-q)(\alpha-2q)(\alpha-3q)(\alpha-4q)]\mu^4$$
$$+\left[-180p^4+60(6q-7\alpha-2)p^3-90(\alpha-q)(3\alpha-4q+2)p^2-30(\alpha-q)\right.$$
$$\cdot(\alpha-2q)(2\alpha+2-3q)p-(\alpha-q)(\alpha-2q)(\alpha-3q)(4\alpha+5-6q)]\mu^3$$
$$+\left[70p^4+20(10\alpha-7q+6)p^3+10\left(-30q\alpha+18\alpha^2+24\alpha+3+14q^2-18q\right)\right.$$
$$\cdot p^2+10(\alpha-q)\left(6\alpha^2+12\alpha-12q\alpha+7q^2-12q+3\right)p+\left(6\alpha^2\right.$$
$$-12q\alpha+15\alpha+5+7q^2-15q)(\alpha-2q)(\alpha-q)]\mu^2+\left[-5p^4+10(q-2-2\alpha)\right.$$
$$\cdot p^3+\left(30q\alpha-30\alpha^2-60\alpha-15-10q^2+30q\right)p^2-52\alpha+(2-q)\left(2\alpha^2\right.$$
$$-2q\alpha+4\alpha-2q+q^2-1)p-(\alpha-q)\left(4\alpha^3-6q\alpha^2+15\alpha^2-15q\alpha+10\alpha\right.$$
$$+4q^2\alpha-5+5q^2-q^3-5q)]\mu+(\alpha-1)(\alpha+1)(\alpha+2)(\alpha+3)\}\cdot\frac{x^5}{5!}+\cdots,$$

where $\alpha=\frac{\lambda}{1-\lambda}$.

Remark 6 For $q=p$ we get the first terms of the series of ${}^{\mathcal{G}(\lambda)}\mathcal{P}_{p;\mu}(1,1-x)$, while for $q=1$ we have also the first terms of the series of ${}^{\mathcal{G}(\lambda)}\mathcal{C}_{p;\mu}(1,1-x)$.

Using the above results, the following property was proved in [12].

Theorem 6 *The relation*

$${}^{\mathcal{G}(\lambda)}\mathcal{S}_{p,q;\mu}=\mathcal{S}_{r,s;\nu}$$

holds if and only if we are in one of the following cases: (4)*,* (5)*,* (6)*,* (7)*,* (13)*,* (14)*, or*

$${}^{\mathcal{G}}\mathcal{S}_{p,q;\mu}=\mathcal{S}_{-p,-q;1-\mu}. \tag{17}$$

Remark 7 Of course, we have also some other equivalent cases, taking into account the property $\mathcal{S}_{s,r;\nu}=\mathcal{S}_{r,s;\nu}$. We have in view this property in all the results that follows.

7 Complementariness with Respect to Weighted Power Means

Basic results related to complementariness with respect to power means were given in [8]. Denote the $\mathcal{P}_{m;\lambda}-$ complementary of M by ${}^{\mathcal{P}(m;\lambda)}M$, or by ${}^{\mathcal{P}(m)}M$ if $\lambda=1/2$.

Corollary 2 *If the mean M has the series expansion*

$$M(1,1-x)=1+\sum_{n=0}^{\infty}a_nx^n,$$

then the first terms of the series expansion of ${}^{\mathcal{P}(m;\lambda)}M$, for $m\neq 0$ and $\lambda\neq 0,1$ are

$$
\begin{aligned}
{}^{\mathcal{P}(m;\lambda)}M(1,1-x) &= 1-(1+\alpha\cdot a_1)\cdot x+\frac{\alpha}{2}\left[(1-m)\left(2a_1+a_1^2+\alpha a_1^2\right)-2a_2\right]\cdot x^2\\
&+\frac{\alpha}{6}\left\{(1-m)\left[\left(2\alpha^2 m-\alpha^2+3\alpha m-3\alpha+m-2\right)a_1^3+3\left(2\alpha m-\alpha+m-1\right)a_1^2\right.\right.\\
&\left.\left.+3ma_1+6a_2+6\left(\alpha+1\right)a_1a_2\right]-6a_3\right\}\cdot x^3+\frac{\alpha}{24}\left\{(1-m)\cdot\left[\left(6\alpha^3m^2-5\alpha^3 m\right.\right.\right.\\
&\left.+\alpha^3+12\alpha^2m^2-18\alpha^2m+6\alpha^2+7\alpha m^2-18\alpha m+11\alpha+m^2-5m+6\right)a_1^4\\
&+4\left(6\alpha^2m^2-5\alpha^2m+\alpha^2+6\alpha m^2-9\alpha m+3\alpha+m^2-3m+2\right)a_1^3+6\left(4\alpha m^2\right.\\
&\left.-2\alpha m+m^2-m\right)a_1^2+4m\left(m+1\right)a_1+12ma_2+24\left(2\alpha m-\alpha+m-1\right)a_1a_2\\
&+12\left(2\alpha^2m-\alpha^2+3\alpha m-3\alpha+m-2\right)a_1^2a_2+24\left(\alpha+1\right)a_1a_3+24a_3\\
&\left.\left.+12\left(\alpha+1\right)a_2^2\right]-24a_4\right\}\cdot x^4+\cdots,
\end{aligned}
$$

where

$$
\alpha=\frac{\lambda}{1-\lambda}.
$$

Using them, the following consequence was proved in [10].

Theorem 7 *The first terms of the series expansion of the* $\mathcal{P}_{m;\lambda}-$ *complementary of* $\mathcal{P}_{p;\mu}$ *are*

$$
\begin{aligned}
{}^{\mathcal{P}(m;\lambda)}\mathcal{P}_{p;\mu}(1,1-x) &= 1-(\alpha\mu-\alpha+1)\,x-\frac{\alpha}{2}\,(\mu-1)\,(\mu m-\alpha\mu+\alpha\mu m\\
&-\mu p+\alpha-1-\alpha m+m)\,x^2-\frac{\alpha}{6}\,(\mu-1)\left(3\alpha\mu^2m^2-3\mu^2mp-3\alpha\mu^2m\right.\\
&-3\alpha^2\mu^2m-3\alpha\mu^2mp+3\alpha\mu^2p+2\alpha^2\mu^2m^2+\alpha^2\mu^2+\mu^2m^2+2\mu^2p^2\\
&+3\alpha\mu mp-\mu p^2-4\alpha^2\mu m^2-3\alpha\mu p+6\alpha^2\mu m-2\alpha^2\mu+\mu m^2-1-3\alpha^2m\\
&\left.+\alpha^2-3\alpha m^2+3\alpha m+2\alpha^2m^2+m^2\right)x^3-\frac{\alpha}{24}\,(\mu-1)\left(-2-\alpha-\alpha m\mu+6\alpha^2m\mu\right.\\
&+6\alpha^2m\mu^3-18\alpha^2m^2\mu^3-6\alpha\mu p-12\alpha^2m\mu^2+10\alpha^2m^2\mu-33\alpha^3m^2\mu-6\alpha m\mu^2\\
&+3\alpha m^2\mu^2-6m\mu^2p+6\alpha\mu^2p+22\alpha^2m^2\mu^2+33\alpha^3m^2\mu^2+3\alpha m^2\mu-11\alpha^3m^2\mu^3\\
&-7\alpha m^2\mu^3+18\alpha^3m\mu-18\alpha^3m\mu^2+6\alpha^3m\mu^3-6\alpha^2\mu p+12\alpha^2\mu^2p-6\alpha^2\mu^3p\\
&+2\alpha^2-11\alpha\mu^3p^2+15\alpha\mu^2p^2-4\alpha\mu p^2+7\alpha m-m+18m\alpha\mu^3p-18m\alpha\mu^2p\\
&+2\alpha^2\mu^2-4\alpha^2\mu+\alpha\mu-m\mu+\mu p+\alpha m^2-14\alpha^2m^2+2m^2+2m^2\mu^2+2m^2\mu\\
&-2\mu p^2+4\mu^2p^2+18\alpha^2m\mu p+18\alpha^2m\mu^3p-36\alpha^2m\mu^2p+\alpha^3+11\alpha^3m^2\\
&-6\alpha^3m-\alpha^3\mu^3+3\alpha^3\mu^2-3\alpha^3\mu+6m^2\alpha\mu p-18m^2\alpha\mu^3p+12m^2\alpha\mu^2p\\
&-12\alpha^2m^2\mu p-12\alpha^2m^2\mu^3p+24\alpha^2m^2\mu^2p+11m\alpha\mu^3p^2-15m\alpha\mu^2p^2\\
&+4m\alpha\mu p^2-7\alpha m^3+12\alpha^2m^3-6\alpha^3m^3-\mu p^3+6\mu^2p^3-6\mu^3p^3
\end{aligned}
$$

$$+12\alpha^2 m^3\mu^3 - 6m^2\mu^3 p - 12\alpha^2 m^3\mu + 18\alpha^3 m^3\mu + 3\alpha m^3\mu^2 - 18\alpha^3 m^3\mu^2$$
$$-3\alpha m^3\mu + 6\alpha^3 m^3\mu^3 + 7\alpha m^3\mu^3 - 7m\mu^2 p^2 + 11m\mu^3 p^2 + m^3\mu$$
$$+m^3\mu^2 + m^3\mu^3 + m^3 - 12\mu^2\alpha^2 m^3)x^4 + \cdots,$$

where

$$\alpha = \frac{\lambda}{1-\lambda}.$$

The problem of invariance in the family of weighted power means was solved in [9].

Theorem 8 *We have*

$$^{\mathcal{P}(m;\lambda)}\mathcal{P}_{p;\mu} = \mathcal{P}_{q;\nu}, m \neq 0,$$

if and only if we are in one of the non-trivial cases:

$$^{\mathcal{P}(m;\lambda)}\mathcal{P}_{m;\mu} = \mathcal{P}_{m;\frac{\lambda(1-\mu)}{1-\lambda}}; \tag{18}$$

$$^{\mathcal{P}(m;\lambda)}\Pi_2 = \mathcal{P}_{m;\frac{\lambda}{1-\lambda}}; \tag{19}$$

$$^{\mathcal{P}(m;\lambda)}\mathcal{P}_{m;\frac{2\lambda-1}{\lambda}} = \Pi_1; \tag{20}$$

$$^{\mathcal{P}(m;\lambda)}\mathcal{P}_{\frac{m}{2};2\lambda-1} = \mathcal{P}_{\frac{m}{2};2\lambda}; \tag{21}$$

$$^{\mathcal{P}(m;1/5)}\mathcal{P}_{\frac{m}{2};-1} = \mathcal{G}; \tag{22}$$

$$^{\mathcal{P}(m;4/5)}\mathcal{G} = \mathcal{P}_{\frac{m}{2};2}. \tag{23}$$

Remark 8 Some of the complementaries in the above theorem are only pre-means.

Remark 9 The problem of invariance in the class of weighted quasi-arithmetic means was solved by other method in [27] and [26]. Of course, the weighted power means are weighted quasi-arithmetic means, but the above results include pre-means as complementaries. The problem of invariance in the class of (symmetric) power means was solved in [28]. The problem of reproducing identities for power means,

$$\mathcal{P}_m\left(\mathcal{P}_p, \mathcal{P}_q\right) = \mathcal{P}_r,$$

was solved in [4]. Only the trivial solution,

$$\mathcal{P}_m\left(\mathcal{P}_p, \mathcal{P}_p\right) = \mathcal{P}_p,$$

and the solutions of the invariance problem,

$$\mathcal{P}_m\left(\mathcal{P}_p, \mathcal{P}_q\right) = \mathcal{P}_m,$$

exist.

Remark 10 The problem of invariance of a weighted power mean with respect to the set of weighted Gini means was studied in [14]. The following result was proved.

Theorem 9 *We have*

$$^{\mathcal{P}(m;\lambda)}\mathcal{S}_{r,s;\mu} = \mathcal{S}_{u,w;\nu}, m \neq 0,$$

if we are in one of the non-trivial cases: (19), (20), (21), (22),

$$^{\mathcal{P}(m;\lambda)}\mathcal{P}_{m;\frac{3\lambda-1}{2}} = \mathcal{P}_m \tag{24}$$

and

$$^{\mathcal{P}(m)}\mathcal{S}_{r,r+m;\mu} = \mathcal{S}_{-r,m-r;1-\mu}; \tag{25}$$

including its special cases

$$^{\mathcal{P}(m)}\mathcal{S}_{r-m,r;\mu} = \mathcal{S}_{m-r,2m-r;1-\mu}; \tag{26}$$

$$^{\mathcal{P}(m)}\mathcal{S}_{r,r+m} = \mathcal{S}_{-r,m-r}; \tag{27}$$

$$^{\mathcal{P}(m)}\mathcal{S}_{r-m,r} = \mathcal{S}_{m-r,2m-r}; \tag{28}$$

$$^{\mathcal{P}(2r)}\mathcal{S}_{r,3r} = \mathcal{G}; \tag{29}$$

$$^{\mathcal{P}(2r)}\mathcal{G} = \mathcal{S}_{r,3r}. \tag{30}$$

Remark 11 Taking into account the warning that "solutions may have been lost" in solving some systems of equations using the computer algebra Maple, it is not sure that "if" in the enunciation of the previous theorem can be replaced by "if and only if".

8 Complementariness with Respect to Weighted Lehmer Means

Denote the $\mathcal{C}_{p;\lambda}-$ complementary of the mean M by $^{\mathcal{C}(p;\lambda)}M$, or by $^{\mathcal{C}(p)}M$ if $\lambda = 1/2$. Using Euler's formula, the following result was established in [36].

Theorem 10 *If the mean M has the series expansion*

$$M(1, 1-x) = 1 + \sum_{n=0}^{\infty} a_n x^n,$$

then the first terms of the series expansion of $^{\mathcal{C}(p;\lambda)}M$*, for* $\lambda \neq 0, 1$*, are*

$$^{\mathcal{C}(p;\lambda)}M(1, 1-x) = 1 - \frac{1-\lambda+\lambda a_1}{1-\lambda}x - \frac{\lambda}{(1-\lambda)^2}\left[(p-1)\, a_1\, (a_1 + 2(1-\lambda))\right.$$
$$\left. + a_2 (1-\lambda)\right] \cdot x^2 - \frac{\lambda}{2(1-\lambda)^3}\left[a_1 (p-1)\left(2\lambda^3 p - \lambda^2 (p+2) - 4\lambda (p-1) + 3p\right.\right.$$

$$-2) + a_1^2 (p-1) \left(2\lambda^2 (1-3p) + \lambda (3p+2) + 3p - 4\right) + a_1^3 (p-1)(2\lambda p + p - 2)$$
$$+4a_2 (p-1)(1-\lambda)^2 + 4a_1 a_2 (p-1)(1-\lambda) + 2a_3 (1-\lambda)^2\big] \cdot x^3 + \cdots$$
$$M^{\mathcal{C}(q;\nu)}(1, 1-x) = 1 - (1 + \alpha \cdot a_1) \cdot x - \alpha \left[a_1^2 (\alpha q - \alpha + q - 1) - 2a_1 + a_2\right] \cdot x^2$$
$$-\frac{\alpha}{2(1+\alpha)} \big[a_1 \left(2\alpha - 5q - 7\alpha q + 2 + 3q^2 + 5\alpha q^2\right)$$
$$+a_1^2 \left(10\alpha - 15q\alpha^2 - 10\alpha q + 6\alpha^2 + 4 - 7q - 12q\alpha + 9q^2\alpha^2 + 12q^2\alpha + 3q^2\right)$$
$$+a_1^3 (2 + 6\alpha + 6\alpha^2 + \alpha^3 - 3q - 11q\alpha - 13q\alpha^2 - 5q\alpha^3 + 5q^2\alpha + 7q^2\alpha^2$$
$$+3q^2\alpha^3 + q^2\big) + 2a_2(1+\alpha)(2q - r - 1) + 2a_1 a_2 (1+\alpha)^2 (2q - r - 1)$$
$$+2a_3(1+\alpha)] \cdot x^3 + \cdots,$$

where $\alpha = \nu/(1-\nu)$.

Using this formula, in [13] is deduced the following results.

Corollary 3 *The first terms of the series expansion of* ${}^{\mathcal{C}(p;\lambda)}\mathcal{C}_{r;\mu}$ *are*

$${}^{\mathcal{C}(p;\lambda)}\mathcal{C}_{r;\mu}(1, 1-x) = 1 - \frac{1 - 2\lambda + \lambda\mu}{1-\mu} x + \frac{\lambda (1-\mu)}{(1-\lambda)^2} [p (1 - 2\lambda + \mu)$$
$$+\mu r (\lambda - 1) - 1 + 2\lambda - \lambda\mu] x^2 +$$
$$\frac{\lambda (1-\mu)}{(1-\lambda)^3} \left[p^2 \left(2\lambda^3 + 2\lambda\mu^2 - 6\lambda^2\mu - \lambda\mu + 5\lambda^2 + \mu^2 + \mu - 5\lambda + 1\right)\right.$$
$$+4pr \left(\lambda\mu^2 + \lambda\mu - \lambda^2\mu - \mu^2\right) + r^2 \left(2\lambda\mu - 4\lambda\mu^2 - \lambda^2\mu - \mu + 2\mu^2\right)$$
$$+p \left(2\lambda^2\mu^2 + 12\lambda^2\mu - 6\lambda\mu^2 - 2\lambda^3 - 9\lambda^2\right.$$
$$\left.+\mu^2 - \lambda\mu + 7\lambda - \mu - 1\right) + r \left(5\lambda^2\mu - 4\lambda^2\mu^2\right.$$
$$\left.+4\lambda\mu^2 - 6\lambda\mu + \mu\right) + 2\lambda^2\mu^2 + 4\lambda^2 - 6\lambda^2\mu + 2\lambda\mu - 2\lambda] x^3 + \cdots$$

Corollary 4 *We have*

$${}^{\mathcal{C}(p;\lambda)}\mathcal{C}_{r;\mu} = \mathcal{C}_{u;\nu}$$

if we are in one of the following non-trivial cases:

$${}^{\mathcal{C}(1;\lambda)}\mathcal{C}_{1;\frac{2\lambda-1}{\lambda}} = \mathcal{C}_{u;1}; \tag{31}$$
$${}^{\mathcal{C}(0;\lambda)}\mathcal{C}_{0;\frac{2\lambda-1}{\lambda}} = \mathcal{C}_{u;1}; \tag{32}$$
$${}^{\mathcal{C}(1)}\mathcal{C}_{r;\mu} = \mathcal{C}_{2-r;1-\mu}; \tag{33}$$
$${}^{\mathcal{C}(1/2)}\mathcal{C}_{r;\mu} = \mathcal{C}_{1-r;1-\mu}; \tag{34}$$
$${}^{\mathcal{C}(0)}\mathcal{C}_{r;\mu} = \mathcal{C}_{-r;1-\mu}; \tag{35}$$
$${}^{\mathcal{C}(1;\lambda)}\mathcal{C}_{1;\frac{3\lambda-1}{2\lambda}} = \mathcal{C}_{1}; \tag{36}$$
$${}^{\mathcal{C}(0;\lambda)}\mathcal{C}_{0;\frac{3\lambda-1}{2\lambda}} = \mathcal{C}_{0}; \tag{37}$$

$$^{\mathcal{C}(1;1/3)}\mathcal{C}_{r;0} = \mathcal{C}_1; \tag{38}$$

$$^{\mathcal{C}(0;1/3)}\mathcal{C}_{r;0} = \mathcal{C}_0; \tag{39}$$

$$^{\mathcal{C}(1;\lambda)}\mathcal{C}_1 = \mathcal{C}_{1;\frac{\lambda}{2-2\lambda}}; \tag{40}$$

$$^{\mathcal{C}(0;\lambda)}\mathcal{C}_0 = \mathcal{C}_{0;\frac{\lambda}{2-2\lambda}}; \tag{41}$$

$$^{\mathcal{C}(2;1/4)}\mathcal{C}_{1;-1/2} = \mathcal{C}_1; \tag{42}$$

$$^{\mathcal{C}(2;3/4)}\mathcal{C}_1 = \mathcal{C}_{1;3/2}; \tag{43}$$

$$^{\mathcal{C}(-1;1/4)}\mathcal{C}_{0;-1/2} = \mathcal{C}_0; \tag{44}$$

$$^{\mathcal{C}(-1;3/4)}\mathcal{C}_0 = \mathcal{C}_{0;3/2}. \tag{45}$$

Remark 12 We are not sure that these are the only solutions of the above problem. It is easy to verify that the solutions (31), (32), (33), (34), (35), (36), (37), (38), (39), (40), (41) are special cases of (20), (20), (25), (17), (25), (18), (18), (19), (19), (18) respectively (18).

Remark 13 The cases involving $\mathcal{C}_{1;\lambda} = \mathcal{A}_\lambda$ and $\mathcal{C}_{0;\lambda} = \mathcal{H}_\lambda$, have no similar for $\mathcal{C}_{1/2;\lambda}$. Instead, the following results:

$$^{\mathcal{G}(\lambda)}\mathcal{G}_{\frac{2\lambda-1}{\lambda}} = \Pi_1, {}^{\mathcal{G}(1/3)}\,\Pi_2 = \mathcal{G}, {}^{\mathcal{G}(\lambda)}\mathcal{G}_{\frac{3\lambda-1}{2\lambda}} = \mathcal{G}, {}^{\mathcal{G}(\lambda)}\mathcal{G} = \mathcal{G}_{\frac{\lambda}{2(1-\lambda)}},$$

are valid, but $\mathcal{G}_\lambda$ is not a weighted Lehmer mean.

Corollary 5 *For symmetric means we have*

$$\mathcal{C}_p(\mathcal{C}_r, \mathcal{C}_u) = \mathcal{C}_p$$

if and only if we are in the following non-trivial cases:

$$i)\ \mathcal{C}_0(\mathcal{C}_r, \mathcal{C}_{-r}) = \mathcal{C}_0;$$

$$ii)\ \mathcal{C}_{1/2}(\mathcal{C}_r, \mathcal{C}_{1-r}) = \mathcal{C}_{1/2};$$

$$iii)\ \mathcal{C}_1(\mathcal{C}_r, \mathcal{C}_{2-r}) = \mathcal{C}_1.$$

Remark 14 This problem of invariance was solved in [28]. The problem of reproducing identities,

$$\mathcal{C}_p(\mathcal{C}_r, \mathcal{C}_u) = \mathcal{C}_v,$$

was solved in [4]. The solution contains the above cases i)-iii) and the trivial case

$$\mathcal{C}_p(\mathcal{C}_r, \mathcal{C}_r) = \mathcal{C}_r.$$

Remark 15 The problem of invariance of a weighted Lehmer mean with respect to the set of weighted Gini means was studied in [15]. The following result was proved.

Theorem 11 *We have*

$$^{\mathcal{C}(p;\lambda)}\mathcal{S}_{r,q;\mu} = \mathcal{S}_{u,t;\nu}$$

if $^{\mathcal{C}(p;\lambda)}\mathcal{C}_{r;\mu} = \mathcal{C}_{u;\nu}$ *(with* $q = r - 1$ *and* $t = u - 1$*), or*

$$^{\mathcal{C}(1)}\mathcal{S}_{\frac{3}{2},\frac{1}{2}} = \mathcal{S}_{u,-u}; \tag{46}$$

$$^{\mathcal{C}(1;1/5)}\mathcal{S}_{\frac{1}{2},0;-1} = \mathcal{S}_{u,-u}; \tag{47}$$

$$^{\mathcal{C}(1;\lambda)}\mathcal{S}_{1,0;\frac{2\lambda-1}{\lambda}} = \Pi_1; \tag{48}$$

$$^{\mathcal{C}(1)}\mathcal{S}_{r+1,r;\mu} = \mathcal{S}_{1-r,-r;1-\mu}; \tag{49}$$

$$^{\mathcal{C}(1;4/5)}\mathcal{S}_{r,-r} = \mathcal{S}_{0,1/2;2}; \tag{50}$$

$$^{\mathcal{C}(1;\lambda)}\Pi_2 = \mathcal{S}_{1,0;\frac{\lambda}{1-\lambda}}; \tag{51}$$

$$^{\mathcal{C}(1/2)}\mathcal{S}_{r,s;\mu} = \mathcal{S}_{-r,-s;1-\mu}; \tag{52}$$

$$^{\mathcal{C}(0)}\mathcal{S}_{-\frac{3}{2},-\frac{1}{2}} = \mathcal{S}_{u,-u}; \tag{53}$$

$$^{\mathcal{C}(0;1/5)}\mathcal{S}_{-\frac{1}{2},0;-1} = \mathcal{S}_{u,-u}; \tag{54}$$

$$^{\mathcal{C}(0;\lambda)}\mathcal{S}_{-1,0;\frac{2\lambda-1}{\lambda}} = \Pi_1; \tag{55}$$

$$^{\mathcal{C}(0)}\mathcal{S}_{r+1,r;\mu} = \mathcal{S}_{-r-1,-r-2;1-\mu}; \tag{56}$$

$$^{\mathcal{C}(0;4/5)}\mathcal{S}_{r,-r} = \mathcal{S}_{0,-1/2;2}; \tag{57}$$

respectively

$$^{\mathcal{C}(0;\lambda)}\Pi_2 = \mathcal{S}_{0,-1;\frac{\lambda}{1-\lambda}}. \tag{58}$$

Remark 16 In fact, (46), (47), (48), (49), (50), (51), (52), (53), (54), (55), (56), (57), and (58) are special cases of (29), (25), (20), (33), (23), (19), (17), (29), (25), (20), (35), (23), respectively (19).

9 Complementariness with Respect to Weighted Gini Means

In [2] it was solved the problem of invariance in the family of Gini means:

Theorem 12 *We have*

$$^{\mathcal{S}(p,q)}\mathcal{S}_{r,s} = \mathcal{S}_{u,w},$$

if and only if (4)*,* (17)*,* (27)*,* (29) *or* (30) *hold.*

We pass now to the complementariness with respect to the weighted Gini means. Denote the $\mathcal{S}_{q,q-r;\nu}-$ complementary of the mean M by ${}^{\mathcal{S}(q,q-r;\nu)}M$, and by ${}^{\mathcal{S}(q,q-r)}M$ if $\nu = 1/2$.

Theorem 13 *If the mean M has the series expansion*

$$M(1, 1-x) = 1 + \sum_{n=1}^{\infty} a_n x^n,$$

then ${}^{\mathcal{S}(q,q-r;\nu)}M$ has, for $r \neq 0$ and $\nu \neq 0, 1$, the series expansion

$${}^{\mathcal{S}(q,q-r;\nu)}M(1, 1-x) = 1 + \sum_{n=1}^{\infty} d_n x^n,$$

where

$$d_0 = 1,\ d_1 = \frac{e_1}{r},$$

$$d_n = -\frac{1}{nr} \sum_{k=0}^{n-1} [k\,(r+1) - n] \cdot d_k \cdot e_{n-k}\ ,\ n \geq 2,$$

with

$$e_1 = (\alpha + 1)\,\beta_1 - \alpha b_1, \alpha = \frac{\nu}{1-\nu},$$

$$e_n = \beta_n - \sum_{k=1}^{n-1} f_k\,(e_{n-k} - \beta_{n-k}) + \alpha \left[\beta_n - b_n + \sum_{k=1}^{n-1} c_k\,(\beta_{n-k} - b_{n-k}) \right],\ n \geq 2,$$

b_n, c_n, f_n and β_n denoting the coefficients of the reduced series expansion of M^r, M^{q-r}, N^{q-r} respectively $\mathcal{S}^r_{q,q-r;\nu}$.

Proof Denoting ${}^{\mathcal{S}(q,q-r;\nu)}M = N$, the condition $\mathcal{S}_{q,q-r;\nu}(M, N) = \mathcal{S}_{q,q-r;\nu}$ gives

$$N^{q-r}\left(N^r - \mathcal{S}^r_{q,q-r;\nu}\right) = \alpha M^{q-r}\left(\mathcal{S}^r_{q,q-r;\nu} - M^r\right).$$

Taking the values $a = 1$ and $b = 1 - x$ and denoting the coefficients of the reduced series expansion of M^r, M^{q-r}, N^r, N^{q-r} and $\mathcal{S}^r_{q,q-r;\nu}$ by b_n, c_n, e_n, f_n respectively β_n, we get

$$\left[1 + \sum_{n=1}^{\infty} f_n x^n\right]\left[\sum_{n=1}^{\infty} (e_n - \beta_n)\,x^n\right] = \alpha \left[1 + \sum_{n=1}^{\infty} c_n x^n\right]\left[\sum_{n=1}^{\infty} (\beta_n - b_n)\,x^n\right].$$

This gives

$$e_1 - \beta_1 = \alpha\,(\beta_1 - b_1)\,,$$

$$e_n - \beta_n + \sum_{k=1}^{n-1} f_k \left(e_{n-k} - \beta_{n-k}\right) = \alpha \left[\beta_n - b_n + \sum_{k=1}^{n-1} c_k \left(\beta_{n-k} - b_{n-k}\right)\right],$$

for $n \geq 2$. Therefore, we have a recurrence relation for e_n and using Euler's formula (16) we can deduce the expression of d_n. □

Corollary 6 *If the mean M has the series expansion*

$$M(1, 1-x) = 1 + \sum_{n=0}^{\infty} a_n x^n,$$

then the first terms of the series expansion of ${}^{\mathcal{S}(q,q-r;\nu)}M$, for $r \neq 0$ and $\nu \neq 0, 1$, are

$$\begin{aligned}
{}^{\mathcal{S}(q,q-r;\nu)}M(1,1-x) &= 1 - (1+\alpha\cdot a_1)\cdot x - \frac{\alpha}{2}\left[a_1\left(2q-r-1\right)\left(2+\alpha a_1 + a_1\right) + 2a_2\right] \\
&\cdot x^2 - \frac{\alpha}{6(1+\alpha)}\left[3a_1\left(\alpha r - 3rq - 2\alpha q + \alpha r^2 - 5r\alpha q - 2q + r^2 + 3q^2 + r + 5\alpha q^2\right)\right. \\
&+3a_1^2\left(3r^2\alpha - 6q\alpha^2 - 10\alpha q + 2r^2\alpha^2 + 3r\alpha^2 + 5r\alpha + 1 + 2r + 2\alpha + \alpha^2 + r^2\right. \\
&\left.-4q - 12qr\alpha - 9qr\alpha^2 - 3qr + 9q^2\alpha^2 + 12q^2\alpha + 3q^2\right) + a_1^3\left(2 + r^2 + 3r\right. \\
&+5\alpha + 4\alpha^2 - 15qr\alpha - 21qr\alpha^2 - 9qr\alpha^3 + \alpha^3 + 5r^2\alpha^2 + 2r^2\alpha^3 + 9r\alpha^2 + 3r\alpha^3 \\
&\left.+4r^2\alpha + 9r\alpha - 6q\alpha^3 - 18q\alpha^2 - 18\alpha q - 3qr + 15q^2\alpha + 21q^2\alpha^2 + 9q^2\alpha^3 - 6q + 3q^2\right) \\
&\left.+6a_2(1+\alpha)\left(2q-r-1\right) + 6a_1a_2(1+\alpha)^2\left(2q-r-1\right) + 6a_3(1+\alpha)\right]\cdot x^3\cdots,
\end{aligned}$$

where $\alpha = \frac{\nu}{1-\nu}$.

As a consequence we obtain:

Corollary 7 *The first terms of the series expansion of the $\mathcal{S}_{p,p-q;\lambda}-$ complementary of $\mathcal{S}_{r,r-s;\mu}$ are*

$$\begin{aligned}
{}^{\mathcal{S}(p,p-q;\lambda)}\mathcal{S}_{r,r-s;\mu}(1,1-x) &= 1 - \frac{1-2\lambda+\lambda\mu}{1-\lambda}x \\
&-\frac{\lambda(1-\mu)}{2(1-\lambda)^2}(\lambda\mu - 2r\mu\lambda + s\mu\lambda + q\mu - 2p\mu - s\mu + 2r\mu - 2\lambda + 1 + 4p\lambda - 2p - 2q\lambda \\
&+q)\cdot x^2 + \frac{\lambda(1-\mu)}{6(1-\lambda)^3}(-1 + 15pq\lambda + q^2 - 3p\mu^2 q + 3\mu^2 p^2 + 6q\lambda^2 + 6q^2\lambda^2 - 15q\lambda^2 p \\
&+3p^2 - 18p^2\mu\lambda^2 + 18p\mu q\lambda^2 - 6p\mu^2 q\lambda + 3\mu qs\lambda - 6\mu qr\lambda - 6q\mu^2 r\lambda + 3rs\mu \\
&-5q^2\lambda - 6q\lambda^3 p + 6p^2\mu^2\lambda + 6p^2\lambda^3 - s^2\mu - 3r^2\mu + 6r^2\mu^2 - 6s\mu^2 r + q^2\mu + 2s^2\mu^2 + \mu^2\lambda^2 \\
&-3p\mu q - 3q^2\mu\lambda^2 + q^2\mu^2\lambda + 2\lambda + 12\mu pr\lambda - 6\mu ps\lambda + 12p\mu^2 r\lambda - 6p\mu^2 s\lambda \\
&+3q\mu^2 s\lambda - 3p^2\mu\lambda - 12p\lambda^2 - 15p^2\lambda + 15p^2\lambda^2 - 12p\lambda^2 r\mu + 6p\lambda^2 s\mu + 6q\lambda^2 r\mu \\
&-3q\lambda^2 s\mu - 3qp - 3q\lambda + 6p\lambda + 3rs\mu\lambda^2 - 6s\mu^2 r\lambda^2 - 6rs\mu\lambda + 12s\mu^2 r\lambda + 6r^2\mu\lambda \\
&-12r^2\mu^2\lambda - 4s^2\mu^2\lambda - s^2\mu\lambda^2 - 3r^2\mu\lambda^2 + 6r^2\mu^2\lambda^2 + 2s^2\mu^2\lambda^2 + 2s^2\mu\lambda + 6q\mu^2 r \\
&-3q\mu^2 s + q^2\mu^2 + 3\mu p^2 - 12p\mu^2 r + 6p\mu^2 s - 2q^2\mu\lambda + 3q\mu^2\lambda - 6q\mu\lambda^2 - 2\lambda^2\mu
\end{aligned}$$

$$-6p\mu^2\lambda + 12p\mu\lambda^2 + 3p\mu q\lambda - 6r\mu\lambda + 3s\mu\lambda + 6r\mu^2\lambda - 3s\mu^2\lambda + 6r\mu\lambda^2 - 3s\mu\lambda^2$$
$$-6r^2\mu\lambda^2 + 3s\mu^2\lambda^2) \cdot x^3 + \cdots$$

Remark 17 The next coefficient needs two pages for printing.

We can study the problem of invariance in the family of weighted Gini means.

Theorem 14 *We have*

$$^{\mathcal{S}(p,m;\lambda)}\mathcal{S}_{r,k;\mu} = \mathcal{S}_{u,t;\nu},$$

if we are in one of the following non-trivial cases:

$$^{\mathcal{S}(p,-p)}\mathcal{S}_{r,t;\mu} = \mathcal{S}_{-r,-t;1-\mu}; \tag{59}$$
$$^{\mathcal{S}(0,0;\lambda)}\mathcal{S}_{0,0;(3\lambda-1)/2\lambda} = \mathcal{S}_{u,-u}; \tag{60}$$
$$^{\mathcal{S}(0,0;1/3)}\Pi_2 = \mathcal{S}_{u,-u}; \tag{61}$$
$$^{\mathcal{S}(0,0;2/3)}\mathcal{S}_{r,-r} = \Pi_1; \tag{62}$$
$$^{\mathcal{S}(p,0)}\mathcal{S}_{r,r+p;\mu} = \mathcal{S}_{-r,-r+p;1-\mu}, \tag{63}$$
$$^{\mathcal{S}(p,0)}\mathcal{S}_{r,r-p;\mu} = \mathcal{S}_{2p-r,p-r;1-\mu}, \tag{64}$$
$$^{\mathcal{S}(p,0;\lambda)}\Pi_2 = \mathcal{S}_{p,0;\lambda/(1-\lambda)}; \tag{65}$$
$$^{\mathcal{S}(p,0;\lambda)}\mathcal{S}_{p,0;(2\lambda-1)} = \Pi_1; \tag{66}$$
$$^{\mathcal{S}(2p,0;1/5)}\mathcal{S}_{p,0;-1} = \mathcal{S}_{u,-u}; \tag{67}$$

or

$$^{\mathcal{S}(p,-p)}\mathcal{S}_{r,t;\mu} = \mathcal{S}_{-r,-t;1-\mu}. \tag{68}$$

Proof Denote $m = p - q, k = r - s, t = u - w$. We have to determine the set of nine parameters $(p, q, r, s, u, w, \lambda, \mu, \nu)$ such that

$$^{\mathcal{S}(p,p-q;\lambda)}\mathcal{S}_{r,r-s;\mu}(1, x) = \mathcal{S}_{u,u-w;\nu}(1, x), \text{for all} x > 0. \tag{69}$$

We do this in more rounds. In each one we choose a fixed n and solve the system of equations obtained by equating the coefficients of x^j in the two members of the equality (69), for $j = 1, \ldots, n$.

I) For $n = 1$, the equality of the coefficients of x gives

$$\nu = (1 + \nu - \mu)\,\lambda.$$

We have the following cases:

1) $\lambda = 0$, implying $\nu = 0$, thus (6);
2) $\nu = 0$, implying $\lambda = 0$, thus again (6), or $\mu = 1$ giving (5);

3) $\mu = 1$, implying $\nu = 0$, thus (5), or $\lambda = 1$ giving (7);
4) $\lambda = 1$ implying $\mu = 1$ thus (7);
5) $1 + \nu - \mu = 0$, implying $\nu = 0$ and then $\mu = 1$, thus (5);
6) $\mu = 0$ and $\nu = \lambda/(1-\lambda)$;
7) $\nu = 1$ and $\mu = 2 - 1/\lambda$;
8) $\nu = \lambda\,(1-\mu)\,/\,(1-\lambda)$.

The first five cases give only trivial solutions. To solve the last three cases, we have to go further. The equations being more and more complicated, for the following cases we used the computer algebra Maple (see [25]).

II) For $n = 2$, we get the special case 6.1) $\mu = 0, \nu = 1, \lambda = 1/2$ giving (8) and the relation $2p = q - s + 2r$ in the case 7).

III) For $n = 3$, we get the special cases:

7.1) $\nu = 1, \mu = 1/2, \lambda = 2/3, p = 0, q = s - 2r$;

7.2) $\nu = 1, \mu = 2 - 1/\lambda, p = r = 0, q = s$, giving (66);

8.1) $\lambda = \mu = \nu = 1/2, 2u = 4p - 2q - 2r + s + w$;

8.2) $\nu = 1/2, \mu = (3\lambda - 1)\,/\,(2\lambda)$.

IV) For $n = 6$, we get the special cases:

7.1) $\nu = 1, \mu = 1/2, \lambda = 2/3, p = 0, q = 0, s = 2r$, thus (62);

7.3.1) $\nu = 1$, $\mu = 2 - 1/\lambda, p = q = r = s$, thus (66);

7.3.2) $\nu = 1$, $\mu = 2 - 1/\lambda, p = q, r = 0, s = -p$, thus (66);

7.3.3) $\nu = 1$, $\mu = 2 - 1/\lambda, p = 0, -q = r = s$, thus (66);

8) $15\lambda^4 - 27\lambda^3 + 24\lambda^2 - 11\lambda + 2 = 0$, but this equation has no solution. Unfortunately we get also the warning that solutions may have been lost. That is why we have considered some more rounds.

V) For $\nu = 1$ and $n = 7$, we get only 6.1), 7.1), 7.2), 7.3.1), 7.3.2) and 7.3.3). Thus the case 7) is completely solved.

VI) For $\nu = 1/2$ and $n = 7$, we get:

6.2.1) $\mu = 0, \lambda = 1/3, p = q = 0, w = 2u$, thus (61);

8.1.1) $\lambda = \mu = 1/2, q = 2p, s = 2r, w = 2u$, thus (4);

8.1.2) $\lambda = \mu = 1/2$, $p = 0$, $s = q, q = -2r, w = 2u$, thus (68);

8.1.3) $\lambda = \mu = 1/2$, $p = 0$, $s = -q, 3q = -2r, w = 2u$, thus (68);

8.1.4) $\lambda = \mu = 1/2$, $p = q$, $s = q, 3q = 2r, w = 2u$, thus (68);

8.1.5) $\lambda = \mu = 1/2$, $p = q$, $s = -q, q = 2r, w = 2u$, thus (68);

8.2.1) $\mu = (3\lambda - 1)\,/\,(2\lambda)\,, p = q = r = s = 0, w = 2u$, thus (60);

8.2.2) $\lambda = 1/5$, $\mu = -1, p = 0$, $2s = -q, s = r, w = 2u$, thus (67);

8.2.3) $\lambda = 1/5$, $\mu = -1, p = r = 0, q = 2s, w = 2u$, thus (67);

8.2.4) $\lambda = 1/5$, $\mu = -1, p = q = 2s, s = r, w = 2u$, thus (67);

8.2.5) $\lambda = 1/5$, $\mu = -1, p = q = -2s, r = 0, w = 2u$, thus (67).

VII) For $\mu = 0$ and $n = 7$, we get again the cases 1), 6.1), 6.2.1), and the new cases:

6.3.1) $\nu = \lambda/(1-\lambda)$, $p = 0$, $u = w = -q$, thus (65);

6.3.2) $\nu = \lambda/(1-\lambda)$, $p = q$, $u = 0$, $w = -q$, thus (65);

6.3.3) $\nu = \lambda/(1-\lambda)$, $p = u = 0$, $w = q$, thus (65). So the case 6) is also completely solved.

VIII) For $\lambda = 1/2$ and $n = 7$, we get the new cases:
8.1.6) $\mu = \nu = 1/2$, $p = 0, s = q = w, u = -r$ thus (63);
8.1.7) $\mu = \nu = 1/2$, $p = u = r, s = q = w$, thus (4);
8.1.8) $\mu = \nu = 1/2$, $p = s = q = w, u = 2q - r$ thus (64);
8.1.9) $\mu = \nu = 1/2$, $p = 0, s = q, u = -q - r, w = -q$ thus (63);
8.1.10) $\mu = \nu = 1/2$, $p = r, s = q = -w, u = r - q$ thus (4);
8.1.11) $\mu = \nu = 1/2$, $p = s = q = -w, u = q - r$ thus (64);
8.1.12) $\mu = \nu = 1/2$, $p = 0, s = -q = w, u = -r - 2q$ thus (64);
8.1.13) $\mu = \nu = 1/2$, $p = r + q, s = -q = w, u = r$ thus (4);
8.1.14) $\mu = \nu = 1/2$, $p = -s = q = -w, u = -r$ thus (63);
8.1.15) $\mu = \nu = 1/2, q = 2p, s = w, u = s - r$ thus (59);
8.1.16) $\mu = \nu = 1/2$, $p = 0, s = -q = -w, u = -q - r$ thus (64);
8.1.17) $\mu = \nu = 1/2$, $p = r + q, u = q + r, w = q$ thus (4);
8.1.18) $\mu = \nu = 1/2$, $p = q = -s = w, u = q - r$ thus (63);
8.1.19) $\mu = \nu = 1/2, 2p = q, s = -w, u = -r$ thus (59);
8.3.1) $\nu = 1 - \mu, 2p = q, s = w, u = s - r$ thus (59);
8.3.2) $\nu = 1 - \mu, 2p = q, s = -w, u = -r$ thus (59);
8.3.3) $\nu = 1 - \mu$, $p = q = s = w, u = 2s - r$ thus (64);
8.3.4) $\nu = 1 - \mu$, $p = q = -s = -w, u = -r$ thus (63);
8.3.5) $\nu = 1 - \mu$, $p = q = s = w, u = 2s - r$ thus (64);
8.3.6) $\nu = 1 - \mu$, $p = 0, s = -q = w, u = 2s - r$ thus (64);
8.3.7) $\nu = 1 - \mu$, $p = 0, q = s = w, u = -r$ thus (63);
8.3.8) $\nu = 1 - \mu$, $p = q = -s = w, u = -r - s$ thus (63);
8.3.9) $\nu = 1 - \mu$, $p = q = s = -w$, $u = s - r$ thus (64);
8.3.10) $\nu = 1 - \mu$, $p = 0, q = -s = w, u = s - r$ thus (64);
8.3.11) $\nu = 1 - \mu$, $p = 0, q = s = -w, u = -r - s$ thus (63);

IX) For $p = q = 0$ we get the results from Proposition 1, Remarks 3 and 5, Theorem 6.

X) For $p = q \neq 0$ we get the results from Theorems 8 and 9.

XI) For $q = 1$ we get the results from Theorem 11 and Corollary 4. □

Remark 18 We are in a case indicated by one of the following items: Proposition 1, Remarks 3 and 5, Theorems 6, 8, 9, 11, and 12, or Corollary 4. Taking into account the warning that solutions may have been lost in solving the round IV), we cannot be sure that "if" in the enunciation of the previous theorem can be replaced by "if and only if".

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Functional Inequalities and Analysis of Contagion in the Financial Networks

P. Daniele, S. Giuffè, M. Lorino, A. Maugeri and C. Mirabella

Abstract In very recent papers, using delicate tools of functional analysis, a general equilibrium model of financial flows and prices is studied. In particular, without using a technical language, but using the universal language of mathematics, some significant laws, such as the Deficit formula, the Balance law and the Liability formula for the management of the world economy are provided. Further a simple but useful economical indicator $E(t)$ is considered. In this paper, considering the Lagrange dual formulation of the financial model, the Lagrange variables called "deficit" and "surplus" variables are considered. By means of these variables, we study the possible insolvencies related to the financial instruments and their propagation to the entire system, producing a "financial contagion".

Keywords Financial networks · Deficit and surplus variables · Shadow market · Balance law · Financial contagion

1 Introduction

In the papers [4–7], the authors study a general model of financial flows and prices related to individual entities called sectors. They are able to provide the equilibrium conditions and to express them in terms of a variational inequality. Then, they study

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the governing variational inequality and provide existence theorems, develop the Lagrange duality theory, and introduce an appropriate Evaluation Index $E(t)$. As a byproduct of the Lagrange duality, they get a dual formulation of the financial equilibrium in which the significance Lagrange functions $\rho_j^{*1}(t)$ and $\rho_j^{*2}(t)$ appear. These functions $\rho_j^{*1}(t), \rho_j^{*2}(t)$, $j = 1, \ldots, n$ represent the deficit and the surplus, respectively, for the financial instrument j shared by the sectors. Studying the balance of all sectors given by

$$\sum_{j=1}^{n} \rho_j^{*1}(t) - \sum_{j=1}^{n} \rho_j^{*2}(t)$$

and the single difference

$$\rho_j^{*1}(t) - \rho_j^{*2}(t) \quad j = 1, \ldots, n$$

we are able to study the possible insolvencies related to the financial instruments and to understand when they propagate to the entire system, producing a “financial contagion”.

2 The Financial Network and the Equilibrium Flows and Prices

The first authors to develop a multi-sector, multi-instrument financial equilibrium model using the variational inequality theory were Nagurney et al. [34]. These results were, subsequently, extended by Nagurney in [30, 31] to include more general utility functions and by Nagurney and Siokos in [32, 33] to the international domain (see also [24, 36] for related papers). In [18], the authors apply for the first time the methodology of projected dynamical systems to develop a multi-sector, multi-instrument financial model, whose set of stationary points coincided with the set of solutions to the variational inequality model developed in [30], and then to study it qualitatively, providing stability analysis results.

Now, we describe in detail the model we are dealing with. We consider a financial economy consisting of m sectors, for example, households, domestic businesses, banks and other financial institutions, as well as state and local governments, with a typical sector denoted by i, and of n instruments, for example mortgages, mutual funds, saving deposits, money market funds, with a typical financial instrument denoted by j, in the time interval $[0, T]$. Let $s_i(t)$ denote the total financial volume held by sector i at time t as assets, and let $l_i(t)$ be the total financial volume held by sector i at time t as liabilities. Then, unlike previous papers (see [9–13] and [15]), we allow markets of assets and liabilities to have different investments $s_i(t)$ and $l_i(t)$, respectively. Since we are working in the presence of uncertainty and of risk perspectives, the volumes $s_i(t)$ and $l_i(t)$ held by each sector cannot be considered stable with respect to time and may decrease or increase. For example, depending on the crisis periods, a sector may decide not to invest on instruments and to buy goods

as gold and silver. At time t, we denote the amount of instrument j held as an asset in sector i's portfolio by $x_{ij}(t)$ and the amount of instrument j held as a liability in sector i's portfolio by $y_{ij}(t)$. The assets and liabilities in all the sectors are grouped into the matrices

$$x(t)=\begin{bmatrix} x_1(t) \\ \dots \\ x_i(t) \\ \dots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) & \dots & x_{1j}(t) & \dots & x_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ x_{i1}(t) & \dots & x_{ij}(t) & \dots & x_{in}(t) \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1}(t) & \dots & x_{mj}(t) & \dots & x_{mn}(t) \end{bmatrix}$$

and

$$y(t)=\begin{bmatrix} y_1(t) \\ \dots \\ y_i(t) \\ \dots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} y_{11}(t) & \dots & y_{1j}(t) & \dots & y_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ y_{i1}(t) & \dots & y_{ij}(t) & \dots & y_{in}(t) \\ \dots & \dots & \dots & \dots & \dots \\ y_{m1}(t) & \dots & y_{mj}(t) & \dots & y_{mn}(t) \end{bmatrix}.$$

We denote the price of instrument j held as an asset at time t by $r_j(t)$ and the price of instrument j held as a liability at time t by $(1+h_j(t))r_j(t)$, where h_j is a nonnegative function defined into $[0,T]$ and belonging to $L^\infty([0,T])$. We introduce the term $h_j(t)$ because the prices of liabilities are generally greater than or equal to the prices of assets in order to describe, in a more realistic way, the behaviour of the markets for which the liabilities are more expensive than the assets. In such a way, this paper appears as an improvement in various directions of the previous ones ([9–13] and [15]). We group the instrument prices held as assets into the vector $r(t)=[r_1(t),r_2(t),\dots,r_i(t),\dots,r_n(t)]^T$ and the instrument prices held as liabilities into the vector $(1+h(t))r(t)=[(1+h_1(t))r_1(t),(1+h_2(t))r_2(t),\dots,(1+h_i(t))r_i(t),\dots,(1+h_n(t))r_n(t)]^T$. In our problem, the prices of each instrument appear as unknown variables. Under the assumption of perfect competition, each sector will behave as if it has no influence on the instrument prices or on the behaviour of the other sectors, whereas the instrument prices depend on the total amount of the investments and the liabilities of each sector. In order to express the time dependent equilibrium conditions by means of an evolutionary variational inequality, we choose as a functional setting the very general Lebesgue space $L^2([0,T],\mathbb{R}^p)=\{f:[0,T]\to\mathbb{R}^p : \int_0^T \|f(t)\|_p^2 dt < +\infty\}$. Then, the set of feasible assets and liabilities for each sector $i=1,\dots,m$, becomes

$$P_i=\Big\{(x_i(t),y_i(t))\in L^2\left([0,T],\mathbb{R}^{2n}\right): \sum_{j=1}^n x_{ij}(t)=s_i(t),\ \sum_{j=1}^n y_{ij}(t)=l_i(t)$$
$$\text{a.e. in } [0,T],\ x_i(t)\ge 0,\ y_i(t)\ge 0,\ \text{a.e. in } [0,T]\Big\}\quad \forall i=1,\dots,m.$$

In such a way, the set of all feasible assets and liabilities becomes

$$P = \Big\{(x(t), y(t)) \in L^2\left([0,T], \mathbb{R}^{2mn}\right) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t), \sum_{j=1}^{n} y_{ij}(t) = l_i(t),$$

$$\forall\, i = 1, \dots, m, \text{ a.e. in } [0,T], \; x_i(t) \geq 0, \; y_i(t) \geq 0, \; \forall\, i = 1, \dots, m, \text{ a.e. in } [0,T]\Big\}.$$

Now, in order to improve the model of competitive financial equilibrium described in [4], which represents a significant but still partial approach to the complex problem of financial equilibrium, we consider the possibility of policy interventions in the financial equilibrium conditions and incorporate them in the form of taxes and price controls and, mainly, we consider a more complete definition of equilibrium prices $r(t)$, based on the demand–supply law, imposing that the equilibrium prices vary between floor and ceiling prices.

To this aim, denote the ceiling price associated with instrument j by $\overline{r}_j$ and the nonnegative floor price associated with instrument j by $\underline{r}_j$, with $\overline{r}_j(t) > \underline{r}_j(t)$, a.e. in $[0,T]$. The floor price $\underline{r}_j(t)$ is determined on the basis of the official interest rate fixed by the central banks, which in turn take into account the consumer price inflation. Then, the equilibrium prices $r_j^*(t)$ cannot be less than these floor prices. The ceiling price $\overline{r}_j(t)$ derives from the financial need to control the national debt arising from the amount of public bonds and of the rise in inflation. It is a sign of the difficulty on the recovery of the economy. However, it should be not overestimated because it produced an availability of money.

In detail, the meaning of the lower and upper bounds is that to each investor a minimal price $\underline{r}_j$ for the assets held in the instrument j is guaranteed, whereas each investor is requested to pay for the liabilities in any case a minimal price $\left(1 + h_j\right)\underline{r}_j$. Analogously each investor cannot obtain for an asset a price greater than $\overline{r}_j$ and as a liability the price cannot exceed the maximum price $\left(1 + h_j\right)\overline{r}_j$.

Denote the given tax rate levied on sector i's net yield on financial instrument j, as τ_{ij}. Assume that the tax rates lie in the interval $[0, 1)$ and belong to $L^\infty([0,T])$. Therefore, the government in this model has the flexibility of levying a distinct tax rate across both sectors and instruments.

Let us group the instrument ceiling prices $\overline{r}_j$ into the column vector $\overline{r}_j(t) = [\overline{r}_1(t), \dots, \overline{r}_i(t), \dots, \overline{r}_n(t)]^T$, the instrument floor prices $\underline{r}_j$ into the column vector $\underline{r}_j(t) = [\underline{r}_1(t), \dots, \underline{r}_i(t), \dots, \underline{r}_n(t)]^T$, and the tax rates τ_{ij} into the matrix

$$\tau(t) = \begin{bmatrix} \tau_{11}(t) & \dots & \tau_{1j}(t) & \dots & \tau_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{i1}(t) & \dots & \tau_{ij}(t) & \dots & \tau_{in}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{m1}(t) & \dots & \tau_{mj}(t) & \dots & \tau_{mn}(t) \end{bmatrix}.$$

The set of feasible instrument prices becomes:

$$\mathcal{R} = \left\{ r \in L^2([0,T],\mathbb{R}^n) : \underline{r}_j(t) \le r_j(t) \le \overline{r}_j(t), \quad j = 1,\dots,n, \text{ a.e. in } [0,T] \right\},$$

where $\underline{r}$ and $\overline{r}$ are assumed to belong to $L^2([0,T],\mathbb{R}^n)$.

In order to determine for each sector i, the optimal composition of instruments held as assets and as liabilities, we consider, as usual, the influence due to risk-aversion and the process of optimization of each sector in the financial economy, namely, the desire to maximize the value of the asset holdings while minimizing the value of liabilities. An example of risk aversion is given by the well-known Markowitz quadratic function based on the variance–covariance matrix denoting the sector's assessment of the standard deviation of prices for each instrument (see [25, 26]).

In our case, however, the Markowitz utility or other more general ones are considered time-dependent in order to incorporate the adjustment in time which depends on the previous equilibrium states. A way in order to obtain the adjustments is to introduce a memory term as it happens in other deterministic models (see [1–3, 8, 20–22, 29]). Then, we introduce the utility function $U_i(t, x_i(t), y_i(t), r(t))$, for each sector i, defined as follows

$$\begin{aligned} U_i(t, x_i(t), y_i(t), r(t)) &= u_i(t, x_i(t), y_i(t)) \\ &+ \sum_{j=1}^{n} r_j(t)(1-\tau_{ij}(t)) \left[x_{ij}(t) - \left(1 + h_j(t)\right) y_{ij}(t)\right], \end{aligned}$$

where the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent and $r_j(t)\left(1-\tau_{ij}(t)\right)\left[x_i(t) - \left(1+h_j(t)\right) y_i(t)\right]$ represents the value of the difference between the asset holdings and the value of liabilities. We suppose that the sector's utility function $U_i(t, x_i(t), y_i(t))$ is defined on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, is measurable in t and is continuous with respect to x_i and y_i. Moreover, we assume that $\partial u_i / \partial x_{ij}$ and $\partial u_i / \partial y_{ij}$ exist and that they are measurable in t and continuous with respect to x_i and y_i. Further, we require that $\forall i = 1,\dots,m$, $\forall j = 1,\dots,n$, and a.e. in $[0,T]$ the following growth conditions hold true:

$$|u_i(t,x,y)| \le \alpha_i(t)\|x\|\|y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{1}$$

and

$$\left|\frac{\partial u_i(t,x,y)}{\partial x_{ij}}\right| \le \beta_{ij}(t)\|y\|, \quad \left|\frac{\partial u_i(t,x,y)}{\partial y_{ij}}\right| \le \gamma_{ij}(t)\|x\|, \tag{2}$$

where α_i, β_{ij}, γ_{ij} are nonnegative functions of $L^\infty([0,T])$. Finally, we suppose that the function $u_i(t,x,y)$ is concave.

We remind that the Markowitz utility function verifies conditions (1) and (2).

In order to determine the equilibrium prices, we establish the equilibrium condition which expresses the equilibration of the total assets, the total liabilities and

the portion of financial transactions per unit F_j employed to cover the expenses of the financial institutions including possible dividends and manager bonus, as in [4]. Hence, the equilibrium condition for the price r_j of instrument j is the following:

$$\sum_{i=1}^{m} \left(1 - \tau_{ij}(t)\right) \left[x_{ij}^*(t) - \left(1 + h_j(t)\right) y_{ij}^*(t)\right] + F_j(t) \begin{cases} \geq 0 & \text{if } r_j^*(t) = \underline{r}_j(t) \\ = 0 & \text{if } \underline{r}_j(t) < r_j^*(t) < \overline{r}_j(t) \\ \leq 0 & \text{if } r_j^*(t) = \overline{r}_j(t) \end{cases} \tag{3}$$

where (x^*, y^*, r^*) is the equilibrium solution for the investments as assets and as liabilities and for the prices.

In other words, the prices are determined taking into account the amount of the supply, the demand of an instrument and the charges F_j, namely, if there is an actual supply excess of an instrument as assets and of the charges F_j in the economy, then its price must be the floor price. If the price of an instrument is positive, but not at the ceiling, then the market of that instrument must clear. Finally, if there is an actual demand excess of an instrument as liabilities and of the charges F_j in the economy, then the price must be at the ceiling.

Now, we can give different but equivalent equilibrium conditions, each of which is useful to illustrate the particular features of the equilibrium.

Definition 1 A vector of sector assets, liabilities and instrument prices $(x^*(t), y^*(t), r^*(t)) \in P \times \mathcal{R}$ is an equilibrium of the dynamic financial model if and only if $\forall i = 1, \ldots, m, \forall j = 1, \ldots, n$, and a.e. in $[0, T]$, it satisfies the system of inequalities

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - \left(1 - \tau_{ij}(t)\right) r_j^*(t) - \mu_i^{(1)*}(t) \geq 0, \tag{4}$$

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + \left(1 - \tau_{ij}(t)\right) \left(1 + h_j(t)\right) r_j^*(t) - \mu_i^{(2)*}(t) \geq 0, \tag{5}$$

and equalities

$$x_{ij}^*(t) \left[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - \left(1 - \tau_{ij}(t)\right) r_j^*(t) - \mu_i^{(1)*}(t)\right] = 0, \tag{6}$$

$$y_{ij}^*(t) \left[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} + \left(1 - \tau_{ij}(t)\right) \left(1 + h_j(t)\right) r_j^*(t) - \mu_i^{(2)*}(t)\right] = 0, \tag{7}$$

where $\mu_i^{(1)*}(t), \mu_i^{(2)*}(t) \in L^2([0, T])$ are Lagrange multipliers, and verifies condition (3) a.e. in $[0, T]$.

Let us explain the meaning of the above conditions. To each financial volumes s_i and l_i held by sector i, we associate the functions $\mu_i^{(1)*}(t), \mu_i^{(2)*}(t)$, related, respectively, to the assets and to the liabilities, and which represent the "equilibrium

disutilities" per unit of the sector i. Then, (4) and (6) mean that the financial volume invested in instrument j as assets x^*_{ij} is greater than or equal to zero if the jth component $-\dfrac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}} - (1-\tau_{ij}(t))r^*_j(t)$ of the disutility is equal to $\mu^{(1)*}_i(t)$, whereas if $-\dfrac{\partial u_i(t,x^*,y^*)}{\partial x_{ij}} - (1-\tau_{ij}(t))r^*_j(t) > \mu^{(1)*}_i(t)$, then $x^*_{ij}(t) = 0$. The same occurs for the liabilities and the meaning of (3) is already illustrated.

The functions $\mu^{(1)*}_i(t)$ and $\mu^{(2)*}_i(t)$ are Lagrange multipliers associated a.e. in $[0,T]$ with the constraints $\sum_{j=1}^n x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^n y_{ij}(t) - l_i(t) = 0$, respectively. They are unknown a priori, but this fact has no influence because we will prove in the following theorem that Definition 1 is equivalent to a variational inequality in which $\mu^{(1)*}_i(t)$ and $\mu^{(2)*}_i(t)$ do not appear.

The following Theorem is proved in [6] (see Theorem 2.1).

Theorem 1 *A vector $\left(x^*,y^*,r^*\right) \in P \times \mathcal{R}$ is a dynamic financial equilibrium if and only if it satisfies the following variational inequality:*

Find $(x^*,y^*,r^*) \in P \times \mathcal{R}$:

$$\sum_{i=1}^m \int_0^T \left\{ \sum_{j=1}^n \left[-\frac{\partial u_i(t,x^*_i(t),y^*_i(t))}{\partial x_{ij}} - (1-\tau_{ij}(t))r^*_j(t) \right] \times \left[x_{ij}(t) - x^*_{ij}(t)\right] \right.$$
$$\left. + \sum_{j=1}^n \left[-\frac{\partial u_i(t,x^*_i(t),y^*_i(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))r^*_j(t)(1+h_j(t)) \right] \times \left[y_{ij}(t) - y^*_{ij}(t)\right] \right\} dt$$
$$+ \sum_{j=1}^n \int_0^T \sum_{i=1}^m \left\{ \left(1-\tau_{ij}(t)\right)\left[x^*_{ij}(t) - (1+h_j(t))y^*_{ij}(t)\right] + F_j(t) \right\}$$
$$\times [r_j(t) - r^*_j(t)]dt \geq 0, \qquad \forall (x,y,r) \in P \times \mathcal{R}. \tag{8}$$

We are also able to provide existence theorems for the variational inequality (8).

To this end, we remind some definitions (see [27, 35]). Let X be a reflexive Banach space and let $\mathbb{K}$ be a subset of X and X^* be the dual space of X.

Definition 2 A mapping $A : \mathbb{K} \to X^*$ is pseudomonotone in the sense of Brezis (B-pseudomonotone) iff

1. For each sequence u_n weakly converging to u (in short $u_n \rightharpoonup u$) in $\mathbb{K}$ and such that $\limsup_n \langle Au_n, u_n - v\rangle \leq 0$, it results that:
$$\liminf_n \langle Au_n, u_n - v\rangle \geq \langle Au, u-v\rangle, \quad \forall v \in \mathbb{K}.$$
2. For each $v \in \mathbb{K}$, the function $u \mapsto \langle Au, u-v\rangle$ is lower bounded on the bounded subset of $\mathbb{K}$.

Definition 3 A mapping $A : \mathbb{K} \to X^*$ is hemicontinuous in the sense of Fan (F-hemicontinuous) iff for all $v \in \mathbb{K}$ the function $u \mapsto \langle Au, u - v\rangle$ is weakly lower semicontinuous on $\mathbb{K}$.

Now, we recall the following hemicontinuity definition, which will be used together with some kinds of monotonicity assumptions.

Definition 4 A mapping $A : \mathbb{K} \to X^*$ is lower hemicontinuous along line segments, iff the function $\xi \mapsto \langle A\xi, u - v\rangle$ is lower semicontinuous for all $u, v \in \mathbb{K}$ on the line segments $[u, v]$.

Definition 5 The map $A : \mathbb{K} \to X^*$ is said to be pseudomonotone in the sense of Karamardian (K-pseudomonotone) iff for all $u, v \in \mathbb{K}$

$$\langle Av, u - v\rangle \geq 0 \implies \langle Au, u - v\rangle \geq 0.$$

Then, the following existence theorems hold (see [27]). The first one does not require any kind of monotonicity assumptions.

Theorem 2 *Let $\mathbb{K} \subset X$ be a nonempty closed convex bounded set and let $A : \mathbb{K} \subset E \to X^*$ be B-pseudomonotone or F-hemicontinuous. Then, the variational inequality*

$$\langle Au, v - u\rangle \geq 0 \quad \forall v \in \mathbb{K} \tag{9}$$

admits a solution.

The next theorem requires the K-pseudomonotonicity assumption.

Theorem 3 *Let $\mathbb{K} \subset X$ be a closed convex bounded set and let $A : \mathbb{K} \to X^*$ be a K-pseudomonotone map which is lower hemicontinuous along line segments. Then, variational inequality (9) admits solutions.*

We can apply such theorems to our model, setting:

$$v = \left(\left(x_{ij}\right)_{i=1,\dots,m \quad j=1,\dots,n}, \left(y_{ij}\right)_{i=1,\dots,m \quad j=1,\dots,n}, \left(r_j\right)_{j=1,\dots,n}\right);$$

$$A : L^2\left([0,T], \mathbb{R}^{2mn+n}\right) \to L^2\left([0,T], \mathbb{R}^{2mn+n}\right),$$

$$\begin{aligned} A(v) = \Bigg(& \left[-\frac{\partial u_i(x,y)}{\partial x_{ij}} - (1-\tau_{ij})r_j\right]_{i=1,\dots,m \quad j=1,\dots,n}, \\ & \left[-\frac{\partial u_i(x,y)}{\partial y_{ij}} + (1-\tau_{ij})(1+h_j)r_j\right]_{i=1,\dots,m \quad j=1,\dots,n}, \\ & \left[\sum_{i=1}^{m}(1-\tau_{ij})\left(x_{ij} - (1+h_j)y_{ij}\right)\right]_{j=1,\dots,n}\Bigg); \end{aligned}$$

$$\mathbb{K} = P \times \mathcal{R} = \Big\{ v \in L^2\left([0,T], \mathbb{R}^{2mn+n}\right) : x_i(t) \geq 0,\ y_i(t) \geq 0, \text{a.e. in } [0,T],$$

$$\sum_{j=1}^{n} x_{ij}(t) = s_i(t), \ \ \sum_{j=1}^{n} y_{ij}(t) = l_i(t) \text{ a.e. in } [0,T], \ \forall i = 1, \dots, m,$$

$$\left. \underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t), \text{a.e. in } [0,T], \forall j = 1, \dots, n \right\}.$$

Hence, evolutionary variational inequality (8) becomes (9) and we can apply Theorems 2 and 3, assuming that A is B-pseudomonotone or K-hemicontinuous, or assuming that A is K-pseudomonotone, lower hemicontinuous along line segments and noting that $\mathbb{K}$ is a nonempty closed convex and bounded set.

Moreover, we recall that condition (2) is sufficient to guarantee that the operator A is lower hemicontinuous along line segments (see [19]).

3 The Lagrange Dual Problem. The Deficit and Surplus Variables

First, let us present the infinite dimensional Lagrange duality, which represents an important and very recent achievement (see [14, 16, 17, 28]) and which we will use.

First, we recall the definition of the tangent cone. If X denote a real normed space and C is a subset of X, given an element $x \in X$, the set:

$$T_C(x) = \Big\{ h \in X :$$

$$h = \lim_{n\to\infty} \lambda_n (x_n - x), \ \lambda_n \in \mathbb{R}, \ \lambda_n > 0, \ \forall n \in \mathbb{N}, \ x_n \in C \ \forall n \in \mathbb{N}, \ \lim_{n\to\infty} x_n = x \Big\}$$

is called the tangent cone to C at x (see [23]).

Now, let us present the new duality principles for a convex optimization problem. Let X be a real normed space and S a nonempty convex subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C, with $C^* = \{\lambda \in Y^* : \langle \lambda, y \rangle \geq 0 \, \forall y \in C\}$ the dual cone of C, Y^* topological dual of Y, and let $(Z, \|\cdot\|_Z)$ be a real normed space with topological dual Z^*. Let us set $-C = \{-x \in Y : \ x \in C\}$. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two convex functions and let $h : S \to Z$ be an affine-linear function.

Let us consider the problem

$$\min_{x \in \mathbb{K}} f(x) \tag{10}$$

where $\mathbb{K} = \{x \in S : \ g(x) \in -C, \ h(x) = \theta_Z\}$ and the dual problem

$$\max_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} \{ f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle \}. \tag{11}$$

Remember that λ and μ are the so-called Lagrange multipliers, associated to the sign constraints and to equality constraints, respectively. They play a fundamental

role to better understand the behaviour of the financial equilibrium. Moreover, as it is well known, it always results:

$$\min_{x \in \mathbb{K}} f(x) \leq \max_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} \{f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle\}, \tag{12}$$

and, if problem (10) is solvable, and in (12), the equality holds, we say that the strong duality between primal problem (10) and dual problem (11) holds. When we have the strong duality, we may consider the so-called "shadow market", namely, the dual Lagrange problem associated to the primal problem.

In order to obtain the strong duality, we need that delicate conditions, called "constraint qualification conditions", hold. In the infinite dimensional settings, the next assumption, the so-called *Assumption S*, results to be a necessary and sufficient condition for the strong duality (see [14, 16, 17, 28]).

Definition of Assumption S We shall say that *Assumption S* is fulfilled at a point $x_0 \in \mathbb{K}$, if it results to be

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap \big(]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\}\big) = \emptyset, \tag{13}$$

where

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : \ x \in S \setminus \mathbb{K}, \ \alpha \geq 0, \ y \in C\}.$$

The following theorem holds (see Theorem 1.1 in [17] for the proof).

Theorem 4 *Under the above assumptions on f, g, h and C, if problem (10) is solvable and Assumption S is fulfilled at the extremal solution $x_0 \in \mathbb{K}$, then also problem (11) is solvable, the extreme values of both problems are equal, namely, if $(x_0, \lambda^*, \mu^*) \in \mathbb{K} \times C^* \times Z^*$ is the optimal point of problem (11),*

$$\begin{aligned} f(x_0) = \min_{x \in \mathbb{K}} f(x) &= f(x_0) + \langle \lambda^*, g(x_0) \rangle + \langle \mu^*, h(x_0) \rangle \\ &= \max_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} \{f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle\} \end{aligned} \tag{14}$$

and, it results to be:

$$\langle \lambda^*, g(x_0) \rangle = 0.$$

Let us recall that the following one is the so-called Lagrange functional

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle, \quad \forall x \in S, \ \forall \lambda \in C^*, \ \forall \mu \in Z^*. \tag{15}$$

Using the Lagrange functional, (14) may be rewritten as

$$f(x_0) = \min_{x \in \mathbb{K}} f(x) = \mathcal{L}(x_0, \lambda^*, \mu^*) = \max_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} \mathcal{L}(x, \lambda, \mu).$$

By means of Theorem 4, it is possible to show the usual relationship between a saddle point of the Lagrange functional and the solution of the constraint optimization problem (10) (see Theorem 5 in [16] for the proof).

Theorem 5 *Let us assume that the assumptions of Theorem 4 are satisfied. Then, $x_0 \in \mathbb{K}$ is a minimal solution to problem (10) if and only if there exist $\overline{\lambda} \in C^*$ and $\overline{\mu} \in Z^*$ such that $(x_0, \overline{\lambda}, \overline{\mu})$ is a saddle point of the Lagrange functional (15), namely,*

$$\mathcal{L}(x_0, \lambda, \mu) \leq \mathcal{L}(x_0, \lambda^*, \mu^*) \leq \mathcal{L}(x, \lambda^*, \mu^*), \quad \forall x \in S,\ \lambda \in C^*,\ \mu \in Z^*$$

and, moreover, it results that

$$\langle \lambda^*, g(x_0) \rangle = 0. \tag{16}$$

Now, we apply the infinite dimensional duality theory to our general model. To this end, as usual, let us set

$$\begin{aligned}
f(x,y,r) = \int_0^T \Bigg\{ & \sum_{i=1}^m \sum_{j=1}^n \left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j^*(t) \right] \\
& \times [x_{ij}(t) - x_{ij}^*(t)] \\
& + \sum_{i=1}^m \sum_{j=1}^n \left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))(1+h_j(t)) r_j^*(t) \right] \times [y_{ij}(t) - y_{ij}^*(t)] \\
& + \sum_{j=1}^n \left[\sum_{i=1}^m (1-\tau_{ij}(t)) \left[x_{ij}^*(t) - (1+h_j(t)) y_{ij}^*(t) \right] + F_j(t) \right] \times \left[r_j(t) - r_j^*(t) \right] \Bigg\} \, dt.
\end{aligned}$$

Then, the Lagrange functional is

$$\begin{aligned}
\mathcal{L}\left(x, y, r, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}\right) &= f(x,y,r) - \sum_{i=1}^m \sum_{j=1}^n \int_0^T \lambda_{ij}^{(1)}(t) x_{ij}(t)\, dt \\
& - \sum_{i=1}^m \sum_{j=1}^n \int_0^T \lambda_{ij}^{(2)} y_{ij}(t)\, dt - \sum_{i=1}^m \int_0^T \mu_i^{(1)}(t) \left(\sum_{j=1}^n x_{ij}(t) - s_i(t) \right) dt \\
& - \sum_{i=1}^m \int_0^T \mu_i^{(2)}(t) \left(\sum_{j=1}^n y_{ij}(t) - l_i(t) \right) dt + \sum_{j=1}^n \int_0^T \rho_j^{(1)}(t)(\underline{r}_j(t) - r_j(t))\, dt \\
& + \sum_{j=1}^n \int_0^T \rho_j^{(2)}(t)(r_j(t) - \overline{r}_j(t))\, dt,
\end{aligned} \tag{17}$$

where $(x, y, r) \in L^2([0,T], \mathbb{R}^{2mn+n})$, $\lambda^{(1)}, \lambda^{(2)} \in L^2([0,T], \mathbb{R}_+^{mn})$, $\mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}_+^n)$.

Remember that $\lambda^{(1)}$, $\lambda^{(2)}$, $\rho^{(1)}$, $\rho^{(2)}$ are the Lagrange multipliers associated, a.e. in $[0,T]$, to the sign constraints $x_i(t) \geq 0$, $y_i(t) \geq 0$, $r_j(t) - \underline{r}_j(t) \geq 0$, $\overline{r}_j(t) - r_j(t) \geq 0$, respectively. The functions $\mu^{(1)}(t)$ and $\mu^{(2)}(t)$ are the Lagrange multipliers

associated, a.e. in $[0,T]$, to the equality constraints $\sum_{j=1}^{n} x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^{n} y_{ij}(t) - l_i(t) = 0$, respectively.

The following theorem holds (see [6] Theorem 6.1).

Theorem 6 *Let $(x^*, y^*, r^*) \in P \times \mathcal{R}$ be a solution to variational inequality (8) and let us consider the associated Lagrange functional (17). Then, Assumption S is satisfied and the strong duality holds and there exist $\lambda^{(1)*}, \lambda^{(2)*} \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\mu^{(1)*}, \mu^{(2)*} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)*}, \rho^{(2)*} \in L^2([0,T], \mathbb{R}^n_+)$ such that $(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})$ is a saddle point of the Lagrange functional, namely,*

$$\begin{aligned} &\mathcal{L}\left(x^*, y^*, r^*, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}\right) \\ &\leq \mathcal{L}(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) \\ &\leq \mathcal{L}(x, y, r, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) \end{aligned} \tag{18}$$

$\forall (x, y, r) \in L^2\left([0,T], \mathbb{R}^{2mn+n}\right)$, $\forall \lambda^{(1)}, \lambda^{(2)} \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\forall \mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\forall \rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}^n_+)$ and, a.e. in $[0,T]$,

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) - \lambda_{ij}^{(1)*}(t) - \mu_i^{(1)*}(t) = 0,$$

$$\forall i = 1, \ldots, m, \ \forall j = 1 \ldots, n;$$

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t)) r_j^*(t) - \lambda_{ij}^{(2)*}(t) - \mu_i^{(2)*}(t) = 0,$$

$$\forall i = 1, \ldots, m, \ \forall j = 1 \ldots, n;$$

$$\sum_{i=1}^{m} (1 - \tau_{ij}(t)) \left[x_{ij}^*(t) - (1 + h_j(t)) y_{ij}^*(t)\right] + F_j(t) + \rho_j^{(2)*}(t) = \rho_j^{(1)*}(t),$$

$$\forall j = 1, \ldots, n; \tag{19}$$

$$\lambda_{ij}^{(1)*}(t) x_{ij}^*(t) = 0, \quad \lambda_{ij}^{(2)*}(t) y_{ij}^*(t) = 0, \quad \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n \tag{20}$$

$$\mu_i^{(1)*}(t) \left(\sum_{j=1}^{n} x_{ij}^*(t) - s_i(t)\right) = 0, \quad \mu_i^{(2)*}(t) \left(\sum_{j=1}^{n} y_{ij}^*(t) - l_i(t)\right) = 0,$$

$$\forall i = 1, \ldots, m \tag{21}$$

$$\rho_j^{(1)*}(t)(\underline{r}_j(t) - r_j^*(t)) = 0, \quad \rho_j^{(2)*}(t)(r_j^*(t) - \overline{r}_j(t)) = 0, \quad \forall j = 1, \dots, n. \tag{22}$$

Let us now call Balance Law the following one

$$\sum_{i=1}^{m} l_i(t) = \sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right] - \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t)$$
$$+ \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t).$$

The following theorem holds.

Theorem 7 *Let* $(x^*, y^*, r^*) \in P \times \mathcal{R}$ *be the dynamic equilibrium solution to variational inequality* (8)*, then the Balance Law*

$$\sum_{i=1}^{m} l_i(t) = \sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right] - \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t)$$
$$+ \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t) \tag{23}$$

is verified.

Remark 1 Let us recall that from the Liability Formula we get the following index $E(t)$, called "Evaluation Index", that is very useful for the rating procedure:

$$E(t) = \frac{\sum_{i=1}^{m} l_i(t)}{\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t)},$$

where we set

$$\tilde{s}_i(t) = \frac{s_i(t)}{1 + i(t)}, \quad \tilde{F}_j(t) = \frac{F_j(t)}{1 + i(t) - \theta(t) - \theta(t)i(t)}.$$

From the Liability Formula, we obtain

$$E(t) = 1 - \frac{\sum_{j=1}^{n} \rho_j^{(1)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t)\right)}$$

$$+\frac{\sum_{j=1}^{n}\rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)\right)} \tag{24}$$

4 Analysis of Financial Contagion

Let us consider (19), namely, the Deficit Formula for the generic instrument j

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1+h_j(t))y_{ij}^*(t)\right]+F_j(t)+\rho_j^{(2)*}(t)=\rho_j^{(1)*}(t),$$
$$\forall j=1,\ldots,n \quad \text{a.e. in } [0,T]$$

together with the complementary Eq. (22)

$$\rho_j^{(1)*}(t)(\underline{r}_j(t)-r_j^*(t))=0,\ \rho_j^{(2)*}(t)(r_j^*(t)-\overline{r}_j(t))=0,\ \rho_j^{(1)*}(t)\cdot\rho_j^{(2)*}(t)=0$$
$$\forall j=1,\ldots,n \quad \text{a.e. in } [0,T].$$

Let us note that if $\rho_j^{(1)*}(t)>0$

$$r_j^*(t)=\underline{r}_j(t)$$

and hence, $\rho_j^{(2)*}(t)=0$. From (19), we get

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))x_{ij}^*(t)>\sum_{i=1}^{m}(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^*(t)+F_j(t),$$

namely, the amount of the assets exceeds the one of the liabilities and of the expenses $F_j(t)$. Then, all the individual entities i, $i=1,\ldots,m$, have the deficit

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))x_{ij}^*(t)\overline{\rho}_j^{(1)*}(t)-\sum_{i=1}^{m}(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^*(t)\underline{r}_j(t)-F_j(t)\underline{r}_j^*(t)$$
$$=\rho_j^{(1)*}(t)\underline{r}_j(t)>0$$

because for the sectors, the quantity

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))x_{ij}^*(t)\rho_j^{(1)}(t)$$

represents the outcome, whereas

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^*(t)\underline{r}_j(t)-F_j(t)r_j^*(t)$$

represents the income.

Then, when $\rho_j^{*(1)}(t)$ is positive, formula (19) represents the deficit, whereas when $\rho_j^{*(2)}(t) > 0$, formula (19) represents the surplus. From formula (19), the Balance Law is derived as

$$\sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m} l_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right] - \sum_{i=1}^{m}\sum_{j=1}^{n}\left(1 - \tau_{ij}(t)\right) h_j(t) y_{ij}^*(t)$$

$$+ \sum_{j=1}^{n} F_j(t) = \sum_{j=1}^{n} \rho_j^{(1)*}(t) - \sum_{j=1}^{n} \rho_j^{(2)*}(t)$$

and we see that the balance of all the financial entities depends on the difference

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) - \sum_{j=1}^{n} \rho_j^{(2)*}(t).$$

If

$$\sum_{j=1}^{n} \rho_j^{(1)*}(t) > \sum_{j=1}^{n} \rho_j^{(2)*}(t), \tag{25}$$

the balance is negative, the whole deficit exceeds the sum of all the surplus and a negative contagion appears and the insolvencies of individual entities propagate through the entire system. As we can see, it is sufficient that only one deficit $\rho_j^{(1)*}(t)$ is large to obtain, even if the other $\rho_j^{(2)*}(t)$ are lightly positive, a negative balance for all the system. Moreover, we can obtain $\rho_j^*(t) > 0$ even if for only a sector has a big insolvency.

Remark 2 When condition (25) is verified, we get $E(t) \leq 1$ and, hence, also $E(t)$ is a significant indicator that the financial contagion happens.

5 The "Shadow Financial Market"

We remark that the financial problem can be considered from two different perspectives: one from the *Point of View of the Sectors* which try to maximize the utility and a second point of view, that we can call *System Point of View*, which regards the whole equilibrium, namely, the respect of the previous laws. For example, from the point of view of the sectors, $l_i(t)$, for $i = 1, \dots, m$, are liabilities, whereas for the economic system they are investments and, hence, the Liability Formula, from the system point of view, can be called *Investments Formula*. The system point of view coincides with the dual Lagrange problem (the so-called "shadow market") in which $\rho_j^{(1)}(t)$ and $\rho_j^{(2)}(t)$ are the dual multipliers, representing the deficit and the surplus per unit arising from instrument j. Formally, the dual problem is given as follows.

Find $\left(\rho^{(1)*}, \rho^{(2)*}\right) \in L^2([0,T], \mathbb{R}^{2n}_+)$ such that

$$\sum_{j=1}^{n} \int_0^T \left(\rho_j^{(1)}(t) - \rho_j^{(1)*}(t)\right) \left(\underline{r}_j(t) - r_j^*(t)\right) dt + \sum_{j=1}^{n} \int_0^T \left(\rho_j^{(2)}(t) - \rho_j^{(2)*}(t)\right)$$
$$\left(r_j^*(t) - \overline{r}_j(t)\right) dt \leq 0, \quad \forall \left(\rho^{(1)}, \rho^{(2)}\right) \in L^2\left([0,T], \mathbb{R}^{2n}_+\right). \tag{26}$$

In fact, taking into account the inequality in the left hand side of (18), we get

$$- \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \left(\lambda_{ij}^{(1)}(t) - \lambda_{ij}^{(1)*}(t)\right) x_{ij}^*(t)\, dt - \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \left(\lambda_{ij}^{(2)} - \lambda_{ij}^{(2)*}\right) y_{ij}^*(t)\, dt$$
$$- \sum_{i=1}^{m} \int_0^T \left(\mu_i^{(1)}(t) - \mu_i^{(1)*}(t)\right) \left(\sum_{j=1}^{n} x_{ij}^*(t) - s_i(t)\right) dt$$
$$- \sum_{i=1}^{m} \int_0^T \left(\mu_i^{(2)}(t) - \mu_i^{(2)*}(t)\right) \left(\sum_{j=1}^{n} y_{ij}^*(t) - l_i(t)\right) dt$$
$$+ \sum_{j=1}^{n} \int_0^T \left(\rho_j^{(1)}(t) - \rho_j^{(1)*}(t)\right) \left(\underline{r}_j(t) - r_j^*(t)\right) dt$$
$$+ \sum_{j=1}^{n} \int_0^T \left(\rho_j^{(2)}(t) - \rho_j^{(2)*}(t)\right) \left(r_j^*(t) - \overline{r}_j(t)\right) dt \leq 0$$

$\forall \lambda^{(1)}, \lambda^{(2)} \in L^2\left([0,T], \mathbb{R}^{mn}_+\right)$, $\mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}^n_+)$.

Choosing $\lambda^{(1)} = \lambda^{(1)*}$, $\lambda^{(2)} = \lambda^{(2)*}$, $\mu^{(1)} = \mu^{(1)*}$, $\mu^{(2)} = \mu^{(2)*}$, we obtain the dual problem (26).

Note that, from the *System Point of View*, also the expenses of the institutions $F_j(t)$ are supported by the liabilities of the sectors.

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Comparisons of Means and Related Functional Inequalities

Włodzimierz Fechner

Abstract We provide a survey of several results on functional inequalities stemming from inequalities between classical means. Further, we recall a few problems in this field which according to the best of author's knowledge remain open. Last section of this paper is devoted to a new, more general functional inequality and a joint generalization of several earlier results is obtained.

Keywords Functional inequality · Mean · Quasi-arithmetic mean · Inequalities between means · Recurrence equation · Schur stability

1 Preliminaries

Throughout the paper it is assumed that the symbol $\mathbb{C}$ stands for the complex plane, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Q}$ is the set of rationals, $\mathbb{N}$ stands for the set of nonnegative integers, and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Further, we will denote the set of positive reals by $\mathbb{R}^+$ and the set of nonnegative reals by $\mathbb{R}_0^+$. Moreover, for $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ or for $a, b \in \mathbb{R}$ open and closed intervals with endpoints a and b are denoted by (a, b) and $[a, b]$, respectively.

Now, let us denote the arithmetic, geometric, and logarithmic mean of two numbers by the respective letters A, G, and L:

$$A(s,t) = \frac{s+t}{2},$$

$$G(s,t) = \sqrt{s \cdot t},$$

$$L(s,t) = \frac{t-s}{\log t - \log s} \quad \text{for } s \neq t \quad \text{and} \quad L(s,s) = s$$

for $s, t \in \mathbb{R}$ for the arithmetic mean and for $s, t \in \mathbb{R}_0^+$ for the geometric and the logarithmic means.

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Next section of the paper contains a brief review of several inequalities involving the three above-mentioned means. In Sect. 3, we deal with functional inequalities stemming from estimates presented in Sect. 2 and we overview known results about these functional inequalities. Also, we recall some unsolved problems connected with them. In the last section of this paper, we introduce a more general functional inequality. We will state and prove a result which yields a joint generalization to a number of earlier results mentioned in Sect. 3. In the last corollary of this paper, we establish a connection between a special case of this inequality and Schur stable polynomials. In particular, we show that the Routh–Hurwitz stability criterion can be applied to deal with this functional inequality.

2 Inequalities Between Means

The following inequality between the arithmetic, geometric, and logarithmic means is well known:

$$G(s,t) \le L(s,t) \le A(s,t), \tag{1}$$

for all $s,t > 0$ (see e.g., Burk [4]). Moreover, the following refinement of (1) holds true:

$$G^{\frac{2}{3}}(s,t) \cdot A^{\frac{1}{3}}(s,t) \le L(s,t) \le \frac{2}{3}G(s,t) + \frac{1}{3}A(s,t) \tag{2}$$

for all $s,t > 0$. Clearly, (1) follows immediately from (2) if we have the estimate

$$G(s,t) \le A(s,t)$$

for all $s,t > 0$, which is elementary. Moreover, the constants $\frac{2}{3}$ and $\frac{1}{3}$ are best possible for both sides of (2).

The first inequality of (2) was proved in 1983 by Leach and Sholander [27], whereas the second inequality of (2) was obtained in 1972 by Carlson [5] (see also Burk [4]), and earlier also by Pólya and Szegő [44]. Finally, let us note that some further refinements of both estimates are due to Chu and Long [6, 28], Leach and Sholander [25, 26], Matejíčka [34], Qian and Zheng [45], Sándor [46, 47], and references therein, among others.

Now, fix arbitrary $x, y \in \mathbb{R}$ such that $x \neq y$, put $s := e^x$ and $t := e^y$ and substitute s and t in (1) and (2). We conclude that the exponential function satisfies the following estimates:

$$e^{\frac{x+y}{2}} \le \frac{e^y - e^x}{y - x} \le \frac{e^x + e^y}{2} \tag{3}$$

and

$$6e^{\frac{2}{3}\cdot\frac{x+y}{2}}\left[\frac{e^x + e^y}{2}\right]^{\frac{1}{3}} \le 6\frac{e^y - e^x}{y - x} \le 4e^{\frac{x+y}{2}} + e^x + e^y \tag{4}$$

for each $x, y \in \mathbb{R}$ such that $x \neq y$. Therefore, inequalities between means have been equivalently transformed into respective inequalities involving the exponential function. In the next section, we will discuss functional inequalities stemming from estimates (3) and (4). We replace the exponential function in (3) and (4) by an unknown mapping f and in this manner we obtain functional inequalities which will be of our interest in the present paper. Our aim is to provide a characterization of all solutions of these functional inequalities.

3 Functional Inequalities

A well-known characterization of the exponential function by means of the functional equations and inequalities is due to Kuczma [22]; see also Kuczma, Choczewski and Ger [24, Chap. 10.2B]. He proved that without any additional regularity assumptions the map $\varphi = \exp$ is the only real-to-real solution of the following system of functional equations and inequalities of a single variable:

$$\begin{aligned} \varphi(x) &> 0, \\ \varphi(x) &\geq 1 + x, \\ \varphi(2x) &= [\varphi(x)]^2, \\ \varphi(-x) &= [\varphi(x)]^{-1}, \end{aligned}$$

postulated for all $x \in \mathbb{R}$. An earlier result of Kuczma [21] (see also M. Kuczma [23, Chap. VI, § 12]) states that all the solutions of a related functional equation of a single variable, which satisfy some additional smoothness, are of the form $\varphi = c \cdot \exp$ with some real c.

In 1988, Poonen answering a problem proposed by Shelupsky [44] proved that the general solution $f : \mathbb{R} \to \mathbb{R}$ of the double inequality:

$$\min\{f(x), f(y)\} \leq \frac{f(y) - f(x)}{y - x} \leq \max\{f(x), f(y)\} \qquad (x \neq y) \tag{5}$$

is of the form $f = c \cdot \exp$, where $c \geq 0$ is an arbitrary constant.

Note that if we insert $f = \exp$ into (5), then we obtain an estimate which is essentially weaker than (3) and thus also weaker than (4). Therefore, in subsequent studies, we need to focus on single functional inequalities rather than on systems.

The above mentioned result of Shelupsky and Poonen was an inspiration for the research of Alsina and Garcia Roig published in [2] in 1990. They studied the following two functional inequalities which are motivated by the second part of the estimate (3):

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(x) + f(y)}{2} \qquad (x \neq y), \tag{6}$$

and

$$0 \le \frac{f(y) - f(x)}{y - x} \le \frac{f(x) + f(y)}{2} \qquad (x \ne y). \tag{7}$$

Among others, they have proved the following two theorems.

Theorem 1 [2, Theorem 1] *A function $f : \mathbb{R} \to \mathbb{R}$ satisfies (6) if and only if there exists a nonincreasing function $d : \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for all $x \in \mathbb{R}$.*

Theorem 2 [2, Theorem 2] *A function $f : \mathbb{R} \to \mathbb{R}$ satisfies (7) if and only if there exists a continuous nonincreasing function $d : \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in \mathbb{R}$ and $d(x + t) \ge e^{-t}d(x)$ for all $x \in \mathbb{R}$ and $t > 0$.*

Remark 1 At the beginning of the proof of Theorem 1, the authors observed that the inequality (6) can be rewritten equivalently in the following form:

$$f(x + h) \le \frac{2 + h}{2 - h} f(x) \quad (\text{for all } x \in \mathbb{R} \text{ and } h \in (0, 2)). \tag{8}$$

It should be clear that (8) is a particular case of a more general functional inequality (23), which will be studied in Sect. 4.

Moreover, an inspection of the original proofs of the two foregoing theorems shows that as the domain of mapping f one can take an arbitrary nonempty open interval instead of the whole real line.

The following functional inequality, which corresponds to the first part of the estimate (3):

$$f\left(\frac{x + y}{2}\right) \le \frac{f(y) - f(x)}{y - x} \qquad (x \ne y) \tag{9}$$

was considered by Alsina and Ger [3] and later by Fechner [14]. It turns out that the two functional inequalities (6) and (9) do not behave in a fully symmetric way. Namely, (9) is more difficult to deal with. However, under some additional assumptions a result analogous to Theorem 1 holds true. We should expect that all solutions of (9) on an open interval I, which enjoy some regularity properties, are of the form $f(x) = i(x)e^x$ for all $x \in I$ with a nondecreasing map i. The following theorem, which generalizes some earlier results of Alsina and Ger from [3], is published in [14].

Theorem 3 [14, Theorem 1] *Assume that I is an open nonvoid interval, $f : I \to \mathbb{R}$ satisfies (9), and*

$$\limsup_{h \to 0+} f(x + h) \ge f(x) \qquad (\text{for all } x \in I). \tag{10}$$

Then, there exists a nondecreasing map $i : I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for all $x \in I$.

Remark 2 In the proof of Theorem 3, it is observed that the inequality (9) can be rewritten equivalently in the following form:

$$f(x+2\,h) \geq 2hf(x+h) + f(x), \tag{11}$$

with $x \in I$ and $h > 0$ such that $x + 2\,h \in I$ (see [14, formula (11)]). Therefore, similarly, like in the case of (8), we conclude that (11) is a particular case of the functional inequality (23), which will be discussed in Sect. 4 (one needs to replace f by $-f$ to obtain the converse inequality to (11), which is precisely a special case of (23)).

Let us recall the following two open problems connected with Theorem 3.

Problem 1 [16, Problem 1] The converse of Theorem 3 is not true (see [14, Remark 1]). For example, take $I = \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = -e^x \quad \text{(for all } x \in \mathbb{R}\text{)}.$$

It is clear that f is of the form

$$f(x) = i(x)e^x \tag{12}$$

with $i(x) = -1$ for all $x \in \mathbb{R}$. Moreover, f as a continuous mapping satisfies condition (10). However, one can see that inequality (9) fails to hold.

Find and prove an additional condition upon mapping i from Theorem 3 to obtain the "if and only if" result, i.e., to get that each function which is of the form (12) solves functional inequality (9).

Problem 2 [16, Problem 2] Is it possible to drop or weaken the assumption (10) in Theorem 3? Compare this also with assumption (26) which appears in Theorem 8.

One more result from [14] shows that solutions of (9) satisfy some functional-integral inequality.

Theorem 4 [14, Theorem 2] *If $f : I \to \mathbb{R}$ is a Riemann-integrable solution of* (9)*, then it satisfies the following functional-integral inequality:*

$$\frac{1}{y-x}\int_x^y f(t)dt \leq \frac{f(y)-f(x)}{y-x} \quad (\textit{for all } x, y \in I \textit{ such that } x < y). \tag{13}$$

There is also an analogue of this theorem for functional inequality (6).

Theorem 5 [14, Theorem 4] *If $f : I \to \mathbb{R}$ is a Riemann-integrable solution of* (6) *then f satisfies the following functional-integral inequality:*

$$\frac{f(y)-f(x)}{y-x} \leq \frac{1}{y-x}\int_x^y f(t)dt \quad (\textit{for all } x, y \in I \textit{ such that } x < y). \tag{14}$$

A more general result in this spirit for continuous solutions of some more general functional inequality was proved in [15].

Theorem 6 [15, Theorem 1] *Assume that $I \subset \mathbb{R}$ is a nonempty open interval and mappings $M_1, M_2 : I \times I \to \mathbb{R}$ and $N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are arbitrary means, i.e.:*

$$\min\{x, y\} \le M_i(x, y) \le \max\{x, y\} \quad (\textit{for all } x, y \in I \textit{ and } i = 1, 2),$$
$$\min\{x, y\} \le N(x, y) \le \max\{x, y\} \quad (\textit{for all } x, y \in \mathbb{R}).$$

Further, assume that $f : I \to \mathbb{R}$ is arbitrary, $g : I \to \mathbb{R}$ is continuous and the following functional inequality

$$\frac{f(y) - f(x)}{y - x} \le N\left(g(M_1(x, y)), g(M_2(x, y))\right) \quad (\textit{for all } x, y \in I), \tag{15}$$

is fulfilled. Then

$$f(y) - f(x) \le \int_x^y g(t)dt \quad (\textit{for all } x, y \in I \textit{ such that } x \le y). \tag{16}$$

Finally, we will quote a result which describes solutions of the following functional inequality:

$$6\frac{f(y) - f(x)}{y - x} \le 4 f\left(\frac{x + y}{2}\right) + f(x) + f(y) \tag{17}$$
$$(\text{for all } x, y \in I \text{ such that } x \ne y),$$

which is motivated by the second part of estimate (4).

Theorem 7 [15, Theorem 2] *Assume that $I \subset \mathbb{R}$ is a nonempty open interval and $f : I \to \mathbb{R}$ is a solution of* (17) *which satisfies*

$$\liminf_{h \to 0+} f(x + h) \le f(x) \quad (\textit{for all } x \in I). \tag{18}$$

Then, there exists a nonincreasing map $d : I \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for all $x \in I$.

Remark 3 The assertion of the foregoing theorem and of Theorem 1 of Alsina and Garcia Roig can be rewritten equivalently in the form of the following inequality:

$$f(y) \ge e^{y-x} f(x) \qquad (x \le y). \tag{19}$$

Moreover, assertion of Theorem 3 is equivalent to the converse inequality to (19).

Remark 4 In the proof of Theorem 7, it is observed that the inequality (17) can be rewritten equivalently in the following form:

$$f(x + 2\,h) \ge \alpha(h) f(x + h) + \beta(h) f(x), \tag{20}$$

for all $x \in I$ and $h > 0$ such that $x + 2\,h \in I$, where functions α and β are given by

$$\alpha(h) = \frac{4\,h}{3 - \,h}, \qquad \beta(h) = \frac{3 + h}{3 - \,h}.$$

(see [15, formula (15)]). Therefore, similar to functional inequalities (8) and (11), we see that, after replacing f by $-f$, (20) is a particular case of the general functional inequality (23).

One can ask about a functional inequality motivated by the first part of estimate (4). We do not know affirmative results in this direction which can be viewed as a counterpart to Theorem 7. Therefore, let us formulate the following open problem.

Problem 3 Assume that $I \subset \mathbb{R}$ is a nonempty open interval and assume that $f : I \to \mathbb{R}$ is a solution of the following functional inequality:

$$f\left(\frac{x+y}{2}\right)^2 \cdot \frac{f(x)+f(y)}{2} \leq \left[\frac{f(y)-f(x)}{y-x}\right]^3, \tag{21}$$

which is postulated to hold for all $x, y \in I$ such that $x \neq y$. Is it true that (under some regularity conditions) there exists a nondecreasing map $i : I \to \mathbb{R}$ such that f is of the form (12) for all $x \in I$?

Now, let us mention a possible application of the foregoing results in the theory of Hyers–Ulam stability of functional equations.

Remark 5 Using the abovementioned results, it is possible to obtain Hyers–Ulam stability results for some functional equations related to Riedel–Sahoo functional equations, for details see [17]. Moreover, in [17], the stability of the functional-differential equation $f = f'$ is investigated for mapping f having values in a reflexive normed linear space.

Next, we will discuss the relation of the abovementioned theorems with other known results concerning comparisons of quasi-arithmetic means. We begin with the definition of a quasi-arithmetic mean.

Definition 1 Assume that $f : I \to \mathbb{R}$ is a continuous and strictly increasing mapping. Then, the following formula defines a mean on $I \times I$:

$$M(s,t) = f^{-1}\left(\frac{f(s)+f(t)}{2}\right) \quad \text{(for all } s,t \in I).$$

Mean M of the above form is called a quasi-arithmetic mean.

For a detailed discussion of the topic of quasi-arithmetic means, the reader is referred to the monograph of Aczél and Dhombres [1, Chaps. 15 and 17]. In particular, it is known that the only two quasi-arithmetic means which are homogeneous (with respect to each variable) are the geometric mean G and the power means M_p given by

$$M_p(s,t) = (s^p + t^p)^{\frac{1}{p}} \quad \text{(for all } s,t \in \mathbb{R}^+)$$

with a real parameter p. On the other hand, the logarithmic mean L is homogeneous. Therefore, we conclude that L is not a quasi-arithmetic mean. A much deeper result in this direction is due to Ger and Kochanek [19]. They studied the following functional equation:

$$f(M(x,y)) = N(f(x), f(y)), \tag{22}$$

where M and N are abstract means, and one of them is quasi-arithmetic and one is not. They showed that every solution of the Eq. (22) is equal to a constant function

(with no regularity assumptions). Therefore, applying the result of Ger and Kochanek for $M = L$ and $L = A$, we deduce that L cannot be a quasi-arithmetic mean.

To conclude the section, let us mention some papers which are devoted to various functional inequalities and related problems which are motivated by comparisons of means. Daróczy in [11] dealt with a general inequality for means defined with the aid of deviations. Further results in this direction were obtained by Daróczy and Páles in [12, 13, 36, 42], among others. Minkowski-type and Hölder-type inequalities for means were studied by Losonczi, Páles, and Czinder in [7–10, 29–33, 35, 37–41] among others. A one more related result in this field is due to Páles [43].

4 A General Functional Inequality

Let I be a nonvoid open interval, $k \in \mathbb{N}$ and let $c \in \mathbb{R}^+ \cup \{+\infty\}$ be arbitrarily fixed and denote $U = (0, c)$. Further, assume that we are given some mappings $\alpha_0, \alpha_1, \dots, \alpha_k : U \to \mathbb{R}_0^+$ and $f : I \to \mathbb{R}$ is an unknown function. We are interested in the following functional inequality:

$$f(x + (k+1)h) \le \sum_{i=0}^{k} \alpha_i(h) f(x + ih), \tag{23}$$

which is assumed to be satisfied for all $x \in I$ and $h \in U$ such that $x + (k+1)h \in I$.

As we have already noticed in Sect. 3, previously discussed functional inequalities (6), (9), and (17) are special cases of (23) with given constants c and k and with specified mappings α_i.

Let us introduce an auxiliary double sequence of mappings $\xi_{j,n} : U \to \mathbb{R}$, where $j \in \{0, 1, \dots, k\}$ and $n \in \mathbb{N}$. For $j, n \in \{0, 1, \dots, k\}$ and for $h \in U$, we put

$$\xi_{j,n}(h) = \delta_{j,n}, \tag{24}$$

where $\delta_{j,n}$ denotes the Kronecker delta (equals to 1 if $j = n$ and equals to 0 otherwise). Further, we define

$$\xi_{j,n+k+1}(h) = \sum_{i=0}^{k} \alpha_i(h) \xi_{j,n+i}(h) \tag{25}$$

for $j \in \{0, 1, \dots, k\}$, $n \in \mathbb{N}$ and for $h \in U$.

It is clear that for each $j = 0, 1, \dots, k$, the sequence $(\xi_{j,n} : n \in \mathbb{N})$ is well defined recursively. Moreover, we can see that $\xi_{j,n}(h) \ge 0$ and $\alpha_j(h) = \xi_{j,k+1}(h)$ for each $h \in U$ and for all $j \in \{0, 1, \dots, k\}$ and all $n \in \mathbb{N}$.

Our main result concerning (23) reads as follows.

Theorem 8 *Assume that $f : I \to \mathbb{R}$ satisfies functional inequality (23) jointly with*

$$\limsup_{h \to 0+} f(x + h) \le f(x) \quad (\textit{for all } x \in I). \tag{26}$$

If for every $h \in U$, there exists a strictly increasing sequence of positive integers $(N_n : n \in \mathbb{N}^+)$ such that the following limit exists:

$$\Lambda(h) = \lim_{n\to+\infty} \sum_{i=0}^{k} \xi_{i,N_n}\left(\frac{h}{N_n}\right), \tag{27}$$

then the following inequality holds true:

$$f(x+h) \le \Lambda(h) f(x) \tag{28}$$

for all $x \in I$ and $h \in U$ such that $x + h \in I$.

Proof We will verify inductively the following auxiliary inequality:

$$f(x+(n+k)h) \le \sum_{i=0}^{k} \xi_{i,n+k}(h) f(x+ih). \tag{29}$$

We claim that (29) is valid for all $n \in \mathbb{N}^+$, $x \in I$, and $h \in U$ such that $x+(n+k)h \in I$. It is clear that for $n = 1$, inequality (29) is identical with (23). Next, assume that $n \in \mathbb{N}$ is arbitrary and the estimate (29) is valid for all positive integers not greater than n and for all $x \in I$ and $h \in U$ such that $x + (n + k)h \in I$. Fix $x \in I$ arbitrarily and $h \in U$ such that $x + (n + k + 1)h \in I$. Using inequality (23) and in the second line inequality (29), we obtain:

$$\begin{aligned}
f(x+(n+k+1)h) &= f(x+nh+(k+1)h) \\
&\le \sum_{i=0}^{k} \alpha_i(h) f(x+(n+i)h) \\
&\le \sum_{i=0}^{k} \alpha_i(h) \sum_{j=0}^{k} \xi_{j,n+i}(h) f(x+jh) \\
&= \sum_{j=0}^{k} \sum_{i=0}^{k} \alpha_i(h)\xi_{j,n+i}(h) f(x+jh) \\
&= \sum_{j=0}^{k} \xi_{j,n+k+1}(h) f(x+jh).
\end{aligned}$$

Next step is to replace h by $(n + k)^{-1}h$ in inequality (29) to derive the following estimation:

$$f(x+h) \le \sum_{i=0}^{k} \xi_{i,n+k}\left(\frac{h}{n+k}\right) f\left(x + \frac{i}{n+k}h\right). \tag{30}$$

Observe that inequality (30) is valid for every $n \in \mathbb{N}^+$ and for all $x \in I$ and $h \in U$ such that $x + h \in I$.

Now, fix for a moment $x \in I$ and $h \in U$ in such a way that $x + h \in I$. Condition (26) says that for arbitrarily fixed $\varepsilon \in \mathbb{R}^+$ we can find a sufficiently large $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $i \in \{0, 1, \dots, k\}$, we have the following estimate

$$f\left(x + \frac{i}{n+k}h\right) \leq f(x) + \varepsilon.$$

Therefore, we obtain

$$f(x+h) \leq \sum_{i=0}^{k} \xi_{i,n+k}\left(\frac{h}{n+k}\right)(f(x) + \varepsilon)$$

for $n \geq N$. From this, we deduce that

$$f(x+h) \leq \lim_{n \to +\infty} \sum_{i=0}^{k} \xi_{i,M_n+k}\left(\frac{h}{M_n+k}\right)(f(x) + \varepsilon) = \Lambda(h)(f(x) + \varepsilon),$$

where $M_n = N_n - k$ for $n \in \mathbb{N}^+$ and the sequence $(N_n : n \in \mathbb{N}^+)$ is postulated in assumption (27). This eventually leads to estimation (28).

Observe that if for a fixed $h \in U$, the numbers $\alpha_i(h)$ for $i = 0, 1, \dots, k$ are explicitly known, then it may be possible to calculate the exact formula of the limit $\Lambda(h)$. To visualize this, observe that (25) is homogeneous linear recurrence with constant coefficients (in a sense that the coefficients do not depend upon n, but they can be dependent upon h). Let us consider the characteristic equation of this recurrence:

$$w(z) = z^{k+1} - \sum_{i=0}^{k} \alpha_i(h) z^i = 0. \tag{31}$$

Note that all the roots of this characteristic equation are in fact functions of the variable $h \in U$.

It is well known that if some $\lambda \in \mathbb{C}$ is a root of the (complex) polynomial w of order $d \in \{1, 2, \dots, k+1\}$, then every following sequence:

$$(\lambda^n : n \in \mathbb{N}), \quad (n\lambda^n : n \in \mathbb{N}), \quad \dots, \quad (n^{d-1}\lambda^n : n \in \mathbb{N})$$

provides a solution of the recurrence (25) and moreover every solution of (25) is a linear combination of the foregoing sequences for all complex roots of (31) (see e.g., the book of Greene and Knuth [20]). Next, using the initial conditions (24), one is able to derive the exact formula of the sequences $(\xi_{j,n}(h) : n \in \mathbb{N})$ for $j = 0, 1, \dots, k$. The final step is to employ these formulas to calculate the limit (27).

In what follows, we will exhibit a special case of the foregoing discussion. Namely, we will consider the situation when the coefficients α_i do not depend upon $h \in U$ and we provide an easy to verify condition which implies that the limit (27) is equal to zero (and thus, due to Theorem 8, every solution of (23) is nonpositive on I).

Therefore, let us assume that we are given constants $\alpha_0, \alpha_1, \dots, \alpha_k \in \mathbb{R}_0^+$ and we want to solve the following functional inequality:

$$f(x + (k+1)h) \leq \sum_{i=0}^{k} \alpha_i f(x + ih), \tag{32}$$

with unknown mapping $f : I \to \mathbb{R}$, which is assumed to be satisfied for all $x \in I$ and $h \in U$ such that $x + (k+1)h \in I$. Note that for every $j \in \{0, 1, \dots, k\}$, the sequences $(\xi_{j,n} : n \in \mathbb{N})$ defined by (24) and (25) do not depend upon h. Therefore, the limit (27), if it exists, does not depend upon h as well.

The following notions and facts regarding the stability of polynomials can be conferred with the monograph of Gantmacher [18]. A complex polynomial is called *Hurwitz stable* if all its roots lie in the open left halfplane. Moreover, a complex polynomial is called *Schur stable* if all its roots lie in the open unit ball. The two notions are related by the fact that the Möbius transform

$$\mathbb{C} \ni z \to \frac{z+1}{z-1} \in \mathbb{C}$$

maps the left halfplane into the unit ball. Therefore, polynomial w of degree $d \in \mathbb{N}^+$ is Schur stable if and only if the polynomial p (of the same degree) given by

$$p(z) = (z-1)^d w\left(\frac{z+1}{z-1}\right) \quad \text{(for all } z \in \mathbb{C}) \tag{33}$$

is Hurwitz stable. Further, a necessary condition for a polynomial to be Hurwitz stable is that all its coefficients are of the same sign. Moreover, a sufficient condition for this fact is that the coefficients are positive and they form a strictly increasing sequence. A more elaborated result is the Routh–Hurwitz stability criterion, which provides a necessary and sufficient condition for the Hurwitz stability. Assume that for some $n \in \mathbb{N}^+$ we are given a polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

with $a_n \neq 0$ and $a_0 > 0$. Moreover, agree that $a_m = 0$ whenever $m > n$. The Routh–Hurwitz criterion says that p is Hurwitz stable if and only if every principal minor of the following $n \times n$ matrix:

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \dots & a_n \end{pmatrix} \tag{34}$$

is positive.

Our last observation that if the characteristic polynomial (31) of the recurrence (25) is Schur stable, then regardless of the initial conditions for every $j \in \{0, 1, \dots, k\}$, the sequence $(\xi_{j,n} : n \in \mathbb{N})$ is a linear combination of the sequences of the form

$$(n^{d_j} t_j{}^n : n \in \mathbb{N})$$

with $|t_j| < 1$ and with some $d_j \in \mathbb{N}^+$. Consequently,

$$\lim_{n \to +\infty} \xi_{j,n} = 0$$

for every $j = 0, 1, \dots k$. This easily implies that the limit in (27) exists and is equal to 0. Therefore, we have proved the following corollary from Theorem 8, which is a criterion for the nonpositivity of all solutions of the functional inequality (32).

Corollary 1 *Assume that* $\alpha_0, \alpha_1, \dots, \alpha_k \in \mathbb{R}_0^+$ *and* $f : I \to \mathbb{R}$ *satisfies functional inequality* (32) *jointly with* (26). *If the polynomial* $w : \mathbb{C} \to \mathbb{C}$ *given by*

$$w(z) = z^{k+1} - \sum_{i=0}^{k} \alpha_i z^i \quad (\textit{for all } z \in \mathbb{C})$$

is Schur stable, then f *is nonpositive on* I.

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Constructions and Extensions of Free and Controlled Additive Relations

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Abstract By using several auxiliary results on relations and their intersection convolutions, we give some necessary and sufficient conditions in order that a certain additive partial selection relation Φ of a relation F of one group X to another Y could be extended to a total, additive selection relation Ψ of the relation $F + \Phi(0)$.

The results obtained extend some Hahn–Banach type extension theorems of B. Rodríguez-Salinas, L. Bou, Z. Gajda, A. Smajdor, W. Smajdor, and the second author. Moreover, they can be used to prove some alternate forms of the Hyers–Ulam type selection theorems of Z. Gajda, R. Ger, R. Badora, Zs. Páles, and the second author.

Keywords Additive and homogeneous relations · Intersection convolutions of relations · Extensions of additive partial selection relations

1 Introduction

The origin of the following generalization of the classical Hahn–Banach extension theorem goes back to Kaufman [28]. It is a particular case of [13, Corollary 1.3] by Fuchssteiner. (For some more readable treatments, see also Fuchssteiner and Lusky [15, Theorem 1.3.2] and Száz [64, Theorem 3.3].)

Theorem 1 *If p is a subadditive function of a commutative semigroup X to $\mathbb{R}$ and φ is an additive function of a subsemigroup V of X to $\mathbb{R}$ such that:*

(1) $\varphi(v) \leq p(v)$ *for all* $v \in V$,
(2) $\varphi(u+v) \leq p(u) + \varphi(v)$ *for all* $u \in X$ *and* $v \in V$ *with* $u+v \in V$,

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then φ can be extended to an additive function ψ of X to $\mathbb{R}$ such that $\psi(x) \le p(x)$ for all $x \in X$.

Remark 1 To see that condition (2) is also necessary, note that if ψ is as above, then for any $u \in X$ and $v \in V$ with $u + v \in V$ we have $\varphi(u+v) = \psi(u+v) = \psi(u) + \psi(v) \le p(u) + \varphi(v)$.

Moreover, it is also worth noticing that, by using the infimal convolution

$$(f * g)(x) = \inf\left\{ f(u) + g(v) : \quad x = u + v, \;\; u \in D_f, \; v \in D_g \right\}$$

of functions f and g studied mainly by Moreau [34], Strömberg [51], and the present authors [21, 64], condition (2) can be briefly expressed by writing that $\varphi(x) \le (p * \varphi)(x)$ for all $x \in V$.

In [20], to have a close analogue of Theorem 1, we have proved the following simple generalization of the classical Hyers–Ulam stability theorem [25]. (For a predecessor and some direct generalizations, see Pólya and Szegő [42, Aufgabe 99], Rätz [44], Székelyhidi [73], Forti [12], Hyers et al. [26], and Száz [59].)

Theorem 2 *If f is an ε-approximately additive function of a commutative semigroup X to a Banach space Y, for some $\varepsilon \ge 0$, in the sense that*

$$\| f(x+y) - f(x) - f(y) \| \le \varepsilon$$

for all $x, y \in X$, and φ is a 2-homogeneous function of a subsemigroup V of X to Y which is δ-near to f, for some $\delta \ge 0$, in the sense that

$$\| f(v) - \varphi(v) \| \le \delta$$

for all $v \in V$, then φ can be extended to an additive function ψ of X to Y that is ε-near to f.

Remark 2 To see that this theorem is somewhat more general than that of Hyers and Ulam, note that if in particular X has a zero element 0, then $\| f(0) \| \le \varepsilon$. Thus, $\varphi = \{(0,0)\}$ is an additive function of the subgroup $\{0\}$ of X to Y such that φ is ε-near to f. Therefore, by the above theorem, there exists an additive function ψ of X to Y which is ε-near to f.

The extensive references of a recent semisurvey paper [70] of the second author show that the Hahn–Banach and the Hyers–Ulam theorems have been generalized by a great number of authors in an enormous variety of directions. However, among these generalizations, we are only interested here in the set-valued ones.

For this, we can note that if p and φ are as in Theorem 1, then by defining a relation F of X to $\mathbb{R}$ such that

$$F(x) = \,]-\infty, \; p(x)]$$

for all $x \in X$, we have $\varphi(v) \in F(v)$ for all $v \in V$.

While, if f and φ are as in Theorem 2, then by defining a relation F of X to Y such that

$$F(x) = f(x) + B_\delta(0), \qquad \text{with} \qquad B_\delta(0) = \{ y \in Y : \;\; \|y\| \le \delta \},$$

for all $x \in X$, we again have $\varphi(v) \in F(v)$ for all $v \in V$.

Therefore, the essence of Theorems 1 and 2 is nothing else but the statement that an additive partial selection function φ of a certain relation F of X to $\mathbb{R}$ and Y, respectively, can be extended to a total, additive selection function of ψ of F.

The corresponding fact in connection with the classical Hahn–Banach extension theorem was already recognized by Rodríguez-Salinas and Bou [46]. (For some further developments, see Ioffe [27], Gajda et al. [18], Smajdor and Szczawińska [50], and Száz [53].)

Moreover, Smajdor [49] and Gajda and Ger [17] observed that the essence of the classical Hyers–Ulam stability theorem is the existence of an additive selection function of a certain relation. (For some further developments, see Gajda [16], Badora [2], Popa [43], Badora et al. [4], Nikodem and Popa [38], Piao [41], Lu and Park [32], and Száz [57, 61].)

The importance of the above set-valued considerations was soon recognized by Fuchssteiner and Horváth [14], Rassias [45], and Czerwik [8]. Moreover, the second author has been motivated to continue his early investigations on additive and linear relations. (See [72] and [53, 57, 61].) In [53], by introducing a particular case the intersection convolution

$$(F * G)(x) = \bigcap \{F(u) + G(v) : \quad x = u + v, \;\; u \in D_F, \;\; v \in D_G\}$$

of relations F and G, the second author has proved the following generalization of [46, Theorem 1] of Rodríguez-Salinas and Bou.

Theorem 3 *If F is a sublinear relation of one vector space X to another Y over K such that $F(x) \in \mathcal{B}$ for all $x \in X$, for some translation-invariant Nachbin system $\mathcal{B}$ of subsets of Y, and Φ is a superlinear relation of a subspace V of X to Y such that $\Phi \subset F$, then Φ can be extended to a linear relation Ψ of X to Y such that $\Psi \subset F + \Phi(0)$.*

Remark 3 Here the sublinearity of F means only that $F(\lambda x) \subset \lambda F(x)$ and $F(x + y) \subset F(x) + F(y)$ for all $\lambda \in K_0$ and $x, y \in X$, where $K_0 = K \setminus \{0\}$. This is a natural weakening of the linearity studied by Cross [7] and his predecessors.

Moreover, a family $\mathcal{B}$ of sets is called here a Nachbin system if for every subfamily $\mathcal{C}$ of $\mathcal{B}$, having the binary intersection property in the sense that $U \cap V \neq \emptyset$ for all $U, V \in \mathcal{C}$, we also have $\bigcap \mathcal{C} \neq \emptyset$.

The primary example for such a Nachbin system is the family of all closed, bounded intervals in $\mathbb{R}$, or more generally the family of all closed balls in the supremum-normed space of all bounded functions of a nonvoid set U to $\mathbb{R}$.

Now, by improving the arguments of [53], we shall prove the following generalization of [18, Theorem 1] of Gajda et al.

Theorem 4 *If F is a subodd, $\mathbb{N}$-subhomogeneous, subadditive relation of a commutative group X to a vector space Y over $\mathbb{Q}$ such that $F(x) \in \mathcal{B}$ for all $x \in X$, for some admissible Nachbin system $\mathcal{B}$ of subsets of Y, and Φ is a superodd, $\mathbb{N}$-subhomogeneous, superadditive relation of a subgroup V of X to Y such that $\Phi \subset F$, then Φ can be extended to a $\mathbb{Z}_0$-homogeneous, additive relation Ψ of X to Y such that $\Psi \subset F + \Phi(0)$.*

Remark 4 Here, in accordance with Remark 3, the superoddness, $\mathbb{N}$-subhomogeneity, and superadditivity of Φ mean only that $-\Phi(v) \subset \Phi(-v)$, $\Phi(nv) \subset n\Phi(v)$, and $\Phi(u) + \Phi(v) \subset \Phi(u+v)$ for all $n \in \mathbb{N}$ and $u, v \in V$, respectively.

Moreover, the Nachbin system $\mathcal{B}$ is called admissible if in addition to its translation invariance, we also have $n^{-1}B \in \mathcal{B}$ for all $n \in \mathbb{N}$ and $B \in \mathcal{B}$. That is, in addition to that $y + \mathcal{B} = \mathcal{B}$ for all $y \in Y$, we also have $n^{-1}\mathcal{B} \subset \mathcal{B}$ for all $n \in \mathbb{N}$, or equivalently $\mathcal{B} \subset n\mathcal{B}$ for all $n \in \mathbb{N}$.

To simplify Theorem 4, one may assume that $\mathcal{B}$ is effective in the sense that every $\mathcal{B}$-valued, odd subadditive relation Ω of a group U to Y is $\mathbb{N}$-subhomogeneous. However, our only example for such $\mathcal{B}$ is the family of all subsets B of Y which are $\mathbb{N}^{-1}$-convex in the sense that $n^{-1}B + \left(1 - n^{-1}\right) B \subset B$ for all $n \in \mathbb{N}$.

Unfortunately, by using the convolutional method of the second author, we have not been able to extend Theorem 3 to commutative semigroups. However, the several auxiliary results leading to Theorem 4 are much more general than those used for the proof Theorem 3. They are mostly formulated in terms of semigroups.

In the next preparatory sections, to keep the paper self-contained, we shall list several basic facts on semigroups, relations, and intersection convolutions which are certainly unfamiliar to the reader. These only slightly improve some earlier observations [70, 63] of the second author. Therefore, the proofs are usually omitted.

2 A Few Basic Facts on Relations and Groupoids

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, with $X^2 = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the identity relation of X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$, the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, the inverse relation F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$.

Moreover, if in addition, G is a relation on Y to Z, then the composition relation $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

Now, a relation F on X may, for instance, be naturally called reflexive, transitive, symmetric, and antisymmetric if $\Delta_X \subset F$, $F \circ F \subset F$, $F^{-1} = F$, and $F \cap F^{-1} \subset \Delta_X$, respectively.

As is customary, a transitive (symmetric) reflexive relation is called a preorder (tolerance) relation. Moreover, a symmetric (antisymmetric) preorder relation is called an equivalence (partial order) relation.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

If F is a relation on X to Y and $A_i \subset X$ for all $i \in I$, then in general we only have $F\left[\bigcup_{i\in I} A_i\right] = \bigcup_{i\in I} F[A_i]$. However, if in particular f is a function, then all set-theoretic operations are preserved under the relation f^{-1}.

If F is a relation on X to Y, then a subset Φ of F is called a partial selection relation of F. Thus, we also have $D_\Phi \subset D_F$. Therefore, a partial selection relation Φ of F may be called total if $D_\Phi = D_F$.

The total selection relations of a relation F will usually be simply called the selection relations of F. Thus, the axiom of choice can be briefly expressed by saying that every relation F has a selection function.

If F is a relation on X to Y and $U \subset D_F$, then the relation $F|U = F \cap (U \times Y)$ is called the restriction of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subset D_G$ and $F = G|D_F$, then G is called an extension of F.

In particular, a function $\star$ of a set X to itself is called an unary operation in X. While, a function $*$ of X^2 to X is called a binary operation in X. And, for any $x, y \in X$, we write $x^\star$ and $x * y$ instead of $\star(x)$ and $*((x, y))$, respectively.

An ordered pair $X(+) = (X, +)$, consisting of a set X and a binary operation $+$ in X, is called a groupoid. Instead of groupoids, it is usually sufficient to consider only semigroups (associative groupoids), or even monoids (semigroups with zero).

However, several definitions on semigroups can be naturally extended to groupoids. For instance, if X is a groupoid, then for any $n \in \mathbb{N}$ and $x \in X$ we may naturally define $nx = x$ if $n = 1$ and $nx = (n-1)x + x$ if $n \neq 1$. Thus, by induction, we can easily prove the following.

Theorem 5 *If X is a semigroup, then for any $x \in X$ and $n, m \in \mathbb{N}$ we have*
(1) $(n+m)x = nx + mx$, (2) $(nm)x = n(mx)$.
Moreover, if in addition $y \in X$ such that $x + y = y + x$, then we also have
(3) $nx + my = my + nx$, (4) $n(x + y) = nx + ny$.

Hint Note that (2) is a consequence of (1). Moreover, (3) and (4) are consequences of the $m = 1$ particular case of (3).

If in particular X is a groupoid with zero, then for any $x \in X$, we may also naturally define $0x = 0$. Moreover, if more specially X is a group, then for any $n \in \mathbb{N}$ and $x \in X$, we may also naturally define $(-n)x = n(-x)$.

Now, by using $-x + x = 0 = x + (-x)$ and Theorem 5, we can at once see that $n(-x) + nx = n(-x + x) = n0 = 0$, and thus $(-n)x = n(-x) = -(nx)$. Moreover, we can also easily prove the following.

Theorem 6 *If X is a group, then for any $x \in X$ and $k, l \in \mathbb{Z}$ we have*
(1) $(k+l)x = kx + lx$, (2) $(kl)x = k(lx)$.
Moreover, if in addition $y \in X$ such that $x + y = y + x$, then we also have
(3) $kx + ly = ly + kx$, (4) $k(x + y) = kx + ky$.

Remark 5 Thus, in particular, a commutative group X is already a module over the ring $\mathbb{Z}$ of all integers. Therefore, instead of commutative groups, we should rather work with modules in the sequel.

If X is a groupoid, then for any $n \in \mathbb{N}$ and $U, V \subset X$, we may also naturally define $nU = \{nu : u \in U\}$ and $U + V = \{u + v : u \in U, v \in V\}$. Thus, for instance, $2U$ can be easily confused with the possibly larger set $U + U$.

If in particular, X has a zero, or more specially X is a group, then we may also quite similarly define $0U$ and kU for all $k \in \mathbb{Z}$, respectively. Moreover, if X is a group, then we may also naturally write $-U = (-1)U$ and $U - V = U + (-V)$.

Thus, by using Theorem 6, we can easily establish several useful properties of the corresponding operations in the family $\mathcal{P}(X)$ of all subsets of a group X. However, in general, $\mathcal{P}(X)$ is only a monoid and $(k + l)U \subset kU + lU$.

A subset U of a groupoid X is called left-translation invariant if $x + U = U$ for all $x \in X$. Note that if in particular X is a group and either $x + U \subset U$ for all $x \in X$ or $U \subset x + U$ for all $x \in X$, then U is already left-translation invariant.

Moreover, a subset U of a groupoid X is called normal if $x + U = U + x$ for all $x \in X$. Note that if in particular X is a group and either $x + U \subset U + x$ for all $x \in X$ or $U + x \subset x + U$ for all $x \in X$, then U is already normal.

Furthermore, a subset U of groupoid X is called subadditive (superadditive) if $U \subset U + U$ $(U + U \subset U)$. Thus, U is a subgruopoid of X if and only if it is a superadditive subset of X.

Moreover, a subset U of a groupoid X is called n-subhomogeneous (n-superhomogeneous), for some $n \in \mathbb{N}$, if $U \subset nU$ $(nU \subset U)$. And U is called A-subhomogeneous, for some $A \subset \mathbb{N}$, if it is n-subhomogeneous for all $n \in A$.

In particular, a subset U of a group X is called symmetric if $-U = U$. Note that if either $-U \subset U$ or $U \subset -U$, then U is already symmetric. Moreover, U is a subgroup of X if and only if it is a nonvoid, symmetric, superadditive subset of X.

In the sequel, for a subset U of a groupoid X with zero, we shall briefly write $U_0 = U \setminus \{0\}$ if $0 \in U$ and $U_0 = U \cup \{0\}$ if $0 \notin U$. Moreover, as is customary, we shall use the common notation $\mathbb{K}$ for the number fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

3 Divisible and Cancellable Subsets of Groupoids

Definition 1 For some $n \in \mathbb{N}$, a subset U of gruopoid X is called

(1) n-divisible if U is n-subhomogeneous,
(2) n-cancellable if $nu = nv$ implies $u = v$ for all $u, v \in U$.

Now, U may be naturally called A-divisible (A-cancellable), for some $A \subset \mathbb{N}$, if it is n-divisible (n-cancellable) for all $n \in A$.

Remark 6 Note that if (1) holds, then $U \subset nU$. Therefore, for each $u \in U$, there exists $v \in U$ such that $u = nv$.

While, if both (1) and (2) hold, then U is uniquely n-divisible in the sense that for each $u \in U$, there exists a unique $v \in U$ such that $u = nv$. Therefore, $n^{-1}u$ can be defined by this v.

By using Theorem 6 and an obvious analogue of Definition 1, we can easily prove the following two theorems.

Theorem 7 *If U is a k-cancellable subset of a group X, for some $k \in \mathbb{Z}$, then U is also $-k$-cancellable. Therefore, if U is $\mathbb{N}$-cancellable, then U is also $\mathbb{Z}_0$-cancellable.*

Theorem 8 *If U is a k-divisible, symmetric subset of a group X, for some $k \in \mathbb{Z}$, then U is also $-k$-divisible. Therefore, if U is $\mathbb{N}$-divisible, then U is also $\mathbb{Z}_0$-divisible.*

Proof If $x \in U$, then by the k-divisibility of U, there exists $y \in U$ such that $x = ky$. Hence, by Theorem 6 and the corresponding definitions, we can see that

$$x = ky = ((-k)(-1))y = (-k)((-1)y) = (-k)(1(-y)) = (-k)(-y).$$

Therefore, since now $-y \in -U = U$ also holds, the required assertion is also true.

Remark 7 If U is an n-cancellable subset of a groupoid X with zero, for some $n \in \mathbb{N}$, such that $0 \in U$, and $u \in U$ such that $nu = 0$, then we also have $nu = n0$, and hence $u = 0$.

In this respect, it is also worth noticing that the following theorem is also true.

Theorem 9 *If X is a commutative group, then for each $k \in \mathbb{Z}$, the following assertions are equivalent:*
(1) *X is k-cancellable;* (2) *$kx = 0$ implies $x = 0$ for all $x \in X$.*
Therefore, if $nx = 0$ implies $x = 0$ for all $n \in \mathbb{N}$ and $x \in X$, then X is already $\mathbb{Z}_0$-cancellable.

Moreover, in addition to this theorem, we can also easily prove the following.

Theorem 10 *If X is an $\mathbb{N}$-cancellable group, then $kx = lx$ implies $k = l$ for all $k, l \in \mathbb{Z}$ and $x \in X_0$. Thus, $kx = 0$ implies $k = 0$ for all $k \in \mathbb{Z}$ and $x \in X_0$.*

Remark 8 It can be shown that if X is a uniquely $\mathbb{N}$-divisible, commutative group, then by defining $(m/n)x = m(n^{-1}x)$ for all $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and $x \in X$ the module X can be turned into a vector space over $\mathbb{Q}$.

Remark 9 Note that if in particular X is a vector space over K, then every subset U is K_0-cancellable.

Moreover, a subset U of X is λ-divisible (uniquely λ-divisible), for some $\lambda \in K_0$, if and only if $\lambda^{-1}U \subset U$.

Remark 10 If X is only a groupoid, then having in mind the case of vector spaces, we may also naturally define $n^{-1}x = \{y \in X : x = ny\}$ and $n^{-1}U = \bigcup_{u \in U} n^{-1}u$ for all $n \in \mathbb{N}$, $x \in X$ and $U \subset X$.

Thus, we can easily prove several remarkable characterizations of divisible and cancellable sets. However, in this more general setting, several useful rules of computation with sets in vector spaces are no longer true. For instance, in general we only have $n(n^{-1}U) \subset U \subset n^{-1}(nU)$ for all $n \in \mathbb{N}$ and $U \subset X$.

4 The Most Important Additivity and Homogeneity Properties of Relations

Definition 2 Let F be a relation on one groupoid X to another Y, and let Ω be a relation on X. Then, F is called

(1) Ω-subadditive if $F(x+y) \subset F(x)+F(y)$ for all $(x,y) \in \Omega$,
(2) Ω-superadditive if $F(x)+F(y) \subset F(x+y)$ for all $(x,y) \in \Omega$.

Remark 11 Now, the relation F may, for instance, be naturally called superadditive if it is X^2-superadditive.

Note that thus F is superadditive if and only if $F+F \subset F$. That is, F is a subgroupoid of the groupoid $X \times Y$.

Remark 12 Moreover, it is also worth mentioning that if in particular F is a reflexive, superadditive relation of X to itself, then F is already a translation relation [54, 55] in the sense that $x+F(y) \subset F(x+y)$ for all $x, y \in X$.

Remark 13 Note also that if F is only D_F^2-superadditive, then F is already superadditive.

However, the corresponding assertion is not true even for $D_F \times X$-subadditivity. Therefore, we shall need the following weakenings of global subadditivity.

Definition 3 A relation F on one groupoid X to another Y is called

(1) semisubadditive if it is D_F^2-subadditive;
(2) left-quasisubadditive if it is $D_F \times X$-subadditive.

Remark 14 Now, the relation F may be naturally called quasisubadditive if it both left-quasisubadditive and right-quasisubadditive. (The latter is defined by the relation $X \times D_F$.)

Moreover, F may be naturally called quasiadditive if it is both quasisubadditive and superadditive. Later, we shall see that quasiadditivity is a quite important additivity property.

In the sequel, by considering some more special ground sets, we shall need some further weakenings of global additivities.

Definition 4 A relation F on a groupoid X with zero to an arbitrary groupoid Y is called

(1) left-zero-subadditive if it is $\{0\} \times X$-subadditive;
(2) left-zero-superadditive if it is $\{0\} \times X$-superadditive.

Remark 15 Note that if F is only $\{0\}\times D_F$-subadditive ($\{0\}\times D_F$-superadditive), then F is already left-zero-subadditive (left-zero-superadditive). Therefore, in contrast to Definition 3, even zero-semisubadditivity need not be defined. Concerning zero-additivities, we can easily establish the following.

Theorem 11 *A relation F on one groupoid X with zero to another Y, then*
(1) *F is zero-subadditive if $0 \in F(0)$,*
(2) *F is zero-superadditive if $F(0) \subset \{0\}$.*

Remark 16 To feel the importance of zero-additivity, note that if for instance F is right-zero-additive, then $F(x) = F(x) + F(0)$ for all $x \in X$. Therefore, F is a left-representing selection relation for F.

Analogously to Definition 4, we may also naturally introduce the following.

Definition 5 A relation F on a group X to a groupoid Y is called

(1) inversion-subadditive if it is $\{(x, -x) : \ x \in X\}$-subadditive,
(2) inversion-superadditive if it is $\{(x, -x) : \ x \in X\}$-superadditive.

Remark 17 Note that if F is only $\{(x, -x) : \ x \in D_F\}$-superadditive, then F is already inversion-superadditive.

However, the corresponding assertion is not true for inversion subadditivity. Therefore, analogously to Definition 3, F may be naturally called inversion-semi-subadditive if it is $\{(x, -x) : \ x \in D_F\}$-subadditive.

Remark 18 Note that if F is inversion-semi-subadditive, then for any $x \in D_F$, we have $F(0) \subset F(x) + F(-x)$. Hence, if $0 \in D_F$, i.e., $F(0) \neq \emptyset$, we can infer that $F(-x) \neq \emptyset$, i.e., $-x \in D_F$. Therefore, we also have $F(0) \subset F(-x) + F(x)$.

Definition 6 For some $n \in \mathbb{N}$, a relation F on one groupoid X to another Y is called

(1) n-subhomogeneous if $F(nx) \subset nF(x)$ for all $x \in X$,
(2) n-superhomogeneous if $nF(x) \subset F(nx)$ for all $x \in X$.

Remark 19 Note that if we only have $nF(x) \subset F(nx)$ for all $x \in D_F$, then F is already n-superhomogeneous.

However, the corresponding assertion is not true for n-subhomogeneity. Therefore, in accordance with Definition 3, F may be naturally called n-semi-subhomogeneous if $F(nx) \subset nF(x)$ for all $x \in D_F$.

Remark 20 Now, F may, for instance, be naturally called n-semihomogeneous if it is both n-semi-subhomogeneous and n-superhomogeneous.

Moreover, for some $A \subset \mathbb{N}$, the relation F may, for instance, be naturally called A-semihomogeneous if it is n-semihomogeneous for all $n \in A$.

By induction, we can easily prove the following.

Theorem 12 *If F is a superadditive relation on one groupoid X to another Y, then D_F is a subgroupoid of X and F is $\mathbb{N}$-superhomogeneous.*

Remark 21 Note that if F is a relation on one groupoid X with zero to another Y such that $0 \in F(0)$, then we have $0F(x) \subset \{0\} \subset F(0) = F(0x)$ for all $x \in X$. Therefore, F is 0-superhomogeneous.

Now, we can also easily prove the following.

Theorem 13 *If f is a superadditive function on one groupoid X to another Y, then D_f is a subgroupoid of X and f is semiadditive and $\mathbb{N}$-semihomogeneous.*

Proof If $n \in \mathbb{N}$, then by Theorem 12 we have $nf(x) \subset f(nx)$ for all $x \in X$. Hence, since $f(x)$ is a singleton for all $x \in D_f$, we can infer that $nf(x) = f(nx)$ for all $x \in D_f$. Therefore, f is n-semihomogeneous.

On the other hand, by the superadditivity of f, we have $f(x) + f(y) \subset f(x+y)$ for all $x, y \in X$. Hence, since $f(x)$ is a singleton for all $x \in D_f$, we can infer that $f(x) + f(y) = f(x+y)$ for all $x, y \in D_f$. Therefore, f is semiadditive.

Remark 22 Note that if f is, for instance, a right-zero-superadditive function on a groupoid X with zero to a groupoid Y such that $0 \in D_f$, then f is actually right-zero-additive.

Moreover, we have $f(0) + f(0) = f(0)$, and thus $f(0) = 0$ if in particular Y is a group. Therefore, we also have $f(0x) = 0f(x)$ for all $x \in D_f$, and thus f is zero-semihomogeneous.

Now, in addition to Theorems 12, we can also easily prove the following

Theorem 14 *If F is a $\mathbb{N}^{-1}$-convex-valued, subadditive (right-quasisubadditive) relation on a groupoid X to a vector space Y over $\mathbb{K}$, then F is $\mathbb{N}$-subhomogeneous ($\mathbb{N}$-semi-subhomogeneous).*

Proof Note that if $n \in \mathbb{N}$ such that $F(nx) \subset nF(x)$ for all $x \in D_F$, then by the right-quasisubadditivity of F and the n^{-1}-convexity of $F(x)$ we also have

$$\begin{aligned} F((n+1)x) &= F(nx + x) \subset F(nx) + F(x) = F(x) + nF(x) \\ &= (n+1)\big((n+1)^{-1}F(x) + (1 - (n+1)^{-1})F(x)\big) \subset (n+1)F(x) \end{aligned}$$

for all $x \in D_F$. Therefore, in the right-quasisubadditive case, F is $\mathbb{N}$-semi-subhomogeneous.

Remark 23 Note that if F is a relation on one groupoid X with zero to another Y such that either $0 \notin D_F$, or $D_F = X$ and $F(0) \subset \{0\}$, then $F(0x) = F(0) \subset 0F(x)$ for all $x \in X$. Therefore, F is zero-subhomogeneous. While, if only $F(0) \subset \{0\}$ is assumed, then we can only state that F is zero-semi-subhomogeneous.

Moreover, in addition to Theorem 13, we can also easily establish the following.

Theorem 15 *If f is a subadditive (right-quasisubadditive) function of one groupoid X to another Y, then f is $\mathbb{N}$-subhomogeneous ($\mathbb{N}$-semi-subhomogeneous).*

Remark 24 Note that if f is, for instance, a right-zero-subadditive function on a groupoid X with zero to a groupoid Y such that $0 \in D_f$, then f is actually right-zero-additive. Therefore, if in particular Y is a group, then by Remark 22, we can see that f is zero-semihomogeneous.

5 Some Further Important Homogeneity Properties of Relations

Definition 7 A relation F on a group X to a set Y is called even if $F(-x) = F(x)$ for all $x \in X$.

While, a relation F on one group X to another Y is called odd if $F(-x) = -F(x)$ for all $x \in X$.

Remark 25 Note that if the above equalities are required to hold only for all $x \in D_F$, then D_F is already symmetric, and thus they also hold for all $x \in X$. Therefore, semieven and semiodd relations need not be introduced.

However, by using the notations of [70], or rather [68], the above definition and the following obvious theorem can be more briefly formulated.

Theorem 16 *If F is a relation on on group X to a set (group) Y, then the following assertions are equivalent:*

(1) F is even (odd),
(2) $F(-x) \subset F(x)$ $(F(-x) \subset -F(x))$ for all $x \in X$,
(3) $F(x) \subset F(-x)$ $(-F(x) \subset F(-x))$ for all $x \in D_F$.

Remark 26 Note that in assertion (3) we can write X in place of D_F, but in assertion (2) we cannot write D_F in place of X.

Therefore, the relation F may be naturally called semi-subeven (semi-subodd) if $F(-x) \subset F(x)$ $(F(-x) \subset -F(x))$ for all $x \in D_F$.

Remark 27 In addition to Theorem 16, it is also worth noticing that the relation F is odd if and only if $-F \subset F$, and thus $-F = F$. That is, F is a symmetric subset of the group $X \times Y$.

Hence, by using that $-(F^{-1}) = (-F)^{-1}$, we can at once see that F^{-1} is odd if and only if F is odd. However, the corresponding assertion is not true for even relations. Namely, we have the following

Theorem 17 *If F is a relation on a group X to a groupoid Y, then the following assertions are equivalent:*
(1) *F^{-1} is even;*
(2) *F is symmetric-valued.*

Proof If $x \in X$ and $y \in -F(x)$, then $-y \in F(x)$, and thus $x \in F^{-1}(-y)$. Hence, if (1) holds, we can infer that $x \in F^{-1}(y)$, and thus $y \in F(x)$. Therefore, $-F(x) \subset F(x)$, and thus $-F(x) = F(x)$. Therefore, (2) also holds.

Corollary 1 *An even relation F on one group X to another Y is odd if and only if its inverse F^{-1} is even.*

Remark 28 Note that if a function f on one group X to another Y is both even and odd, then we have $f(x) = -f(x)$, and hence $2f(x) = 0$ for all $x \in D_f$. Therefore, if in particular Y is 2-cancellable, then $f(x) = 0$ for all $x \in D_f$.

The next obvious theorems, together with the above remark, will show that odd relations are much more important than the even ones.

Theorem 18 *If f is an inversion-semi-subadditive function on one group X to another Y such that $0 \in D_f$, then D_f is symmetric, $f(0) = 0$, and f is odd and inversion-semiadditive.*

Corollary 2 *If f is a nonvoid, inversion-superadditive function on one group X to another Y with a symmetric domain, then $f(0) = 0$ and f is odd and inversion-semiadditive.*

Remark 29 Note that if F is an inversion-semi-subadditive relation on a group to a groupoid Y such that $0 \in D_F$, then D_F is symmetric. Moreover, if in particular F is inversion-subadditive, then $D_F = X$. Thus, F is total.

By using some obvious analogues of Definition 6 and Remark 19, we can also easily prove the following:

Theorem 19 *If F is an odd, k-superhomogeneous (k-subhomogeneous, resp. k-semi-subhomogeneous) relation on one group X to another Y, for some $k \in \mathbb{Z}$, then F is also $-k$-superhomogeneous ($-k$-subhomogeneous, resp. $-k$-semi-subhomogeneous).*

Corollary 3 *If F is an odd, $\mathbb{N}$-superhomogeneous ($\mathbb{N}$-subhomogeneous, resp. $\mathbb{N}$-semi-subhomogeneous) relation on one group X to another Y, then F is $\mathbb{Z}_0$-superhomogeneous ($\mathbb{Z}_0$-subhomogeneous, resp. $\mathbb{Z}_0$-semi-subhomogeneous).*

Now, in addition to Theorem 12 and Corollary 3, we can also easily prove

Theorem 20 *If F is a nonvoid odd, superadditive relation on one group X to another Y, then D_F is a subgroup of X, $0 \in F(0)$, and F is quasiadditive and $\mathbb{Z}$-superhomogeneous.*

Proof Because of $F \neq \emptyset$, we have $D_F \neq \emptyset$. Moreover, since F is odd and superadditive, $-D_F \subset D_F$ and $D_F + D_F \subset D_F$. Therefore, D_F is a subgroup of X. Now, by taking $x \in D_F$, we can see that $0 \in F(x) - F(x) = F(x) + F(-x) \subset F(0)$.

Moreover, if $x \in X$ and $y \in D_F$, then by using that $0 \in F(-y) + F(y)$ we can see that $F(x+y) = F(x+y) + \{0\} \subset F(x+y) + F(-y) + F(y) \subset F(x) + F(y)$. Therefore, F is right-quasisubadditive. The left-quasisubadditivity of F can be proved even more easily.

Moreover, as an immediate consequence of this theorem and Corollary 2, we can also state

Theorem 21 *If f is a nonvoid, superadditive function on one group X to another Y, with a symmetric domain, then D_f is a subgroup of X and $f(0) = 0$, and f is odd, quasiadditive and $\mathbb{Z}$-semihomogeneous.*

On the other hand, as an immediate consequence of Theorem 14 and Corollary 3, we can also state

Theorem 22 *If F is an odd, $\mathbb{N}^{-1}$-convex-valued, subadditive (left or right-quasisubadditive) relation on a group X to a vector space Y over $\mathbb{K}$, then F is $\mathbb{Z}_0$-subhomogeneous ($\mathbb{Z}_0$-semi-subhomogeneous).*

Moreover, as an immediate consequence of Theorems 15 and 18 and Corollary 3, we can also state

Theorem 23 *If f is a subadditive (left or right-quasisubadditive) function on one group X to another Y such that $0 \in D_f$, then f is $\mathbb{Z}_0$-subhomogeneous ($\mathbb{Z}$-semi-subhomogeneous).*

Remark 30 From our former results on additive and homogeneous relations on one groupoid X to another Y, one can easily derive some results on a subset U of X by using the relation $F = U^2 \cup G$, with $G = \emptyset$, $G = \Delta_{U^c}$, $G = (U^c)^2$, $G = U^c \times U$, or $G = U^c \times X$, for instance.

In the theory of generalized uniform spaces, it is quite usual to associate the Davis–Pervin relation $F_U = U^2 \cup U^c \times X$ with the set $U \subset X$, and more generally the Császár–Hunsaker–Lindgren relation $F_{(U,V)} = U \times V \cup U^c \times Y$ with the sets $U \subset X$ and $V \subset Y$. (See [52, 58].)

The latter relations seem to be the most natural totalizations of the box relations $\Gamma_{(U,V)} = U \times V$ studied by the second author in [65–67]. In a later paper [69], the same natural totalization has also been applied to a particular subadditive relation of Zs. Páles published first in Gajda and Ger [17].

6 The Importance of Quasi-odd Relations and Odd-like Selections

Definition 8 A relation F on a group X to a groupoid Y with zero is called quasi-odd if $0 \in F(x) + F(-x)$ for all $x \in D_F$

Remark 31 Thus, an odd relation is, in particular, quasi-odd. Moreover, each semireflexive relation on X, with a symmetric domain, is quasi-odd.

Furthermore, we can also note that if $0 \in F(0)$ and F is inversion-semi-subadditive, then F is quasi-odd. Thus, quasi-oddness a rather weak property.

Now, analogously to Theorem 20, we can also easily prove the following

Theorem 24 *If F is a nonvoid, quasi-odd, superadditive relation on a group X to a monoid Y, then D_F is a subgroup of X, $0 \in F(0)$, and F is quasiadditive and $\mathbb{N}$-superhomogeneous.*

Moreover, as some useful reformulations of a particular case of Definition 8, we can also easily establish the following theorem which can again be more briefly formulated by using the notations of [70], or rather [68].

Theorem 25 *A relation F on one group X to another Y, then the following assertions are equivalent:*

(1) F is quasi-odd,
(2) $-F(x) \cap F(-x) \neq \emptyset$ for all $x \in D_F$,
(3) $F(x) \cap (-F(-x)) \neq \emptyset$ for all $x \in D_F$.

Definition 9 A partial selection relation Φ of a relation F on one group X to another Y is called odd-like if $-\Phi(x) \subset F(-x)$ for all $x \in X$.

Remark 32 Note that if Φ is an odd partial selection relation of F, then $-\Phi(x) = \Phi(-x) \subset F(-x)$ for all $x \in X$. Therefore, Φ is odd-like.

Moreover, if Φ is a partial selection relation of F and F is odd, then $-\Phi(x) \subset -F(x) = F(-x)$ for all $x \in X$. Therefore, Φ is again odd-like. Now, by using Theorem 25 and the axiom of choice, we can also easily establish

Theorem 26 *If F is a relation on one group X to another Y, then the following assertions are equivalent:*
(1) *F is quasi-odd;*
(2) *F has an odd-like selection function.*

Remark 33 In [19], by using Zorn's lemma, we have proved that a relation F on one group X to another Y has an odd selection function φ if and only if F is quasi-odd and for any $x \in D_F$, with $2x = 0$, there exists a $y \in F(x)$ such that $2y = 0$.

Definition 10 A relation Φ on a groupoid X with zero to an arbitrary groupoid Y is called a left representing for a relation F on X to Y if $F(x) = \Phi(x) + F(0)$ for all $x \in X$.

Remark 34 Note that if in particular $F(0)$ is a normal subset of Y, then we also have $F(x) = F(0) + \Phi(x)$ for all $x \in X$. Therefore, Φ is also a right representing, and thus a representing relation for F.

The importance of odd-like selections is also quite obvious from the following

Theorem 27 *If F is a right-zero-superadditive, inversion-superadditive relation on one group X to another Y and Φ is an odd-like selection relation of F, then Φ is a left-representing selection relation of F.*

Proof For any $x \in X$, we have $\Phi(x) + F(0) \subset F(x) + F(0) \subset F(x)$ and

$$F(x) = \{0\} + F(x) \subset \Phi(x) - \Phi(x) + F(x) \subset \Phi(x) + F(-x) + F(x) \subset \Phi(x) + F(0).$$

Therefore, $F(x) = \Phi(x) + F(0)$, and thus the required assertion is also true.

Remark 35 If φ is a selection function of a left-zero-superadditive relation F on a groupoid X to a group Y such that $F(x) \subset \varphi(x) + F(0)$ for all $x \in D_F$ and $-\varphi[D_F] \subset \varphi[D_F]$, then it can be shown that φ is already a representing selection function of F.

However, it is now more important to note that, as an immediate consequence of Theorems 26 and 27, we can also state

Corollary 4 *If F is a quasi-odd, inversion-superadditive relation on one group X to another Y such that $F(0) \subset \{0\}$, then F is already a function.*

Remark 36 Some deeper sufficient conditions, in order that a relation should be a function, have been given by Nikodem and Popa [37].

Remark 37 Because of Corollary 2 and Theorem 27, it seems an important problem to find an additive selection function f of a superadditive relation F.

However, the set-valued generalizations of the classical Hyers–Ulam and Hahn–Banach theorems, mentioned in the Introduction, revealed that the same problem is even more important for subadditive relations.

Actually, they have shown that one has to find some sufficient conditions in order that an additive partial selection function φ of a certain subadditive relation F could be extended to an additive total selection function f of F.

7 Direct Sum Decompositions of Groupoids

Definition 11 If U, V, and W are subsets of a groupoid X such that for every $x \in W$ there exists a unique pair $(u_x, v_x) \in U \times V$ such that $x = u_x + v_x$, then we say that W is the direct sum of U and V, and we write $W = U \oplus V$.

Remark 38 Thus, in particular we have $W = U + V$. Hence, if in addition X has a zero such that $0 \in V$, we can infer that $U \subset W$.

Moreover, in this particular case for any $x \in U$ we have $x = x + 0$. Hence, by using the unicity of u_x and v_x, we can infer that $u_x = x$ and $v_x = 0$.

Remark 39 Therefore, if $W = U \oplus V$, and in particular X has a zero such that $0 \in U \cap V$, then in addition to $W = U + V$, we can also state that $U \cup V \subset W$ and $U \cap V = \{0\}$.

Namely, by Remark 38 and its dual, we have $U \subset W$ and $V \subset W$, and thus $U \cup V \subset W$. Moreover, if $x \in U \cap V$, i.e., $x \in U$ and $x \in V$, then we have $v_x = 0$ and $u_x = 0$, and thus $x = u_x + v_x = 0$.

In this respect, we can also easily prove the following

Theorem 28 *If U and V are subgroups of a monoid X, then the following assertions are equivalent:*

(1) $X = U \oplus V$,
(2) $X = U + V$ *and* $U \cap V = \{0\}$.

Hint If $x \in X$ such that $x = u_1 + v_1$ and $x = u_2 + v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$, then $u_1 + v_1 = u_2 + v_2$, and thus $-u_2 + u_1 = v_2 - v_1$. Moreover, we also have $-u_2 + u_1 \in U$ and $v_2 - v_1 \in V$. Hence, if the second part of (2) holds, we can infer that $-u_2 + u_1 = 0$ and $v_2 - v_1 = 0$. Therefore, $u_1 = u_2$, and $v_1 = v_2$ also hold.

Remark 40 Note that if U and V are subgroups of a monoid X such that $X = U + V$, then for any $x \in X$ there exist $u \in U$ and $v \in V$ such that $x = u + v$. Hence, by taking $y = -v - u$, we can see that $x + y = 0$ and $y + x = 0$. Therefore, $-x = y$, and thus X is also a group.

Now, as a useful consequence of Theorem 28, we can also state

Corollary 5 *If V is an $\mathbb{N}$-divisible subgroup of an $\mathbb{N}$-cancellable group X and $a \in X \setminus V$ such that, under the notation $U = \mathbb{Z}a = \{ka : \ k \in \mathbb{Z}\}$, we have $X = U + V$, then we actually have $X = U \oplus V$.*

Proof To show that $U \cap V \subset \{0\}$, note that if $x \in U$, then there exists $k \in \mathbb{Z}$ such that $x = ka$. Moreover, if $x \neq 0$, then $k \neq 0$. Therefore, if $x \in V$ also holds, then by Theorem 8 there exists $v \in V$ such that $x = kv$. Thus, we have $ka = kv$. Hence, by Theorem 7, it follows that $a = v$, and thus $a \in V$, which is a contradiction.

Moreover, to clarify the origin of the notion of direct sums, we can also state

Example 1 If G is a group, then the Descartes product $X = G^2 = G \times G$, with the coordinatewise addition, is also a group. Moreover, $U = \{(x, 0) : \ x \in G\}$ and $V = \{(0, y) : \ y \in G\}$ are subgroups of X such that $X = U + V$ and $U \cap V = \{(0, 0)\}$. Therefore, by Theorem 28, we have $X = U \oplus V$.

Moreover, it is also worth noticing that U and V are now elementwise commuting in the sense that $u + v = v + u$ for all $u \in U$ and $v \in V$.

The importance of elementwise commuting sets is already apparent from the following two theorems.

Theorem 29 *If U and V are elementwise commuting subgroupoids of a semigroup X such that $X = U \oplus V$, then the mappings $x \mapsto u_x$ and $x \mapsto v_x$, where $x \in X$, are additive. Thus, in particular, they are $\mathbb{N}$-homogeneous.*

Remark 41 Note that if in particular X has a zero such that $0 \in V$, then by Remark 38 the mapping $x \mapsto u_x$, where $x \in X$, is idempotent. Moreover, if $0 \in U$ also holds, then $u_0 = 0$. Thus, the above mapping is also zero-homogeneous.

Remark 42 While, if in particular X, U, and V are groups, then the mappings considered in Theorem 29 are odd. Therefore, by Theorem 29 and Corollary 3 and Remark 41, they are $\mathbb{Z}$-homogeneous.

Theorem 30 *If U and V are subsets of a semigroup X such that $X = U + V$, then the following assertions are equivalent:*

(1) X is commutative;
(2) U and V are commutative and elementwise commuting.

Concerning elementwise commuting sets, we can also easily prove the following

Theorem 31 *If U and V are subsets of a groupoid X such that $X = U \oplus V$, then the following assertions are equivalent:*

(1) U and V are elementwise commuting,
(2) $u + V = V + u$ and $v + U = U + v$ for all $u \in U$ and $v \in V$,
(3) $u + V \subset V + u$ and $v + U \subset U + v$ for all $u \in U$ and $v \in V$.

Remark 43 Note that if U is a subgroup of a monoid X, then for any $V \subset X$, the following assertions are also equivalent:

(1) $u + V = V + u$ for all $u \in U$,

(2) $u + V \subset V + u$ for all $u \in U$.

Remark 44 It is well known that if U is a subspace of a vector space X, then there exists a subspace V of X such that $X = U \oplus V$. (This, in contrast to [74, p. 43], can be proved more easily by using Zorn's lemma, than Hamel bases.)

From this decomposition theorem, by using Remark 8, we can immediately infer that if U is an $\mathbb{N}$-divisible subgroup of a uniquely $\mathbb{N}$-divisible, commutative group X, then there exists an $\mathbb{N}$-divisible subgroup V of X such that $X = U \oplus V$.

To see the necessity of the $\mathbb{N}$-divisibility of the subgroup U in the above statement, note, for instance, that $\mathbb{Z}$ is a subgroup of the vector space $\mathbb{Q}$ such that for any $\mathbb{N}$-superhomogeneous subset V of $\mathbb{Q}$ with $\mathbb{Z} \cap V = \{0\}$, we have $V = \{0\}$.

Remark 45 Much more generally, Baer [5] proved that if U is an $\mathbb{N}$-divisible subgroup of a commutative group X, then there exists a subgroup V of X such that $X = U \oplus V$.

Moreover, Kertész [30] proved that if X is a commutative group such that the order of each element of X is a square-free number, then for every subgroup U of X, there exists a subgroup V of X such that $X = U \oplus V$.

Surprisingly, the above two results were already considered to be well known by R. Baer in 1936 and 1946, respectively. Moreover, it is also worth mentioning that Hall [23], analogously to A. Kertész, also proved an "if and only if result."

8 Constructions of Additive Relations on Cyclic Sets

Theorem 32 *Let X and Y be monoids. Suppose that $a \in X_0$, $b \in Y$ and $\emptyset \neq C \subset Y$ such that*

(1) $C = C + C$ and $b + C = C + b$,
(2) $na = ma$ implies $nb + C = mb + C$ for all $n, m \in \mathbb{N}_0$.

Then, there exists a unique additive relation F of the monoid $U = \mathbb{N}_0 a$ to Y such that $F(0) = C$ and $F(a) = b + C$. Moreover, we have $F(na) = nb + C$ for all $n \in \mathbb{N}_0$.

Proof To prove the existence of F, note that by (2), we may unambiguously define a relation F of U to Y such that $F(na) = nb + C$ for all $n \in \mathbb{N}_0$. Thus, we evidently have $F(0) = C$ and $F(a) = b + C$.

Moreover, from (1) by induction, we can see that $nb + C = C + nb$ for all $n \in \mathbb{N}_0$. Hence, by Theorem 5, it is clear that

$$\begin{aligned} F(na + ma) &= F((n+m)a) = (n+m)b + C \\ &= nb + mb + C + C = nb + C + mb + C = F(na) + F(ma) \end{aligned}$$

for all $n, m \in \mathbb{N}_0$. Therefore, F is additive.

Remark 46 If in particular C is n-divisible for some $n \in \mathbb{N}$, then in addition to $nC \subset C$, we also have $C \subset nC$, and thus $C = nC$.

Moreover, if in particular b commutes with the elements of C, then by using Theorem 5, we can see that $n(mb) + nC = n(mb + C)$ for all $m \in \mathbb{N}_0$.

Therefore, if the above assumptions also hold, then we have $F(n(ma)) = F((nm)a) = (nm)b + C = n(mb) + nC = n(mb + C) = nF(ma)$ for all $m \in \mathbb{N}_0$. Thus, F is also n-homogeneous.

Analogously to the above theorem, we can also prove the following

Theorem 33 *Let X and Y be groups. Suppose that $a \in X_0$, $b \in Y$, and C is a subgroup of Y such that*

(1) $b + C = C + b$,
(2) $na = 0$ implies $nb \in C$ for all $n \in \mathbb{N}$.

Then, there exists a unique odd, additive relation F of the group $U = \mathbb{Z}a$ to Y such that $F(0) = C$ and $F(a) = b + C$. Moreover, we have $F(ka) = kb + C$ for all $k \in \mathbb{Z}$.

Proof If F is as above, then by the proof of Theorem 32, we have $F(na) = nb + C$ for all $n \in \mathbb{N}_0$. Moreover, we can also note that

$$F((-n)a) = F(-na) = -F(na)$$
$$= -(nb + C) = -(C + nb) = -nb - C = (-n)b + C$$

for all $n \in \mathbb{N}$. Therefore, the unicity of F is true.

Quite similarly, we can also note that if $n \in \mathbb{N}$ such that $(-n)a = 0$, then we also have $-na = 0$, and thus $na = 0$. Hence, by (2), it follows that $nb \in C$. Thus, $(-n)b = -nb \in -C = C$ also holds. Moreover, we can note that $0b = 0 \in C$ is also true. Therefore, $ka = 0$ implies $kb \in C$ for all $k \in \mathbb{Z}$.

Now, to prove the existence of F, we can note that if $k, l \in \mathbb{Z}$ such that $ka = la$, then

$$(-l + k)a = (-l)a + ka = -la + ka = 0.$$

Hence, by the above mentioned extension of (2), we can infer that

$$-lb + kb = (-l)b + kb = (-l + k)b \in C.$$

Now, since C is a group, we can also easily see that

$$-lb + kb + C = C, \qquad \text{and thus} \qquad kb + C = lb + C.$$

Therefore, we may unambiguously define a relation F of U to Y such that $F(ka) = kb + C$ for all $k \in \mathbb{Z}$. Thus, we evidently have $F(0) = C$ and $F(a) = b + C$.

Moreover, in addition to our observation on b and C in the proof of Theorem 32, we can see that

$$(-n)b + C = -nb - C = -(C + nb) = -(nb + C) = -C - nb = C + (-n)b$$

for all $n \in \mathbb{N}$. Therefore, $kb + C = C + kb$ holds for all $k \in \mathbb{Z}$. Hence, it is clear that

$$F(-ka) = F((-k)a) = (-k)b + C$$

$$= C + (-k)b = -C - kb = -(kb + C) = -F(ka)$$

for all $k \in \mathbb{Z}$. Therefore, F is odd. Moreover, quite similarly as in the proof of Theorem 32, we can see that $F(ka+la) = F(ka)+F(la)$ for all $k, l \in \mathbb{Z}$. Therefore, F is also additive.

Remark 47 If in particular C is n-divisible, for some $n \in \mathbb{N}$, and b commutes with the elements of C, then analogously to Remark 46 we can see that F is n-homogeneous. Hence, by Theorem 19, we can also state that F is $-n$-homogeneous.

9 Constructions of Additive Relations on Sum Sets

Definition 12 Two relations F and G of some subsets U and V of a set X to a groupoid Y, respectively, are called pointwise commuting (pointwise–elementwise commuting) if the sets $F(u)$ and $G(v)$ are commuting (elementwise-commuting) for all $u \in U$ and $v \in V$.

Remark 48 Note that, in contrast to the above definition, two relations F and G are usually called commuting if $F \circ G = G \circ F$.

Now, in addition to Theorems 32, we can also prove the following

Theorem 34 *Suppose that U and V are elementwise commuting submonoids of a monoid of X such that $X = U \oplus V$.*

Moreover, assume that F and G are pointwise commuting, additive relations of U and V to a semigroup Y, respectively, such that $F(0) = G(0)$.

Then, there exists a unique additive relation H of X to Y that extends both F and G. Moreover, we have $H(u+v) = F(u) + G(v)$ for all $u \in U$ and $v \in V$.

Proof To prove the existence of H, define a relation H of X to Y such that $H(x) = F(u_x) + G(v_x)$ for all $x \in X$. Then, by Theorem 29 and Remark 38 and its dual, for any $s \in U$ and $t \in V$ we have

$$\begin{aligned} H(s+t) &= F(u_{s+t}) + G(v_{s+t}) \\ &= F(u_s + u_t) + G(v_s + v_t) = F(s+0) + F(0+t) = F(s) + G(t). \end{aligned}$$

Hence, it is clear that $H(s) = H(s+0) = F(s) + G(0) = F(s) + F(0) = F(s)$ and $H(t) = H(0+t) = F(0) + G(t) = G(0) + G(t) = G(t)$. Therefore, H extends both F and G.

Moreover, by taking $\omega \in U$ and $w \in V$, we can also easily see that

$$\begin{aligned} H((s+t)+(\omega+w)) &= H((s+\omega)+(t+w)) \\ &= F(s+\omega) + G(t+w) = (F(s)+F(\omega)) + (G(t)+G(w)) \\ &= (, F(s)+G(t)) + (F(\omega)+G(w)) \\ &= H(s+t) + H(\omega+w). \end{aligned}$$

Therefore, H is also additive.

Remark 49 If in particular F and G are n-subhomogeneous for some $n \in \mathbb{N}$, then by Theorem 12 they are actually n-homogeneous.

Therefore, if in addition F and G are pointwise–elementwise commuting, then we have $H(n(s+t)) = H(ns+nt) = F(ns)+G(nt) = nF(s)+nG(t) = n(F(s)+G(t)) = nH(s+t)$ for all $s \in U$ and $t \in V$. Therefore, H is also n-homogeneous.

However, it is now more interesting that in addition to Theorem 33, we can also easily prove the following

Theorem 35 *Suppose that U and V are elementwise commuting subgroups of a group X such that $X = U + V$.*

Moreover, assume that F and G are pointwise commuting, additive relations of U and V to a semigroup Y, respectively, such that $F(x) = G(x)$ for all $x \in U \cap V$.

Then, there exists a unique additive relation H of X to Y that extends both F and G. Moreover, we have $H(u+v) = F(u)+G(v)$ for all $u \in U$ and $v \in V$.

Proof To prove the existence of H, note that if $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that

$$u_1 + v_1 = u_2 + v_2,$$

then $-u_2 + u_1 = v_2 - v_1$. Hence, it is clear that, in addition to $-u_2 + u_1 \in U$ and $v_2 - v_1 \in V$, we also have $-u_2 + u_1 \in V$ and $v_2 - v_1 \in U$. Thus, in particular by the hypothesis $F(v_2 - v_1) = G(-u_2 + u_1)$ also holds. Now, by observing that

$$F(u_1) = F(u_2 + v_2 - v_1) = F(u_2) + F(v_2 - v_1) = F(u_2) + G(-u_2 + u_1),$$

$$G(v_1) = G(-u_1 + u_2 + v_2) = G(-(-u_2 + u_1) + v_2) = G(-(-u_2 + u_1)) + G(v_2),$$

we can already see that

$$\begin{aligned} &F(u_1) + G(v_1) \\ &= F(u_2) + G(-u_2 + u_1) + G(-(-u_2 + u_1)) + G(v_2) = F(u_2) + G(0) + G(v_2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = F(u_2) + G(v_2). \end{aligned}$$

Therefore, we may unambiguously define a relation H of $X = U + V$ to Y such that $H(u+v) = F(u) + G(v)$ for all $u \in U$ and $v \in V$. Now, quite similarly as in the proof of Theorem 34, we can see that H extends both F and G. Moreover, H is also additive.

Remark 50 If in particular Y is also a group, and F and G are odd, then we have $H(-(u+v)) = H(-(v+u)) = H(-u+(-v)) = F(-u)+G(-v) = G(-v)+F(-u) = -G(v)+(-F(u)) = -(F(u)+G(v)) = -H(u+v)$ for all $u \in U$ and $v \in V$. Therefore, H is also odd.

While, if in particular F and G are n-subhomogeneous, for some $n \in \mathbb{N}$, and F and G are pointwise–elementwise commuting, then analogously to Remark 49, we

can see that H is n-homogeneous. Hence, if in addition, Y is also a group, and F and G are odd, then by Theorem 19 we can also state that H is $-n$-homogeneous.

From Theorem 35, by taking $F = U \times G(0)$, we can immediately derive

Corollary 6 *Suppose that U and V are elementwise commuting subgroups of a group X such that $X = U + V$. Moreover, assume that G is an additive relation of V to a semigroup Y such that $G(x) = G(0)$ for all $x \in U \cap V$.*

Then, G can be uniquely extended to an additive relation H of X to Y such that $H(u) = G(0)$ for all $u \in U$. Moreover, we have $H(u + v) = G(v)$ for all $u \in U$ and $V \in V$.

Remark 51 If in particular G is odd, then we can easily see that $F = U \times G(0)$ is also odd. Thus, by Remark 50, H is also odd.

While, if in particular G is n-subhomogeneous, for some $n \in \mathbb{N}$, then we can easily see that H is also n-homogeneous. Hence, if in particular G is odd, then by Theorem 19 we can see that H is also $-n$-homogeneous.

10 One-step Extensions of Additive Relations

Now, by using Theorems 32 and 34, we can easily prove the following

Theorem 36 *Let X and Y be monoids. Suppose that G is an additive relation of a submonoid V of X to Y. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U \oplus V$ holds with $U = \mathbb{N}_0 a$,
(2) $a + v = v + a$ and $b + G(v) = G(v) + b$ for all $v \in V$,
(3) $na = ma$ implies $nb + G(0) = mb + G(0)$ for all $n, m \in \mathbb{N}_0$.

Then, there exists a unique additive relation H of X to Y extending G such that $H(a) = b + G(0)$. Moreover, we have $H(na + v) = nb + G(v)$ for all $n \in \mathbb{N}_0$ and $v \in V$.

Proof To prove the existence of H, note that in particular, we have $a \neq 0$, $G(0) \neq \emptyset$, $G(0) = G(0) + G(0)$, and $b + G(0) = G(0) + b$. Thus, by Theorem 32, there exists an additive relation F of U to Y such that $F(0) = G(0)$ and $F(a) = b + G(0)$. Moreover, we have $F(na) = nb + G(0)$ for all $n \in \mathbb{N}_0$.

On the other hand, from (2) by induction, we can see that $na + v = v + na$ and $nb + G(v) = G(v) + nb$ also hold for all $n \in \mathbb{N}_0$ and $v \in V$. Therefore, U and V are elementwise commuting. Moreover, we can see that

$$\begin{aligned} F(na) + G(v) &= nb + G(0) + G(v) = nb + G(v) \\ &= nb + G(v) + G(0) = G(v) + nb + G(0) = G(v) + F(na) \end{aligned}$$

for all $n \in \mathbb{N}_0$ and $v \in V$. Therefore, F and G are pointwise commuting.

Now, by Theorem 34, we can state that there exists an additive relation H of X to Y that extends both F and G. Moreover, since $a \in U$, we can also note $H(a) = F(a) = b + G(0)$.

Remark 52 If in particular G is n-subhomogeneous for some $n \in \mathbb{N}$, then by Theorem 12 we can state that G is actually n-homogeneous.

Moreover, if in addition, b commutes with the elements of $G(v)$ for all $v \in V$, then by using Theorem 5, we can see that that $H(n(ma+v)) = H(nma+nv) = nmb + G(nv) = nmb + nG(v) = n(mb+G(v)) = nH(ma+v)$ for all $m \in \mathbb{N}_0$. Therefore, H is also n-homogeneous.

Now, by using Theorems 33 and 34, we can also easily prove the following

Theorem 37 *Let X and Y be groups. Suppose that G is an odd, superadditive relation of a subgroup V of X to Y. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U \oplus V$ *holds with* $U = \mathbb{Z}a$,
(2) $na = 0$ *implies* $nb \in G(0)$ *for all* $n \in \mathbb{N}$,
(3) $a + v = v + a$ *and* $b + G(v) = G(v) + b$ *for all* $v \in V$.

Then, there exists a unique odd, additive relation H of X to Y extending G such that $H(a) = b + G(0)$. Moreover, we have $H(ka+v) = kb + G(v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

Proof To prove the existence of H, note that we now have $a \neq 0$, $G(0) \neq \emptyset$, and $G(0) - G(0) = G(0) + G(0) \subset G(0)$. Thus, $G(0)$ is a subgroup of Y. Moreover, by (3), we have $b + G(0) = G(0) + b$. Therefore, by Theorem 33, there exists an odd, additive relation F of U to Y such that $F(0) = G(0)$ and $F(a) = b + G(0)$. Moreover, we have $F(ka) = kb + G(0)$ for all $k \in \mathbb{Z}$.

On the other hand, from Theorem 20, we can see that G is now actually additive. Moreover, in addition to our observations on a and v and b and $G(v)$ made in the proof of Theorem 36, we can now see that

$$(-n)a + v = -na + v = -(-v + na) = -(na - v) = v - na = v + (-n)a$$

and

$$\begin{aligned}(-n)b + G(v) &= -nb + G(v) = -(-G(v) + nb)\\ &= -(G(-v) + nb) = -(nb + G(-v)) = -G(-v) - nb = G(v) + (-n)b\end{aligned}$$

also hold for all $n \in \mathbb{N}$. Therefore, we now have $ka + v = v + ka$ and $kb + G(v) = G(v) + kb$ for all $k \in \mathbb{Z}$ and $v \in V$. Thus, U and V are elementwise commuting. Moreover, quite similarly to the proof of Theorem 36, we can see that $F(ka) + G(v) = G(v) + F(ka)$ for all $k \in \mathbb{Z}$ and $v \in V$. Therefore, F and G are pointwise commuting.

Now, by Theorem 34, we can state that there exists an additive relation H of X to Y that extends both F and G. Moreover, from Remark 50, we can see that H is also odd.

Remark 53 If in particular X is $\mathbb{N}$-cancellable, then by Remark 7, we have $na \neq 0$ for all $n \in \mathbb{N}$. Therefore, (2) automatically holds.

Moreover, if in addition V is $\mathbb{N}$-divisible, then by Corollary 5, the equality $X = U + V$ already implies that $X = U \oplus V$. Therefore, instead of (1) it is enough to assume only that $X = U + V$.

Remark 54 While, if in particular G is n-subhomogeneous, for some $n \in \mathbb{N}$, and b commutes with the elements of $G(v)$ for all $v \in V$, then analogously to Remark 52, we can see that H is n-homogeneous. Hence, by Theorem 19, we can also state that H is $-n$-homogeneous.

However, it is now more interesting that, by using Theorems 33 and 35, we can also prove the following

Theorem 38 *Let X and Y be groups. Suppose that G is an odd, $\mathbb{N}$-subhomogeneous, superadditive relation of a subgroup V of X to Y. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U + V$ holds with $U = \mathbb{Z}a$,
(2) $a + v = v + a$ and $b + G(v) = G(v) + b$ for all $v \in V$,
(3) $nb \in G(na)$ and Y is n-cancellable for some $n \in \mathbb{N}$.

Then, there exists a unique odd, additive relation H of X to Y extending G such that $H(a) = b + G(0)$. Moreover, we have $H(ka + v) = kb + G(v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

Proof To prove the existence of H, define

$$L = \{k \in \mathbb{Z} : \quad ka \in V\}.$$

Then, by using Theorem 6, it can be easily seen that L is an ideal in $\mathbb{Z}$. Moreover, if n is as in (3), then we can note that $na \in V$, and thus $n \in L$.

On the other hand, from Theorem 20, we can see that G is now actually $\mathbb{Z}_0$-homogeneous. Thus, we have

$$n(kb) = k(nb) \in kG(na) = G(k(na)) = G(n(ka)) = nG(ka)$$

for all $k \in L_0$. Hence, by using the n-cancellabilty of Y, we can infer that $kb \in G(ka)$ for all $k \in L_0$. Moreover, from Theorem 20, we can see that $0 \in G(0)$, and thus $0b = 0 \in G(0) = G(0a)$ also holds. Therefore, we actually have $kb \in G(ka)$ for all $k \in L$. Hence, by Remark 32 and Theorem 27, it is clear that $G(ka) = kb + G(0)$ for all $k \in L$.

Moreover, we can note that if $m \in \mathbb{Z}$ such that $ma = 0$, then $m \in L$. Therefore, $mb \in G(ma) = G(0)$. Now, by using Theorem 33, and the corresponding properties of $G(0)$, we can see that there exists an odd, additive relation F of U to Y such that $F(0) = G(0)$ and $F(a) = b + G(0)$. Moreover, we have $F(ka) = kb + G(0)$ for all $k \in \mathbb{Z}$.

Thus, in particular $F(ka) = kb + G(0) = G(ka)$ for all $k \in L$. Hence, by the definition of L, we can infer that $F(x) = G(x)$ for all $x \in U \cap V$. Moreover, from

Theorem 20, we know that G is also additive. And, from the proofs of Theorems 36 and 37, we can see that U and V are elementwise commuting, and F and G are pointwise commuting.

Thus, by Theorem 35 and Remark 50, there exists an odd additive relation H of X to Y that extends both F and G.

Remark 55 If in particular b commutes with the elements of $G(v)$ for all $v \in V$, then analogously to Remark 52, we can see that H is also $\mathbb{Z}_0$-homogeneous.

11 The Intersection Convolution of Relations

Definition 13 If X is a groupoid, then we define a relation Γ on X to X^2 such that

$$\Gamma(x) = \{(u, v) \in X^2 : \quad x = u + v\} \quad \text{for all} \quad x \in X.$$

Remark 56 Thus, it can be easily seen that Γ is just the inverse relation of the operation $+$ in X. Therefore, several properties of Γ can be immediately derived from those of $+$ by some inversion–invariance theorems.

Definition 14 If X is a groupoid, then for any $x \in X$ and $U, V \subset X$, we define

$$\Gamma(x, U, V) = \Gamma(x) \cap \Gamma_{(U,V)}, \quad \text{where} \quad \Gamma_{(U,V)} = U \times V.$$

Remark 57 Thus, the properties of the relation $\Gamma(x, U, V)$ can be easily derived from those of Γ and $\Gamma_{(U,V)}$.

However, in the sequel, we shall rather use that, for any $u, v \in X$, we have $(u, v) \in \Gamma(x, U, V) \iff u \in U, \ v \in V, \ x = u + v.$

Definition 15 If F and G are relations on one groupoid X to another Y, then we define a relation $F * G$ on X to Y such that

$$(F * G)(x) = \bigcap \{F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G)\}$$

for all $x \in X$. The relation $F * G$ is called the intersection convolution of the relations F and G.

Remark 58 This definition has been introduced in [63] to extend the results of [53]. For some closely related notions, see also the infimal convolutions of [34, 51, 62, 64].

The intersection convolution of relations is closely related not only to the infimal convolution of functions [11], but also to the global sum, and the composition and box products of relations [60].

The treatment of [53] has also been closely followed and substantially generalized by Beg [6]. He did not refer to [53], but in a letter he informed the second author that this was not intentional.

In particular, in [63], the second author has proved the following

Theorem 39 *If F and G are relations on a group X to a groupoid Y, then for any $x \in X$ we have*

$$(F * G)(x) = \bigcap \{F(x - v) + G(v) : \;\; v \in (-D_F + x) \cap D_G\} =$$
$$= \bigcap \{F(u) + G(-u + x) : \;\; u \in D_F \cap (x - D_G)\}.$$

Hence, by using that $-X + x = X$ and $x - X = X$, we can immediately derive

Corollary 7 *If F and G are relations on a group X to a groupoid Y, then for any $x \in X$ we have*

(1) $(F * G)(x) = \bigcap_{v \in D_G} (F(x - v) + G(v))$ *whenever F is total,*

(2) $(F * G)(x) = \bigcap_{u \in D_F} (F(u) + G(-u + x))$ *whenever G is total.*

Remark 59 The multiplicative form of the $D_G = X$ particular case of (1) closely resembles to the definition of the ordinary convolution of integrable functions.

By using the corresponding definitions, we can also easily prove the following two theorems.

Theorem 40 *If F and G are pointwise commuting relations on one groupoid X to another Y such that their domains D_F and D_G are elementwise commuting, then $F * G = G * F$.*

Remark 60 To prove the above theorem, one can also note that $\Gamma(x, V, U) = \Gamma(x, U, V)^{-1}$ for all $x \in X$ and elementwise commuting subsets U and V of X.

Theorem 41 *If F and G are odd relations on one group X to another Y, then for any $x \in X$ we have*

$$(F * G)(-x) = -(G * F)(x).$$

Proof If $x \in X$ and $(v, u) \in \Gamma(x, D_G, D_F)$, then $v \in D_G$ and $u \in D_F$ such that $x = v + u$. Hence, by using the symmetry of D_F and D_G, we can infer that $-v \in D_G$ and $-u \in D_F$. Moreover, we can also note that $-x = -u + (-v)$. Therefore, $(-u, -v) \in \Gamma(-x, D_F, D_G)$. Hence, by using the oddness of F and G, we can infer that

$$(F * G)(-x) = \bigcap \{F(s) + G(t) : \quad (s, t) \in \Gamma(-x, D_F, D_G)\}$$
$$\subset F(-u) + G(-v) = -F(u) + (-G(v)) = -(G(v) + F(u)).$$

Hence, since the mapping $y \mapsto -y$, where $y \in Y$, is injective, we can also see that

$$(F * G)(-x) \subset \bigcap \{-(G(v) + F(u)) : \quad (v, u) \in \Gamma(x, D_G, D_F)\}$$
$$= -\bigcap \{G(v) + F(u) : \quad (v, u) \in \Gamma(x, D_G, D_F)\} = -(G * F)(x).$$

Now, by using writing G in place of F, F in place of G, and $-x$ in place of x, we can see that the converse inclusion is also true.

Remark 61 To prove the above theorem, one can also note that $\Gamma(-x,-U,-V) = -\Gamma(x,V,U)^{-1}$ for all $x \in X$ and $U,V \subset X$. Thus, in particular $\Gamma(-x,U,V) = -\Gamma(x,V,U)^{-1}$ whenever U and V are symmetric.

Now, as an immediate consequence of Theorems 40 and 41, we can also state the following generalization of [53, Theorem 4.3].

Corollary 8 *If F and G are odd, pointwise commuting relations on one group X to another Y such that D_F and D_G are elementwise commuting, then $F * G$ is also odd.*

12 Additivity and Homogeneity Properties of the Intersection Convolution

Now, as an extension of [53, Theorem 4.1], we can also prove the following

Theorem 42 *If F is an arbitrary and G is a superadditive relation on a monoid X to a semigroup Y such that D_G is a subgroup of X, then for any $x,y \in X$ we have*

$$(F * G)(x) + G(y) \subset (F * G)(x+y).$$

Proof If $(u,v) \in \Gamma(x+y, D_F, D_G)$, then $u \in D_F$ and $v \in D_G$ such that $x+y = u+v$. Hence, if in particular $G(y) \neq \emptyset$, i.e., $y \in D_G$, we can infer that $x = u+v-y$ and $v-y \in D_G$. Therefore, $(u, v-y) \in \Gamma(x, D_F, D_G)$. Hence, it is clear that

$$(F * G)(x) = \bigcap \{F(s)+G(t) : \quad (s,t) \in \Gamma(x, D_F, D_G)\} \subset F(u)+G(v-y).$$

Therefore, $(F * G)(x) + G(y) \subset F(u) + G(v-y) + G(y) \subset F(u) + G(v)$. Hence, it is clear that

$$\begin{aligned}(F * G)(x) &+ G(y) \\ &\subset \bigcap \{F(u)+G(v) : \quad (u,v) \in \Gamma(x+y, D_F, D_G)\} = (F * G)(x+y).\end{aligned}$$

Simple applications of the above theorem give the following

Corollary 9 *If F is an arbitrary and G is a superadditive relation on one monoid X to another Y such that D_G is a subgroup of X and G is quasi-odd, then for any $x \in X$ and $y \in D_G$ we have*

$$(F * G)(x+y) = (F * G)(x) + G(y).$$

Proof Now, because of $0 \in G(-y) + G(y)$ and Theorem 42, we also have

$$(F * G)(x+y) \subset (F * G)(x+y) + G(-y) + G(y) \subset (F * G)(x) + G(y).$$

Remark 62 Note that if F and G are as above, then in particular we have $(F*G)(x) = (F*G)(x) + G(0)$ for all $x \in X$, and $(F*G)(y) = (F*G)(0) + G(y)$ and $(F*G)(0) = (F*G)(-y) + G(y)$ for all $y \in D_G$.

Moreover, if in particular $0 \in (F*G)(0)$, then from the second equality, we can infer that $G \subset F*G$. However, in general, $F*G$ need not be an extension of G. Namely, because of the third equality, we usually have $(F*G)(0) \neq \{0\}$.

Remark 63 The above theorem and its corollary show that, analogously to continuity properties of pairs of relations studied in [71], additivity and homogeneity properties of pairs of relations should have also been investigated in Sects. 4–6.

Analogously to [53, Theorem 4.4], we can also easily prove the following two theorems.

Theorem 43 *If F and G are n-superhomogeneous relations on an n-cancellable semigroup X to an arbitrary semigroup Y, for some $n \in \mathbb{N}$, such that*

(1) F and G are pointwise-elementwise commuting,
(2) D_F and D_G are n-divisible and elementwise commuting, then $F*G$ is also n-superhomogeneous.

Theorem 44 *If F and G are n-semi-subhomogeneous relations on an arbitrary semigroup X to an n-cancellable semigroup Y, for some $n \in \mathbb{N}$, such that*

(1) F and G are pointwise–elementwise commuting,
(2) D_F and D_G are n-superhomogeneous and elementwise commuting, then $F*G$ is n-subhomogeneous.

Proof If $x \in X$ and $(u, v) \in \Gamma(x, D_F, D_G)$, then $u \in D_F$ and $v \in D_G$ such that $x = u + v$. Hence, by using (2) and Theorem 5, we can infer that $nu \in D_F$, $nv \in D_G$, and $nx = nu + nv$. Therefore, $(nu, nv) \in \Gamma(nx, D_F, D_G)$. Now, by using the n-semi-subhomogeneity of F and G, and condition (1) and Theorem 5, we can see that

$$
\begin{aligned}
(F*G)(nx) = \bigcap \{F(\omega) + G(w) : \quad (\omega, w) \in \Gamma(nx, D_F, D_G)\} \\
\subset F(nu) + G(nv) \subset nF(u) + nG(v) = n(F(u) + G(v)).
\end{aligned}
$$

Hence, we can already infer that

$$
\begin{aligned}
(F*G)(nx) &\subset \bigcap \{n(F(u) + G(v)) : \quad (u, v) \in \Gamma(x, D_F, D_G)\} \\
&= n \bigcap \{F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G)\} = n(F*G)(x).
\end{aligned}
$$

Namely, by the n-cancellability of Y, the mapping $y \mapsto ny$, where $y \in Y$, is injective.

Now, as an immediate consequence of Theorems 43 and 44, we can state

Corollary 10 *If F and G are pointwise–elementwise commuting, n-semihomogeneous relations on one n-cancellable semigroup X to another Y, for some $n \in \mathbb{N}$, such that D_F and D_G are n-divisible and elementwise commuting, then $F * G$ is n-homogeneous.*

Moreover, from Theorems 43 and 44 by using Corollary 8 and Theorem 19, we can also immediately get the following two theorems.

Theorem 45 *If F and G are odd, $\mathbb{N}$-superhomogeneous relations on an $\mathbb{N}$-cancellable group X to an arbitrary group Y such that*

(1) *F and G are pointwise–elementwise commuting,*
(2) *D_F and D_G are $\mathbb{N}$-divisible and elementwise commuting,*

then $F * G$ is $\mathbb{Z}_0$-superhomogeneous.

Theorem 46 *If F and G are odd, $\mathbb{N}$-semi-subhomogeneous relations on an arbitrary group X to an $\mathbb{N}$-cancellable group Y such that*

(1) F and G are pointwise–elementwise commuting,
(2) D_F and D_G are $\mathbb{N}$-superhomogeneous and elementwise commuting,

then $F * G$ is $\mathbb{Z}_0$-subhomogeneous.

Hence, it is clear that in particular we also have

Corollary 11 *If F and G are pointwise–elementwise commuting, odd $\mathbb{N}$-semihomogeneous relations on one $\mathbb{N}$-cancellable group X to another Y such that D_F and D_G are $\mathbb{N}$-divisible and elementwise commuting, then $F*G$ is $\mathbb{Z}_0$-homogeneous.*

Remark 64 To guarantee the 0-superhomogeneity of $F * G$, note by Theorem 39 we have $0 \in (F * G)(0)$ if and only if $0 \in F(x) + G(-x)$ for all $x \in D_F \cap (-D_G)$. Thus, in particular the relation F is quasi-odd if and only if D_F is symmetric and $0 \in (F * F)(0)$.

13 Selection and Inclusion Properties of the Intersection Convolution

In the sequel, we shall also need some consequences of the corresponding results of [9]. A few direct proofs are included here for the reader's convenience.

Theorem 47 *If F is a relation on a monoid X to a groupoid Y, and Φ is a semi-subadditive partial selection relation of F such that D_Φ is a subgroup of X, then $\Phi \subset F * \Phi$.*

Proof If $x \in X$ and $u \in D_F$ and $v \in D_\Phi$ such that $x = u + v$, then since v has an additive inverse $-v$ in D_Φ, we also have $u = x - v$. Moreover, if in particular $\Phi(x) \neq \emptyset$, i.e., $x \in D_\Phi$, we can see that $u \in D_\Phi$. Hence, it is clear that

$$\Phi(x) = \Phi(u + v) \subset \Phi(u) + \Phi(v) \subset F(u) + \Phi(v).$$

Therefore,

$$\Phi(x) \subset \bigcap \{F(u) + \Phi(v) : \quad (u, v) \in \Gamma(x, D_F, D_\Phi)\} = (F * \Phi)(x).$$

Remark 65 By [10, Example 6.1], a semiadditive partial selection relation Φ of a relation F of one group X to another Y can only be, in general, extended to an additive, total selection relation of the relation $F + \Phi(0)$.

Therefore, it is also necessary to prove the following

Theorem 48 *If F is a relation on a groupoid X with zero to an arbitrary groupoid Y and Φ is a right-zero-subadditive partial selection relation of F, then Φ is also a partial selection relation of $F + \Phi(0)$.*

Proof $\Phi(x) \subset \Phi(x) + \Phi(0) \subset F(x) + \Phi(0) = (F + \Phi(0))(x)$ for all $x \in X$.

Now, by Theorem 11, we can also state

Corollary 12 *If F is a relation on one groupoid X with zero to another Y and Φ is a partial selection relation of F such that $0 \in \Phi(0)$, then Φ is also a partial selection relation of $F + \Phi(0)$.*

However, it is now more important to note that in addition to Theorem 47, we can also prove the following

Theorem 49 *If F is a relation on a groupoid X with zero to a semigroup Y, and moreover Φ is a left-zero-superadditive relation on X to Y and Ψ is a $D_F \times D_\Phi$-subadditive partial selection relation of $F + \Phi(0)$ such that $\Psi(v) \subset \Phi(v)$ for all $v \in D_\Phi$, then $\Psi \subset F * \Phi$.*

Proof If $x \in X$ and $u \in D_F$ and $v \in D_\Phi$ such that $x = u + v$, then by the hypotheses

$$\begin{aligned} \Psi(x) &= \Psi(u + v) \subset \Psi(u) + \Psi(v) \\ &\subset (F + \Phi(0))(u) + \Phi(v) = F(u) + \Phi(0) + \Phi(v) \subset F(u) + \Phi(v). \end{aligned}$$

Therefore,

$$\Psi(x) \subset \bigcap \{F(u) + \Phi(v) : \quad (u, v) \in \Gamma(x, D_F, D_\Phi)\} = (F * \Phi)(x).$$

From this theorem, we can immediately derive the following

Corollary 13 *If F is a total and Φ is a left-zero-superadditive relation on a groupoid X with zero to a semigroup Y such that $\Phi(0) \neq \emptyset$ and there exists an $X \times D_\Phi$-subadditive total selection relation Ψ of $F + \Phi(0)$ such that $\Psi(v) \subset \Phi(v)$ for all $v \in D_\Phi$, then $F * \Phi$ is also a total relation on X to Y.*

Remark 66 This corollary gives an important necessary condition in order that a left-zero-additive partial selection relation Φ of an arbitrary relation F of a groupoid X with zero to a semigroup Y could be extended to an $X \times D_\Phi$-subadditive total selection relation Ψ of $F + \Phi(0)$.

In addition to Theorem 48, we can also prove the following

Theorem 50 *If F and G are relations on one groupoid X with zero to another Y, then*
(1) $F \subset F + G(0)$ *if* $0 \in G(0)$,
(2) $F + G(0) \subset F$ *if F is right-zero-superadditive and* $G(0) \subset F(0)$.

Proof If the conditions of (2) hold, then we have $(F + G(0))(x) = F(x) + G(0) \subset F(x) + F(0) \subset F(x)$ for all $x \in X$. Therefore, the conclusion of (2) also holds.

Now, as an immediate consequence of this theorem, we can also state

Corollary 14 *If F is a right-zero-superadditive and G is an arbitrary relation on one groupoid X with zero to another Y such that $0 \in G(0) \subset F(0)$, then $F = F + G(0)$.*

Moreover, in addition to Theorem 47, we can also prove the following

Theorem 51 *If F is a total and G is an arbitrary relation on a groupoid X with zero to an arbitrary groupoid Y such that $G(0) \neq \emptyset$, then $F * G \subset F + G(0)$.*

Proof If $x \in X$, then because of the assumptions $D_F = X$ and $0 \in D_G$ we have $(x, 0) \in \Gamma(x, D_F, D_G)$. Therefore,

$$(F * G)(x) = \bigcap \{F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G)\}$$
$$\subset F(x) + G(0) = (F + G(0))(x).$$

Now, combining Theorems 47 and 51, we can also state

Corollary 15 *If F is a relation of a monoid X to a groupoid Y, and Φ is a semi-subadditive partial selection relation of F such that D_Φ is a subgroup of X, then $\Phi \subset F * \Phi \subset F + \Phi(0)$.*

Moreover, in addition to Theorem 51, we can also prove

Theorem 52 *If F is a superadditive relation on a group X to a semigroup Y and Φ is an inversion-semi-subadditive partial selection relation of F, then $F + \Phi(0) \subset F * \Phi$.*

Proof If $x \in X$, then by Remark 18 we have

$$(F + \Phi(0))(x) = F(x) + \Phi(0)$$
$$\subset F(x) + \Phi(-v) + \Phi(v) \subset F(x) + F(-v) + \Phi(v) \subset F(x - v) + \Phi(v)$$

for all $v \in D_\Phi$. Therefore, by Theorem 39, we also have

$$(F + \Phi(0))(x) \subset \bigcap \{F(x - v) + \Phi(v) : \ v \in (-D_F + x) \cap D_\Phi\} = (F * \Phi)(x).$$

Now, as a consequence of Theorems 51 and 52, we can also state

Corollary 16 *If F is a superadditive relation of a group X to a semigroup Y and Φ is an inversion-semi-subadditive partial selection relation of F such that $\Phi(0) \neq \emptyset$, then $F * \Phi = F + \Phi(0)$.*

Finally, we note that the following theorem is also true.

Theorem 53 *If F and G are relations on a groupoid X with zero to a semigroup Y, then*
(1) $F * G \subset (F + G(0)) * G$ *if* G *is left-zero-subadditive,*
(2) $(F + G(0)) * G \subset F * G$ *if* G *is left-zero-superadditive and* $G(0) \neq \emptyset$.

Hence, it is clear that in particular we also have

Corollary 17 *If F and G are relations on a groupoid X with zero to a semigroup Y such that G is left-zero-additive and $G(0) \neq \emptyset$, then* $F * G = (F + G(0)) * G$.

14 One-step Extensions of Additive Partial Selection Relations

In this section, by using the intersection convolution, we shall prove some partial generalizations of Theorems 36–38.

Theorem 54 *Let F be a relation of one monoid X to another Y. Suppose that Φ is an additive relation of a subgroup V of X to Y such that $\Phi \subset F$. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U \oplus V$ *holds with* $U = \mathbb{N}_0 a$,
(2) $nb \in (F * \Phi)(na)$ *for all* $n \in \mathbb{N}$,
(3) $a + v = v + a$ *and* $b + \Phi(v) = \Phi(v) + b$ *for all* $v \in V$,
(4) $na = ma$ *implies* $nb + \Phi(0) = mb + \Phi(0)$ *for all* $n, m \in \mathbb{N}_0$.

Then, there exists a unique additive selection relation Ψ of $F + \Phi(0)$ extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, we have $\Psi(na + v) = nb + \Phi(v)$ for all $n \in \mathbb{N}_0$ and $v \in V$.

Proof Now, by Theorem 36, there exists a unique additive relation Ψ of X to Y extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, we have $\Psi(na + v) = nb + \Phi(v)$ for all $n \in \mathbb{N}_0$ and $v \in V$.

Thus, we need only to show that $\Psi \subset F + \Phi(0)$ also holds. For this, note that by (2) and Theorems 42 and 51, we have

$$\begin{aligned}\Psi(na + v) = nb + \Phi(v) &\subset (F * \Phi)(na) + \Phi(v)\\ &\subset (F * \Phi)(na + v) \subset (F + \Phi(0))(na + v)\end{aligned}$$

for all $n \in \mathbb{N}$ and $v \in V$. Moreover, by Theorems 47 and 51, we also have

$$\begin{aligned}\Psi(0a + v) = 0b + \Phi(v) = \Phi(v) &\subset (F * \Phi)(v)\\ &\subset (F + \Phi(0))(v) \subset (F + \Phi(0))(0a + v).\end{aligned}$$

Therefore, we have $\Psi(na + v) \subset (F + \Phi(0))(na + v)$ for all $n \in \mathbb{N}_0$ and $v \in V$.

Remark 67 Note that now Φ is superadditive as a relation on X to Y. Thus, by Theorem 12, Φ is $\mathbb{N}$-superhomogeneous.

Therefore, if in particular X is $\mathbb{N}$-cancellable, X and V are $\mathbb{N}$-divisible, V is commutative, F is $\mathbb{N}$-superhomogeneous, and F and Φ are pointwise–elementwise

commuting, then by Theorem 43, $F * \Phi$ is also $\mathbb{N}$-superhomogeneous. Thus, we have $nb \in n(F * \Phi)(a) \subset (F * \Phi)(na)$ for all $n \in \mathbb{N}$ and $b \in (F * \Phi)(a)$.

Now, by using Theorem 37 instead of Theorem 36, we can quite similarly prove

Theorem 55 *Let F be an odd relation of one group X to another Y. Suppose that Φ is an odd, superadditive relation of a commutative subgroup V of X to Y such that $\Phi \subset F$, and moreover F and Φ are pointwise commuting. Furthermore, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $X = U \oplus V$ holds with $U = \mathbb{Z}a$,
(2) $nb \in (F * \Phi)(na)$ for all $n \in \mathbb{N}$,
(3) $na = 0$ implies $nb \in \Phi(0)$ for all $n \in \mathbb{N}$,
(4) $a + v = v + a$ and $b + \Phi(v) = \Phi(v) + b$ for all $v \in V$.

Then, there exists a unique odd, additive selection relation Ψ of $F + \Phi(0)$ extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, we have $\Psi(ka + v) = kb + \Phi(v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

Proof Now, by Theorem 37, there exists a unique odd, additive relation Ψ of X to Y extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, we have $\Psi(ka + v) = kb + \Phi(v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

On the other hand, from (1), the commutativity of V, and the first part of (4), it is clear that now X and V are elementwise commuting. Hence, by Corollary 8, we can see that $F * G$ is also odd. Thus, by (2), we also have

$$(- n)b = -nb \in -(F * \Phi)(na) = (F * \Phi)(- na) = (F * \Phi)((- n)a)$$

for all $n \in \mathbb{N}$. Moreover, since $0 \in -F(v) + \Phi(v) = F(0 - v) + \Phi(v)$ for all $v \in V$, by Corollary 7 we can also see that

$$0b = 0 \in \bigcap_{v \in V} (F(0 - v) + \Phi(v)) = (F * \Phi)(0) = (F * \Phi)(0a).$$

Therefore, now we actually have $kb \in (F * \Phi)(ka)$ for all $k \in \mathbb{Z}$.

Now, quite similarly as in the proof of Theorem 54, we can see that

$$\Psi(ka + v) \subset (F + \Phi(0))(ka + v)$$

for all $k \in \mathbb{Z}$ and $v \in V$.

Remark 68 Note that if (2) holds, then by Theorem 51 we have

$$nb \in (F * \Phi)(na) \subset (F + \Phi(0))(na) = F(na) + \Phi(0)$$

for all $n \in \mathbb{N}$. Thus, if in particular $n \in \mathbb{N}$ such that $na = 0$, and moreover $F(0) = \Phi(0)$, then we also have $nb \in F(0) + \Phi(0) = \Phi(0) + \Phi(0) \subset \Phi(0)$. Therefore, in this particular case, (2) implies (3).

Now, by using Theorem 38 instead of Theorem 37, we can also easily prove

Theorem 56 *Let F be an n-subhomogeneous relation of an arbitrary group X to an n-cancellable group Y for some $n \in \mathbb{N}$. Suppose that Φ is an odd, $\mathbb{N}$-subhomogeneous, superadditive relation of a subgroup V of X to Y such that $\Phi \subset F$. Moreover, assume that $a \in X \setminus V$ and $b \in Y$ such that*

(1) $nb \in \Phi(na)$,
(2) $X = U + V$ holds with $U = \mathbb{Z}a$,
(3) $a + v = v + a$ and $b + w = w + b$ for all $v \in V$ and $w \in \Phi(v)$.

Then, there exists a unique $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of F extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, we have $\Psi(ka+v) = kb+\Phi(v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

Proof Now, by Theorem 38 and Remark 55, there exists a unique $\mathbb{Z}_0$-homogeneous additive relation Ψ of X to Y extending Φ such that $\Psi(a) = b + \Phi(0)$. Moreover, we have $\Psi(ka + v) = kb + \Phi(v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

Now, since kb also commutes with the elements of $\Phi(v)$ for all $k \in \mathbb{Z}$ and $v \in V$, and Φ is also $\mathbb{Z}_0$-homogeneous and additive, we can already see that

$$\begin{aligned} n\Psi(ka + v) &= n(kb + \Phi(v)) = n(kb) + n\Phi(v) = k(nb) + \Phi(nv) \\ &\subset k\Phi(na) + \Phi(nv) = \Phi(k(na)) + \Phi(nv) = \Phi(k(na) + nv) \\ &\subset F(k(na) + nv) = F(n(ka) + nv) = F(n(ka + v)) \subset nF(ka + v) \end{aligned}$$

for all $k \in \mathbb{Z}_0$ and $v \in V$. Hence, by using the n-cancellability of Y, we can infer that

$$\Psi(ka + v) \subset F(ka + v)$$

for all $k \in \mathbb{Z}_0$ and $v \in V$. Moreover, we can also note that

$$\Psi(0a + v) = \Psi(v) = \Phi(v) \subset F(v) = F(0a + v)$$

for all $v \in V$. Therefore, we have $\Psi(ka + v) \subset F(ka + v)$ for all $k \in \mathbb{Z}$ and $v \in V$.

Remark 69 Note that if in particular $X \neq U \oplus V$, then by Theorem 28, we have $U \cap V \neq \{0\}$. Thus, there exists $n \in \mathbb{N}$ such that $na \in V$. Therefore, there exists $y \in Y$ such that $y \in \Phi(na)$.

Now, if in addition, Y is n-divisible, then we can state that there exists $b \in Y$ such that $y = nb$. Hence, we can see that $nb = y \in \Phi(na)$. Therefore, in this particular case, condition (1) automatically holds.

However, note that if in particular V is $\mathbb{N}$-divisible and X is $\mathbb{N}$-cancellable, then by Corollary 5, $X = U + V$ implies that $X = U \oplus V$. Therefore, in this particular case, the above remark and Theorem 56 cannot be applied.

15 Admissible Partial Selection Relations and Functions

Because of the two possibilities occurring in Theorems 55 and 56, it seems necessary to introduce the following

Definition 16 Let F be a relation of one group X to another Y, and suppose that Φ is an odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation of F.

Moreover, denote by $\mathcal{F}$, the family of all odd, $\mathbb{Z}_0$-semihomogeneous, quasiadditive partial selection relations Ψ of $F + \Phi(0)$ that extend Φ.

Then, the above partial selection relation Φ of F will be called admissible if every maximal member Ψ of $\mathcal{F}$ has the following two properties:

(1) for each $a \in X \setminus D_\Psi$, with $\mathbb{N}a \cap D_\Psi \neq \emptyset$, there exist $b \in Y$ and $n \in \mathbb{N}$ such that $nb \in \Psi(na)$,

(2) for each $a \in X \setminus D_\Psi$, with $\mathbb{N}a \cap D_\Psi = \emptyset$, there exists $b \in Y$ such that $nb \in (F * \Psi)(na)$ for all $n \in \mathbb{N}$.

Remark 70 Note that if Ψ is only a nonvoid odd, superadditive relation on X to Y, then by Theorem 20, D_Ψ is a subgroup of X, $0 \in \Psi(0)$, and Ψ is quasiadditive and $\mathbb{Z}$-superhomogeneous.

Remark 71 Therefore, if Ψ is an odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation of $F + \Phi(0)$ extending Φ, then we already have $\Psi \in \mathcal{F}$.

Also by Theorem 20, we have $0 \in \Phi(0)$. Therefore, by Corollary 12, Φ is also a partial selection relation of $F + \Phi(0)$. Thus, in particular we have $\Phi \in \mathcal{F}$.

Remark 72 Note that if Ψ is a relation on X to Y, then for every $a \in X$, with $\mathbb{N}a \cap D_\Psi \neq \emptyset$, there exists $n \in \mathbb{N}$ such that $na \in D_\Psi$. Therefore, $\Psi(na) \neq \emptyset$, and thus there exists $y \in Y$ such that $y \in \Psi(na)$.

Moreover, if in particular, Y is $\mathbb{N}$-divisible, then there exists $b \in Y$ such that $y = nb$, and thus $nb \in \Psi(na)$. Therefore, in this particular case, condition (1) automatically holds. However, the $\mathbb{N}$-divisibility of Y is a too strong restriction.

Remark 73 While, if in particular $\Psi \in \mathcal{F}$, then by using Theorem 47 and Corollary 17, we can see that

$$\Psi \subset (F + \Phi(0)) * \Psi = (F + \Psi(0)) * \Psi = F * \Psi.$$

Hence, by using the $\mathbb{Z}$-superhomogeneity of Ψ, we can already infer that

$$ky \in k\Psi(x) \subset \Psi(kx) \subset (F * \Psi)(kx)$$

for all $k \in \mathbb{Z}$, $x \in D_\Psi$ and $y \in \Psi(x)$.

Remark 74 Therefore, if $\Psi \in \mathcal{F}$ such that condition (2) holds, then for each $x \in X$ there exists $y \in Y$ such that $ny \in (F * \Psi)(nx)$ for all $n \in \mathbb{N}$.

Moreover, if F is odd and X and Y are commutative, then by Corollary 8, $F * \Psi$ is also odd. Therefore, we also have $ky \in (F * \Psi)(kx)$ for all $k \in \mathbb{Z}_0$.

In addition to Remark 71, we can also easily prove the following

Theorem 57 *If $\mathcal{G}$ is a nonvoid chain in the family $\mathcal{F}$ considered in Definition 16, then $\bigcup \mathcal{G} \in \mathcal{F}$.*

Proof Define $\Psi = \bigcup \mathcal{G}$. Then, since $G \subset F + \Phi(0)$ for all $G \in \mathcal{G}$, it is clear that $\Psi \subset F + \Phi(0)$. Thus, Ψ is also a partial selection relation of $F + \Phi(0)$.

Moreover, we can also note that

$$\Psi(x) = \Big(\bigcup_{G \in \mathcal{G}} G\Big)(x) = \bigcup_{G \in \mathcal{G}} G(x)$$

for all $x \in X$. Thus, in particular we also have $D_\Psi = \bigcup_{G \in \mathcal{G}} D_G$.

Furthermore, since each member of $\mathcal{G}$ is an extension of Φ and $\mathcal{G} \neq \emptyset$, we can also see that

$$\Psi(v) = \bigcup_{G \in \mathcal{G}} G(v) = \bigcup_{G \in \mathcal{G}} \Phi(v) = \Phi(v)$$

for all $v \in D_\Phi$. Therefore, Ψ is also an extension of Φ.

On the other hand, since relations preserve unions, we can also see that

$$\Psi(-x) = \bigcup_{G \in \mathcal{G}} G(-x) = \bigcup_{G \in \mathcal{G}} -G(x) = -\bigcup_{G \in \mathcal{G}} G(x) = -\Psi(x)$$

for all $x \in X$. Therefore, Ψ is also odd.

Moreover, if $x, y \in X$ and $z \in \Psi(x)$ and $w \in \Psi(y)$, then by the definition of Ψ, there exist $G_1, G_2 \in \mathcal{G}$ such that $z \in G_1(x)$ and $w \in G_2(y)$. Moreover, since $\mathcal{G}$ is a chain, we have either $G_1 \subset G_2$ or $G_2 \subset G_1$. Hence, it is clear that either

$$z + w \in G_1(x) + G_2(y) \subset G_2(x) + G_2(y) \subset G_2(x + y) \subset \Psi(x + y)$$

or

$$z + w \in G_1(x) + G_2(y) \subset G_1(x) + G_1(y) \subset G_1(x + y) \subset \Psi(x + y)$$

holds. Therefore, we have $\Psi(x) + \Psi(y) \subset \Psi(x + y)$, and thus, Ψ is also superadditive.

Furthermore, if $x \in D_\Psi$, $n \in \mathbb{N}$ and $y \in \Psi(nx)$, then by the definition Ψ there exist $G_1, G_2 \in \mathcal{G}$ such that $x \in D_{G_1}$ and $y \in G_2(nx)$. Moreover, since $\mathcal{G}$ is a chain, we have either $G_1 \subset G_2$ or $G_2 \subset G_1$. If $G_1 \subset G_2$ holds, then $x \in D_{G_1}$ implies $x \in D_{G_2}$. Therefore, we have

$$y \in G_2(nx) \subset nG_2(x) \subset n\Psi(x).$$

While, if $G_2 \subset G_1$ holds, then by using that $x \in D_{G_1}$ we can see that

$$y \in G_2(nx) \subset G_1(nx) \subset nG_1(x) \subset n\Psi(x).$$

Therefore, we have $\Psi(nx) \subset n\Psi(x)$. Thus, Ψ is also $\mathbb{N}$-semi-subhomogeneous. Hence, by Remark 71, we can infer that $\Psi \in \mathcal{F}$.

Remark 75 Note that if in particular the members of $\mathcal{F}$ were supposed to be $\mathbb{N}$-homogeneous, then we could more easily prove that the relation Ψ defined in the above proof is also $\mathbb{N}$-homogeneous.

However, in contrast to the $\mathbb{N}$-semi-subhomogeneity, the assumption of the $\mathbb{N}$-subhomogeneity of the partial selection function Φ is a very strong restriction. Namely, in this case, we have $nx \in X \setminus D_\Phi$ for all $n \in \mathbb{N}$ and $x \in X \setminus D_\Phi$.

Remark 76 On the other hand, it is also worth noticing that if $\Psi \in \mathcal{F}$, then by Remark 32 and Theorem 27 we have $\Psi(x) = \psi(x) + \Psi(0) = \psi(x) + \Phi(0)$ for any $x \in X$ and selection function ψ of Ψ.

Remark 77 Note that if in particular Φ is a function, then because of $0 \in \Phi(0)$, we have $\Phi(0) = \{0\}$.

Therefore, by Remark 76 and the equality $F + \Phi(0) = F$, every member Ψ of $\mathcal{F}$ is a partial selection function F.

Remark 78 Now, we can also note that if φ is only a nonvoid, superadditive function on X to Y, with a symmetric domain, then by Theorem 21 D_f is a subgroup of X, $\varphi(0) = 0$, and φ is odd, quasiadditive and $\mathbb{Z}$-semihomogeneous.

Therefore, by Definition 16, we can speak of the admissibility of φ as well. Moreover, we also have $F + \varphi(0) = F$.

From the latter remarks, it is clear that in particular we also have the following

Theorem 58 *Let F be a relation of one group X to another Y, and suppose that φ is a nonvoid, superadditive partial selection function of F with a symmetric domain.*

Moreover, denote by $\mathcal{F}$ the family of all odd, $\mathbb{Z}$-semihomogeneous, quasiadditive partial selection functions ψ of F that extend φ.

Then, the above φ is admissible, in the sense of Definition 16, if and only if every maximal member ψ of $\mathcal{F}$ has the following two properties:

(1) for each $a \in X \setminus D_\psi$, with $\mathbb{N}a \cap D_\psi \neq \emptyset$, there exist $b \in Y$ and $n \in \mathbb{N}$ such that $nb = \psi(na)$;
(2) for each $a \in X \setminus D_\psi$, with $\mathbb{N}a \cap D_\psi = \emptyset$, there exists $b \in Y$ such that $nb \in (F * \psi)(na)$ for all $n \in \mathbb{N}$.

Remark 79 Note that, because of Corollary 4, we do not need the widely used fact that chained unions of functions are also functions.

16 The Main Extension Theorems of Additive Partial Selection Relations

Now, by using Theorems 55 and 56, we can easily prove the following

Theorem 59 *Suppose that F is an odd, $\mathbb{N}$-subhomogeneous relation of a commutative group X to an $\mathbb{N}$-cancellable, commutative group Y.*

Then, every admissible, nonvoid odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation Φ of F can be extended to a total, $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of $F + \Phi(0)$.

Proof Let $\mathcal{F}$ be as in Definition 16. Then, by Remark 71, we have $\Phi \in \mathcal{F}$ and thus $\mathcal{F} \neq \emptyset$. Moreover, by Theorem 57, we have $\bigcup \mathcal{G} \in \mathcal{F}$ for any nonvoid chain $\mathcal{G}$ in $\mathcal{F}$.

Therefore, by a particular case of Zorn lemma [29, p. 33], there exists a maximal element Ψ of $\mathcal{F}$. Thus, in particular Ψ is an odd, $\mathbb{Z}_0$-semihomogeneous, quasiadditive partial selection relation of $F + \Phi(0)$ extending Φ such that D_Ψ is a subgroup of X and $0 \in \Psi(0)$.

Therefore, to complete the proof, we need only to show that $D_\Psi = X$. For this, assume on the contrary that there exists $a \in X$ such that $a \notin D_\Psi$, and define $Z = U + D_\Psi$ with $U = \mathbb{Z}a$.

Then, since U and D_Ψ are subgroups of X and X is commutative, it is clear that Z is a subgroup of X. Moreover, we can also note that $a \in Z$ and $D_\Psi \subset Z$. Thus, in particular $D_\Psi \neq Z$ since $a \notin D_\Psi$.

Furthermore, by using the oddness and $\mathbb{N}$-semi-subhomogeneity of F and Φ, and the commutativity of Y, we can also easily see that

$$
\begin{aligned}
(F + \Phi(0))(-x) &= F(-x) + \Phi(0) = F(-x) + \Phi(-0) \\
&\subset -F(x) - \Phi(0) = -(F(x) + \Phi(0)) = -(F + \Phi(0))(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(F + \Phi(0))(nx) &= F(nx) + \Phi(0) = F(nx) + \Phi(n0) \\
&\subset nF(x) + n\Phi(0) = n(F(x) + \Phi(0)) = n(F + \Phi(0))(x)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $x \in X$. Therefore, $F + \Phi(0)$ is also odd and $\mathbb{N}$-subhomogeneous.

Now, if $\mathbb{N}a \cap D_\Psi \neq \emptyset$, then by (1) in Definition 16, we can state that there exist $b \in Y$ and $n \in \mathbb{N}$ such that $nb \in \Psi(na)$. Hence, by using Theorem 56 and the commutativity of X and Y, we can see that there exists a $\mathbb{Z}_0$-homogeneous, additive relation Ω of Z to Y extending Ψ such that $\Omega \subset F + \Phi(0)$.

While, if $\mathbb{N}a \cap D_\Psi = \emptyset$, then by using the symmetry of D_Ψ, we can note that $U \cap D_\Psi = \{0\}$. Therefore, by Theorem 28, now we actually have $Z = U \oplus D_\Psi$. Moreover, by (2) in Definition 16, we can state that there exists $b \in Y$ such that $nb \in (F * \Psi)(na)$ for all $n \in \mathbb{N}$. Hence, by using Remark 73, we can infer that

$$nb \in ((F + \Phi(0)) * \Psi)\,(na)$$

also holds for all $n \in \mathbb{N}$. Moreover, now we can also note that $na \neq 0$ for all $n \in \mathbb{N}$. Thus, by Theorem 55 and the commutativity of X and Y, we can state that there exists an odd, additive relation Ω of Z to Y extending Ψ such that

$$\Omega \subset F + \Phi(0) + \Psi(0) = F + \Phi(0) + \Phi(0) = F + \Phi(0).$$

Moreover, by Theorem 55, we also have $\Omega(ka + v) = kb + \Psi(v)$ for all $k \in \mathbb{Z}$ and $v \in D_\Psi$. Hence, by using the $\mathbb{Z}_0$-semihomogeneity of Ψ and the commutativity of X and Y, we can easily see that

$$\Omega\,(l(ka + v)) = \Omega(lka + lv) = lkb + \Psi(lv)$$

$$= lkb + l\Psi(v) = l(ka + \Psi(v)) = l\Omega(ka + v)$$

for all $v \in D_\Psi$ and $k, l \in \mathbb{Z}$ with $l \neq 0$. Therefore, Ω is also $\mathbb{Z}_0$-homogeneous as a relation of Z to Y.

Thus, in both cases, Ω is a $\mathbb{Z}_0$-homogeneous, additive relation of Z to Y extending Ψ such that $\Omega \subset F + \Phi(0)$. Hence, since Z is a subgroup of X, we can easily see that Ω is an odd, $\mathbb{Z}_0$-semihomogeneous and quasiadditive as a relation on X to Y. Thus, since Ω is an extension of Φ too, we can see that $\Omega \in \mathcal{F}$. Hence, by the maximality of Ψ in $\mathcal{F}$, we can infer that $\Omega = \Psi$, and thus $Z = D_\Omega = D_\Psi$. This contradiction proves that $D_\Psi = X$.

Now, from the above theorem, by using Remarks 77 and 78, we can easily get

Corollary 18 *If F is as in Theorem 59, then every admissible, nonvoid, superadditive partial selection function φ of F, with a symmetric domain, can be extended to a total, $\mathbb{Z}$-homogeneous, additive selection function ψ of F.*

Moreover, by using Theorem 59, we can also easily prove the following counterpart of [56, Theorem 9.1].

Theorem 60 *Suppose that F is an odd, $\mathbb{N}$-subhomogeneous, superadditive relation of a commutative group X to an $\mathbb{N}$-cancellable, commutative group Y.*

Then, every nonvoid odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation Φ of F can be extended to a total, $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of F.

Proof Now, by Theorem 20 and the assumption $\Phi \subset F$, we have $0 \in \Phi(0) \subset F(0)$. Therefore, by Corollary 14, we have $F = F + \Phi(0)$. Thus, by Theorem 59, we need only to show that Φ is admissible in the sense of Definition 16.

For this, assume that if Θ is an odd, $\mathbb{Z}_0$-semihomogeneous, quasiadditive partial selection relation of $F + \Phi(0)$ that extends Φ. Then, since $F = F + \Phi(0)$, Θ is also such a partial selection relation of F. Moreover, since $0 \in \Phi(0) = \Theta(0)$, we also have $\Theta(0) \neq \emptyset$. Thus, by Corollary 16,

$$F * \Theta = F + \Theta(0) = F + \Phi(0) = F.$$

Moreover, by Theorem 12, we can see that F is, in particular, $\mathbb{N}$-superhomogeneous. Therefore,

$$ny \in nF(x) \subset F(nx) = (F * \Theta)(nx)$$

for all $n \in \mathbb{N}$, $x \in X$ and $y \in F(x)$. Thus, the conditions (1) and (2) of Definition 16, with Θ in place of Ψ, are substantially satisfied. Therefore, Φ is admissible.

From the above theorem, by taking $\Phi = \{0\} \times \Phi(0)$, we can easily derive

Corollary 19 *Suppose that F is as in Theorem 60, and Z is an $\mathbb{N}$-divisible subgroup of Y such that $Z \subset F(0)$.*

Then, there exists a $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of F such that $\Psi(0) = Z$.

From this corollary, by taking $Z = \{0\}$, we can immediately get

Corollary 20 *If F is as in Theorem 60, then there exists a $\mathbb{Z}$-homogeneous, additive selection function ψ of F.*

17 Some Further Theorems on the Extensions of Additive Partial Selection Relations

Now, by using Corollaries 18 and 20, we can also prove the following

Theorem 61 *Suppose that F is an odd, $\mathbb{N}$-subhomogeneous relation of a commutative group X to an $\mathbb{N}$-cancellable, commutative group Y.*

Moreover, assume that each nonvoid odd, $\mathbb{Z}$-semihomogeneous, quasiadditive partial selection function φ of F is admissible.

Then, each nonvoid odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation Φ of F can be extended to a total, $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of $F + \Phi(0)$.

Proof If Φ is as above, then by Theorem 20 and Corollary 3, we can see that D_Φ is a subgroup of X, $0 \in \Phi(0)$, and Φ is a $\mathbb{Z}_0$-homogeneous, additive relation of D_Φ to Y. Thus, in particular, by Corollary 20, there exists a $\mathbb{Z}$-homogeneous, additive selection function φ of Φ.

Note that now φ is a nonvoid odd, $\mathbb{Z}$-semihomogeneous, quasiadditive partial selection function of F. Namely, the semioddness of φ implies the oddness of φ by Remark 25. And the semiadditivity of φ implies the quasiadditivity of φ by Remark 13 and Theorem 20.

Therefore, by the assumption of the theorem, φ is admissible. Thus, in particular, by Corollary 18, φ can be extended to a total, $\mathbb{Z}$-homogeneous, additive selection function ψ of F.

Define $\Psi = \psi + \Phi(0)$. Then, since

$$\Psi(x) = (\psi + \Phi(0))(x) = \psi(x) + \Phi(0)$$

for all $x \in X$, and $\Phi(0) \neq \emptyset$, it is clear that Ψ is a relation of X to Y.

Moreover, by using the corresponding homogeneity and additivity properties of ψ and Φ, and the commutativity of Y, we can also easily see that

$$\begin{aligned} \Psi(kx) &= \psi(kx) + \Phi(0) = \psi(kx) + \Phi(k0) \\ &= k\psi(x) + k\Phi(0) = k(\psi(x) + \Phi(0)) = k\Psi(x) \end{aligned}$$

and

$$\begin{aligned} \Psi(x+y) &= \psi(x+y) + \Phi(0) = \psi(x+y) + \Phi(0+0) \\ &= \psi(x) + \psi(y) + \Phi(0) + \Phi(0) \\ &= \psi(x) + \Phi(0) + \psi(y) + \Phi(0) = \Psi(x) + \Psi(y) \end{aligned}$$

for all $k \in \mathbb{Z}_0$ and $x, y \in X$. Therefore, Ψ is also $\mathbb{Z}_0$-homogeneous and additive.

On the other hand, since ψ is a selection function of F, it is clear that

$$\Psi(x) = \psi(x) + \Phi(0) \subset F(x) + \Phi(0) = (F + \Phi(0))(x)$$

for all $x \in X$, and thus Ψ is a selection relation of $F + \Phi(0)$.

Moreover, by using the corresponding properties of φ, and Remark 32 and Theorem 27, we can also easily see that

$$\Psi(x) = \psi(x) + \Phi(0) = \varphi(x) + \Phi(0) = \Phi(x)$$

for all $x \in D_\Phi$. Therefore, Ψ is an extension of Φ.

Moreover, as a certain converse to Theorem 61, we can also prove

Theorem 62 *Suppose that F is a relation of one group X to another Y such that every nonvoid odd, $\mathbb{Z}_0$-semihomogeneous, quasiadditive partial selection relation Θ of F can be extended to a total, $\mathbb{N}$-superhomogeneous, subadditive selection relation Ω of $F + \Theta(0)$.*

Then, every nonvoid odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation Φ of F, with $F + \Phi(0) = F$, is admissible.

Proof Suppose that Φ is as above, and Ψ is an odd, $\mathbb{Z}_0$-semihomogeneous, quasiadditive partial selection relation $F + \Phi(0)$ extending Φ.

Then, because of $F + \Psi(0) = F + \Phi(0) = F$ and the assumption of the theorem, Ψ can be extended to a total, $\mathbb{N}$-superhomogeneous, subadditive selection relation Ω of F.

Now, by taking $x \in X$ and $y \in \Omega(x)$, we can see that

$$ny \in n\Omega(x) \subset \Omega(nx) = \Psi(nx)$$

for all $n \in \mathbb{N}$ with $nx \in D_\Psi$.

Moreover, by using Theorem 47 and Corollary 7, we can also see that $\Omega \subset F * \Omega \subset F * \Psi$. Hence, by taking $x \in X$ and $y \in \Omega(x)$, we can see that

$$ny \in n\Omega(x) \subset \Omega(nx) \subset (F * \Psi)(nx)$$

for all $n \in \mathbb{N}$. Thus, the conditions (1) and (2) of Definition 16 are substantially satisfied. Therefore, Φ is admissible.

Remark 80 Note that if in particular Φ is a function, then because of $\Phi(0) = \{0\}$, we have $F + \Phi(0) = F$.

While, if in particular F is right-zero-superadditive, then because of $0 \in \Phi(0) \subset F(0)$ and Corollary 14, we also have $F + \Phi(0) = F$.

Now, as an immediate consequence of Theorems 61 and 62, we can also state

Corollary 21 *If F is as in Theorem 61, then the following are equivalent:*

(1) each nonvoid odd, $\mathbb{Z}$-semihomogeneous, quasiadditive partial selection function φ of F is admissible;
(2) each nonvoid odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation Φ of F can be extended to a total, $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of $F + \Phi(0)$.

18 A Strong Totality Property of the Intersection Convolution

Definition 17 A family $\mathcal{B}$ of sets is said to have the binary intersection property if $U \cap V \neq \emptyset$ for all $U, V \in \mathcal{B}$.

Remark 81 This terminology differs from that of Nachbin [35] and his close followers. But, it is in accordance with the usual definition of the finite intersection property [29, p. 135]. Now, by extending an argument of Gajda et al. [18], we can prove the following counterpart of an improvement [53, Theorem 5.4] of the second author.

Theorem 63 *Suppose that F and G are relations on a group X to a vector space Y over $\mathbb{K}$ such that:*

(1) $F(x) \cap G(x) \neq \emptyset$ for all $x \in D_F \cap D_G$;
(2) F and G are odd, semi-subadditive, and $\mathbb{N}$-semi-subhomogeneous;
(3) D_F and D_G are closed under addition and elementwise commuting with X.

Then, for any $x \in X$, the family

$$\left\{ n^{-1}\big(F(nx - v) + G(v)\big) : \qquad n \in \mathbb{N}, \quad v \in (-D_F + nx) \cap D_G \right\}$$

has the binary intersection property.

Proof Suppose that $n, m \in \mathbb{N}$, and

$$v \in (-D_F + nx) \cap D_G \qquad \text{and} \qquad t \in (-D_F + mx) \cap D_G .$$

Then, $v \in -D_F + nx$ and $t \in -D_F + mx$, and $v, t \in D_G$. Hence, since D_G is symmetric and closed under addition, it is clear that

$$nt - mv \in nD_G - mD_G \subset D_G - D_G = D_G + D_G \subset D_G .$$

Moreover, since D_F is symmetric, closed under addition, and elementwise commutes with X, by using Theorem 6 and the corresponding definitions, we can also see that

$$\begin{aligned} nt - mv &\in n(-D_F + mx) - m(-D_F + nx) \\ &= -nD_F + nmx + mD_F - mnx = -nD_F + mD_F + nmx - nmx \\ &= -nD_F + mD_F \subset -D_F + D_F = D_F + D_F \subset D_F . \end{aligned}$$

Therefore, $nt - mv \in D_F \cap D_G$. Hence, by using (1), we can infer that

$$F(nt - mv) \cap G(nt - mv) \neq \emptyset, \quad \text{and thus} \quad 0 \in F(nt - mv) - G(nt - mv).$$

On the other hand, we can also note that

$$nx - v \in nx - (-D_F + nx) = nx - nx + D_F = D_F$$

and

$$mx - t \in mx - (-D_F + mx) = mx - mx + D_F = D_F.$$

Now, since D_G is symmetric, closed under addition, and elementwise commutes with X, by using Theorem 6, condition (2) and Theorem 19, we can see that

$$\begin{aligned} F(nt - mv) &= F(-mv + nt) = F(mnx - nmx - mv + nt) \\ &= F(mnx - mv - nmx + nt) = F(m(nx - v) - n(mx - t)) \\ &\subset F(m(nx - v)) + F(-n(mx - t)) \subset mF(nx - v) - nF(mx - t) \end{aligned}$$

and

$$G(nt - mv) \subset G(nt) + G(-mv) \subset nG(t) - mG(v).$$

Hence, by the commutativity of Y, it is clear that

$$\begin{aligned} F(nt - mv) - G(nt - mv) &\subset mF(nx - v) - nF(mx - t) - nG(t) + mG(v) \\ &= mF(nx - v) + mG(v) - nF(mx - t) - nG(t) \\ &= m(F(nx - v) + G(v)) - n(F(mx - t) - G(t)) \end{aligned}$$

Now, by using that $0 \in F(nt - mv) - G(nt - mv)$, we can also see that

$$\begin{aligned} 0 &= (nm)^{-1}0 \in (nm)^{-1}(F(nt - mv) - G(nt - mv)) \\ &\subset (nm)^{-1}\,(m(F(nx - v) + G(v)) - n(F(mx - t) - nG(t))) \\ &= n^{-1}(F(nx - v) + G(v)) - m^{-1}(F(mx - t) + G(t)). \end{aligned}$$

Therefore,

$$n^{-1}(F(nx - v) + G(v)) \cap m^{-1}(F(mx - t) + G(t)) \neq \emptyset,$$

and thus the required assertion is also true.

The following plausible terminology was first introduced in [53].

Definition 18 A family $\mathcal{B}$ of subsets of a set Y is called a Nachbin system in Y if for every subfamily $\mathcal{C}$ of $\mathcal{B}$, having the binary intersection property, we have $\bigcap \mathcal{C} \neq \emptyset$.

Remark 82 Quite similarly, a family of subsets of a set may be called a Riesz system if every subfamily of it having the finite intersection property has a nonvoid intersection.

Moreover, a family of subsets of a uniform space may be called a Cantor system if every subfamily of it containing small sets and having the finite intersection property has a nonvoid intersection.

Namely, according to Kelley [29, pp. 136, 193], this terminology allows us to briefly state that a topological (uniform) space is compact (complete) if and only if the family of its closed subsets forms a Riesz (Cantor) system.

Example 2 It can be easily seen that the family $\mathcal{B}$ of all closed balls in $\mathbb{R}$ is a Nachbin system. (This generalization of Cantor's nested interval property was already used

by E. Helly in 1912.) Unfortunately, the same assertion is no longer true in $\mathbb{R}^2$. (The appropriate generalization to convex subsets of $\mathbb{R}^n$ was found in 1913 by E. Helly who could not publish it until 1923.)

However, as a straightforward, but less important generalization of Example 2, one can easily establish the following example. (The results of E. Helly and some more delicate examples for Nachbin systems can be found in the expository paper [14] of Fuchssteiner and Horváth.)

Example 3 If S is a nonvoid set, then the family $\mathcal{B}$ of all closed balls in the supremum-normed space $\mathfrak{B}(X, \mathbb{R})$ of all bounded functions of S to $\mathbb{R}$ is also Nachbin system.

Remark 83 In this respect, it is also worth noticing that the family $\mathcal{B}$ of all closed balls in a normed space Y over $\mathbb{K}$ is invariant under translation by $x \in X$ and multiplication by $\lambda \in \mathbb{K}_0$.

Now, as a useful consequence of Theorems 63 and 39, we can also state

Corollary 22 *If F and G are as in Theorem 63, and there exists a Nachbin system $\mathcal{B}$ in Y such that*
(4) $n^{-1}(F(nx-v)+G(v)) \in \mathcal{B}$ *for all* $n \in \mathbb{N},\ x \in X$ *and* $v \in (-D_F+nx)\cap D_G$,
then we have $\bigcap_{n=1}^{\infty} n^{-1}(F*G)(nx) \neq \emptyset$ *for all* $x \in X$.

Proof Now, by Theorems 39 and 63, it is clear that

$$\begin{aligned}
&\bigcap_{n=1}^{\infty} n^{-1}(F*G)(nx)\\
&=\bigcap_{n=1}^{\infty} n^{-1}\bigcap \{F(nx-v)+G(v): \quad v \in (-D_F+nx)\cap D_G\}\\
&=\bigcap_{n=1}^{\infty} \bigcap \left\{n^{-1}(F(nx-v)+G(v)): \quad v \in (-D_F+nx)\cap D_G\right\}\\
&=\bigcap \left\{n^{-1}(F(nx-v)+G(v)): \quad n \in \mathbb{N}, \quad v \in (-D_F+nx)\cap D_G\right\} \neq \emptyset.
\end{aligned}$$

Remark 84 By using the notation $F^{\star} = \bigcap_{n=1}^{\infty} F_n$ of [70], with $F_n(x) = n^{-1}F(nx)$, the assertion the above theorem can be briefly expressed by saying that $(F*G)^{\star}$ is a total relation on X to Y.

19 A General Hahn–Banach Type Extension Theorem

Because of Remark 83, we may naturally introduce the following

Definition 19 A family $\mathcal{B}$ of subsets of a vector space Y over $\mathbb{K}$ will be called admissible if

(1) $n^{-1}B \in \mathcal{B}$ for all $n \in \mathbb{N}$ and $B \in \mathcal{B}$;
(2) $y + B \in \mathcal{B}$ for all $y \in Y$ and $B \in \mathcal{B}$.

Remark 85 By using our former conventions, the above properties can be briefly expressed by writing that:

(1) $n^{-1}\mathcal{B} \subset \mathcal{B}$ for all $n \in \mathbb{N}$, or equivalently $\mathcal{B} \subset n\mathcal{B}$ for all $n \in \mathbb{N}$;
(2) $y + \mathcal{B} \subset \mathcal{B}$ for all $y \in Y$, or equivalently $y + \mathcal{B} = \mathcal{B}$ for all $y \in Y$.

Therefore, (1) and (2) are certain $\mathbb{N}$-divisibility and translation–invariance properties of the family $\mathcal{B}$ in the space $\mathcal{P}(Y)$ of all subsets of Y.

By using the above terminology, we can now briefly formulate the next useful consequence of Corollary 22.

Theorem 64 *Suppose that F is a relation and g is a function on a group X to a vector space Y over $\mathbb{K}$, and $\mathcal{B}$ is an admissible Nachbin system in Y such that:*

(1) $F(x) \in \mathcal{B}$ for all $x \in D_F$,
(2) $g(x) \in F(x)$ for all $x \in D_F \cap D_g$,
(3) D_F and D_g are subgroups of X and elementwise commuting with X,
(4) F is odd, semi-subadditive and $\mathbb{N}$-semi-subhomogeneous, and g is semi-subadditive.

Then, we have $\bigcap_{n=1}^{\infty} n^{-1}(F * g)(nx) \neq \emptyset$ for all $x \in X$.

Proof If $n \in \mathbb{N}$ and $v \in (-D_F + nx) \cap D_g$, then $v \in -D_F + nx$ and $v \in D_g$. Hence, we can see that $nx - v \in D_F$. Thus, $F(nx - v) \in \mathcal{B}$ and $g(v) \in Y$.

Hence, since $\mathcal{B}$ is admissible, we can already see that

$$n^{-1}(F(nx - v) + g(v)) = n^{-1}F(nx - v) + n^{-1}g(v) \in \mathcal{B}.$$

Thus, Corollary 22 can be applied to get the required assertion.

Namely, now we have not only $g(x + y) \subset g(x) + g(y)$, but also $g(x + y) = g(x) + g(y)$ for all $x, y \in D_g$, since D_g is closed under addition. Thus, by Remark 13, g is superadditive. Moreover, by Theorem 21, g is odd and $\mathbb{Z}$-semihomogeneous, since D_g is now also symmetric.

From the above theorem, it is clear that in particular we also have

Corollary 23 *Suppose that F is an odd, $\mathbb{N}$-subhomogeneous, subadditive relation of a commutative group X to a vector space Y over $\mathbb{K}$, and there exists an admissible Nachbin system $\mathcal{B}$ in Y such that $F(x) \in \mathcal{B}$ for all $x \in X$.*

Then, we have

$$\bigcap_{n=1}^{\infty} n^{-1}(F * \varphi)(nx) \neq \emptyset$$

for any $x \in X$ and semi-subadditive partial selection function φ of F such that D_φ is a subgroup of X.

Now, as an important consequence of Theorems 61, we can also easily establish the following straightforward generalization [18, Theorem 1] of Z. Gajda, A. Smajdor, and W. Smajdor.

Theorem 65 *Suppose that F is an odd, $\mathbb{N}$-subhomogeneous, subadditive relation of a commutative group X to a vector space Y over $\mathbb{K}$, and there exists an admissible Nachbin system $\mathcal{B}$ in Y such that $F(x) \in \mathcal{B}$ for all $x \in X$.*

Then, each nonvoid odd, $\mathbb{N}$-semi-subhomogeneous, superadditive partial selection relation Φ of F can be extended to a total, $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of $F + \Phi(0)$.

Proof By Theorem 61, it is enough to show only that each nonvoid odd, $\mathbb{Z}$-semihomogeneous, quasiadditive partial selection function φ of F is admissible in the sense of Definition 16.

For this, by Theorem 58, we may assume that ψ is an odd, $\mathbb{Z}$-semihomogeneous, quasiadditive partial selection function of F that extends φ. Then, by Theorem 20, D_ψ is a subgroup of X.

Hence, by Corollary 23, we can see that $\bigcap_{n=1}^{\infty} n^{-1}(F * \psi)(nx) \neq \emptyset$ for all $x \in X$. Thus, for each $x \in X$, there exists $y \in Y$ such that $y \in n^{-1}(F * \psi)(nx)$, and thus $ny \in (F * \psi)(nx)$ for all $n \in \mathbb{N}$. Therefore, the condition (2) of Theorem 58 is satisfied.

Moreover, we can also note that if $x \in X$ such that $nx \in D_\psi$ for some $n \in \mathbb{N}$, then there exists $z \in Y$ such that $z = \psi(nx)$. Hence, by taking $y = n^{-1}z$, we can see that $y = n^{-1}z = n^{-1}\psi(nx)$, and thus $ny = \psi(nx)$. Therefore, the condition (1) of Theorem 58 is also satisfied.

Now, from this theorem, by using Remarks 77 and 78, we can derive

Corollary 24 *If F is as in Theorem 65, then every nonvoid, superadditive selection function ϕ of F with a symmetric domain can be extended to a total, $\mathbb{Z}$-homogeneous, additive selection function ψ of F.*

Moreover, by using Theorem 65, we can also easily prove the following

Corollary 25 *Suppose that F is as in Theorem 65. Moreover, assume that Z is a subspace of Y such that $Z \subset F(0)$.*

Then, there exists a $\mathbb{Z}_0$-homogeneous, additive selection relation Ψ of $F + Z$ such that $\Psi(0) = Z$.

Hence, it is clear that in particular we also have

Corollary 26 *If F is as in Theorem 65 and $0 \in F(0)$, then there exists a $\mathbb{Z}$-homogeneous, additive selection function ψ of F.*

Concluding Remarks We note that certain converses of our former results on constructions and extensions of additive relations are also true. Moreover, by using the arguments applied in Kuczma [31, Chap. 8], some of our extension theorems can certainly be subtantially improved.

The existence of additive selections and Hahn–Banach type extension theorems for set-valued functions have formerly been investigated not only by the authors mentioned in the introduction, but also by Godini [22], Nikodem [36], A. Smajdor

[48], Sablik [47]; and Abreu and Etcheberry [1], Meng [33], Peng et al. [40], and Zălinescu [75], respectively.

To prove Hyers–Ulam type stability theorems, in contrast to the direct methods, the techniques of invariant means and fixed point theorems, Hahn–Banach type extension and separation theorems seem to have been used only by Gajda et al. [18], Páles [39], Badora [3], and Huang and Li [24]. Therefore, it would be of some interest to prove some alternate forms of the Hyers–Ulam type stability theorems with the help of Theorem 65.

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Extremal Problems in Polynomials and Entire Functions

N. K. Govil and Q. M. Tariq

Abstract The subject of extremal problems for polynomials and related classes of functions plays an important and crucial role in obtaining inverse theorems in approximation theory. Frequently, the further progress in inverse theorems has depended upon first obtaining the corresponding analogue or generalization of Markov's and Bernstein's inequalities, and these inequalities have been the starting point of a considerable literature in Mathematics.

In this chapter, we begin with the earliest results in the subject (Markov's and Bernstein's inequalities), and present some of their generalizations and refinements. In the process, some of the problems that are still open have also been mentioned. Since there are many results in this subject, we have concentrated here mainly on results concerning Bernstein's inequality.

The chapter contains four sections, with Sect. 1 dealing with introduction to Bernstein's and Markov's inequalities along with some of their generalizations. In Sect. 2, we discuss some constrained Bernstein type inequalities, that is Bernstein type inequalities for some classes of polynomials, while in Sect. 3 the extension to entire functions of exponential type for some of the results of Sect. 2 has been discussed. Finally, Sect. 4 contains some of the open problems, discussed in the text of this chapter, that could be of interest to some of the readers.

Keywords Functions of exponential type· Bernstein's inequality · Polynomials · Inequalities in the complex domain

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1 Introduction

Boas [15] in his paper describes the chemical problem that Mendeleev, the inventor of the periodic table of the elements, was interested in and how he arrived at the question about the upper bound for the first derivative of an algebraic polynomial. In mathematical term, Mendeleev was interested in knowing how large $|f'(x)|$ can be on the interval $[-1, 1]$, where $f(x) = ax^2 + bx + c$ is a quadratic polynomial such that $|f(x)| \le 1$ for $-1 \le x \le 1$. Even though he was a chemist, he was able to prove that $|f'(x)| \le 4$ and even managed to show that the estimate is best possible in a sense that there is a quadratic polynomial $f(x) = 1 - 2x^2$ for which $|f(x)| \le 1$ on $[-1, 1]$ but $|f'(\pm 1)| = 4$. Mendeleev shared his result with his contemporary mathematician A. A. Markov who investigated the more general case of polynomial of degree n, which later came to be known as Markov's Theorem [62]. Markov's result was published in Russian language. The English translation of the paper is prepared by Carl de Boor and Olga Holtz [70]. It is stated below.

Theorem 1 *Let $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ be an algebraic polynomial of degree n such that $|f(x)| \le 1$ for $-1 \le x \le 1$. Then*

$$|f'(x)| \le n^2 \qquad (-1 \le x \le 1). \tag{1}$$

The inequality is sharp. Equality holds only if $f(x) = \gamma T_n(x)$, where γ is a complex number such that $|\gamma| = 1$ and

$$T_n(x) = \cos\left(n \cos^{-1} x\right) = 2^{n-1} \Pi_{\nu=1}^{n} \left\{x - \cos\left(\left(\nu - \frac{1}{2}\right)\pi/n\right)\right\} \tag{2}$$

is the nth degree Tchebycheff polynomial of the first kind. It can be easily verified that $|T_n(x)| \le 1$ for $-1 \le x \le 1$ and $|T_n'(1)| = n^2$.

Once a sharp inequality for the derivative of a polynomial is known, it is quite natural to ask: *What will be the corresponding inequality for the kth order derivative where $k \le n$, the degree of the polynomial?* Since the derivative of an nth degree polynomial is a polynomial of degree $n - 1$, Morkov's theorem can be successively applied to $f, f' \cdots$ to obtain the following estimate for the polynomials considered in Theorem 1.

$$|f^{(k)}(x)| \le n^2(n-1)^2 \ldots (n-k+1)^2.$$

This approach however, does not produce the sharp estimate for $|f^{(k)}(x)|$ on the interval $[-1, 1]$. V. Markov (half brother of A. Markov) in his paper entitled *On functions deviating least from zero in a given interval* proved the extension of Theorem 1 for the higher order derivatives. His original paper was in Russian language but later on it was translated into German, with a short foreword by Bernstein [63].

Theorem 2 *Let $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ be an algebraic polynomial of degree n such that $|f(x)| \le 1$ for $-1 \le x \le 1$. Then*

$$|f^{(k)}(x)| \le \frac{(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1.3\cdots(2k-1)} \qquad (-1 \le x \le 1). \tag{3}$$

The inequality is sharp. Equality holds only if $f(x) = \gamma T_n(x)$ *with* $|\gamma| = 1$. *It can be easily verified that* $|T_n^{(k)}(1)| = (n^2 - 1^2)(n^2 - 2^2)\cdots\left(n^2 - (k-1)^2\right)/1.3\cdots(2k-1)$.

Let $f(x) = x^n + \sum_{\nu=0}^{n-1} a_\nu x^\nu$ be a monic polynomial of degree n. Tchebycheff proved that

$$\|f\| = \|x^n + a_{n-1}x^{n-1} + \cdots a_0\| \geq \frac{1}{2^{n-1}} \tag{4}$$

where $\|f\| = \max_{-1\leq x\leq 1}|f(x)|$ represents the uniform norm of f on the interval $[-1, 1]$.

It is worth mentioning that a special case of Theorem 2 is contained in the above result on monic polynomials which Tchebycheff proved some 38 years [18] before the proof of Theorem 2 was published. Since $\|f^{(n)}\| = n!$, inequality (4) may be written as

$$\|f^{(n)}\| \leq 2^{n-1}n!\|f\|$$

which is nothing but the special case ($k = n$) of Theorem 2.

It was S. Bernstein who recognized the significance of the works of A. Markov and V. Markov when he started his studies in the theory of approximation of functions by polynomials in order to answer the following question posed by de la Vallee Poussin. *Is it possible to approximate every polygonal line by polynomials of degree n with an error of* $o(1/n)$?

In that connection, he proved and made considerable use of the following inequality in answering the question raised by de la Vallee Poussin in the negative.

Theorem 3 *Let* $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ *be an algebraic polynomial of degree n such that* $|f(x)| \leq 1$ *for* $-1 \leq x \leq 1$. *Then*

$$|f'(x)| \leq n(1-x^2)^{-1/2} \qquad (-1 < x < 1). \tag{5}$$

The equality is attained at the points $x = x_\nu = \cos(2\nu - 1)\pi/2n$, $1 \leq \nu \leq n$ *if and only if* $f(x) = \gamma T_n(x)$ *where* γ *is a complex number such that* $|\gamma| = 1$.

Note that the above theorem provides a point-wise estimate of the derivative on the interval $(-1, 1)$. A. Markov's inequality given in Theorem 1 gives a global estimate that is valid on the interval $[-1, 1]$. However, as is easy to see in the neighborhood of origin Theorem 3 gives a sharper bound than the one obtainable from Theorem 1.

Let f be as given in Theorem 3 and $t(\theta) = f(\cos\theta) = \sum_{\nu=0}^{n} a_\nu \cos^\nu\theta$ be a trigonometric cosine polynomial. Applying Theorem 3 on $t(\theta)$, we get

$$|t'(\theta)| \leq n \qquad (-\infty < \theta < \infty). \tag{6}$$

If f is as given in Theorem 1, then $t(\theta) = f(\sin\theta) = \sum_{\nu=0}^{n} a_\nu \sin^\nu\theta$ is a trigonometric sine polynomial. Bernstein proved that (6) holds true for sine polynomials also. However, he did not prove the inequality (6) for the general trigonometric polynomials.

Recall that a trigonometric polynomial $t(\theta)$ of degree n is an expression of the form

$$t(\theta) = \sum_{\nu=0}^{n} a_\nu \cos \nu\theta + b_\nu \sin \nu\theta, \tag{7}$$

where a_ν, b_ν $(0 \leq \nu \leq n)$ are complex numbers. Using Euler's formula, the trigonometric polynomial (7) can be written as

$$t(\theta) = \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta} \tag{8}$$

also, where a_ν $-n \leq \nu \leq n$ are complex numbers.

It was M. Reisz [82] who proved the extension of Theorem 3 for the general trigonometric polynomials. He proved that

Theorem 4 *If $t(\theta) = \sum_{\nu=-n}^{n} a_\nu \mathrm{e}^{\mathrm{i}\nu\theta}$ is a trigonometric polynomial of degree n, then*

$$\max_{0\leq\theta\leq2\pi} |t'(\theta)| \leq n \max_{0\leq\theta\leq2\pi} |t(\theta)|. \tag{9}$$

Equality attains for polynomials $t(\theta) = \sin n(\theta - \theta_0)$ where $\theta_0 \in \mathbb{R}$.

In this chapter, $\mathcal{P}_n$ will denote the class of polynomials $\sum_{\nu=0}^{n} a_\nu z^\nu$ of degree at most n, where a_ν $(0 \leq \nu \leq n)$ are complex numbers, and z a complex variable.

Analogue of Markov's Theorem for polynomials of complex variable with norm on the unit disk has also found applications in many areas of mathematics. It may be formulated as follows:

Let f belong to $\mathcal{P}_n$. How large $|f'(z)|$ can be when z is on the unit disk $\{z : |z| = 1\}$?

The answer of this question is contained in the Theorem 4 which was proved by M. Riesz for the first time.

Theorem 5 *If $f \in \mathcal{P}_n$, then*

$$\max_{|z|=1}|f'(z)| \leq n \max_{|z|=1}|f(z)| \qquad (f \in \mathcal{P}_n)\ . \tag{10}$$

Equality holds only for polynomials of the form λz^n, $\lambda \neq 0$ is a complex number.

Alternate proof of Theorem 5 was given by O'Hara [67], and for some refinements of Theorem 5 we refer to Frappier et al. [26], and the paper of Sharma and Singh [84].

A function of the form $L(z) = \sum_{\nu=-n}^{n} a_\nu z^\nu$, where $a_\nu \in \mathbb{C}$ for $-n \leq \nu \leq n$, is called a Laurent polynomial. Riesz even proved the following result for Laurent polynomials also which contains Theorem 5 as a special case.

Theorem 6 *If $L(z) = \sum_{\nu=-n}^{n} a_\nu z^\nu$ is a Laurent polynomial, then*

$$\max_{|z|=1}|L'(z)| \leq n \max_{|z|=1}|L(z)|. \tag{11}$$

Equality holds if and only if $L(z) = \alpha z^n + \beta z^{-n}$.

Any complex number z on the unit circle $\{z : |z| = 1\}$ can be written as $z = \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in \mathbb{R}$. In view of this, Theorem 6 provides yet another representation of

Theorem 4. For some recent results dealing with inequalities for Laurent polynomials, see Govil et al. [52].

Even though Theorem 5 was proved by M. Reisz, but the resulting inequality goes under the name of Bernstein [82]. Bernstein [10], however proved the following more general result than that given in Theorem 5.

Theorem 7 *Let $F(z) = \sum_{\nu=0}^{n} A_\nu z^\nu$ whose zeros lie in $|z| \le 1$ belong to $\mathcal{P}_n$. Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n$ such that $|f(z)| \le |F(z)|$ for $|z| = 1$. Then*

$$|f'(z)| \le |F'(z)| \qquad (|z| \ge 1). \tag{12}$$

Equality holds in (12), if $f(z) = \gamma F(x)$, where γ is a complex number such that $|\gamma| = 1$.

To see that it is a generalization of Theorem 5, take $F(z) = z^n$ which has a zero of multiplicity n at origin. The condition $|f(z)| \le |F(z)|$ for $|z| = 1$ in the Theorem 7 means $|f(z)| \le 1$ for $|z| = 1$. Then $|f'(z)| \le n|z^{n-1}|$ for $|z| \ge 1$. If we take $|z| = 1$, we have the conclusion of Theorem 5.

Recently, the following generalization of Theorem 7 has been proved by Govil et al. [53].

Theorem 8 *Let $F(z)$ be a polynomial whose zeros lie in $|z| \le 1$. Let $f(z)$ be a polynomial such that degree of $f(z)$ does not exceed that of $F(z)$ and $|f(z)| \le |F(z)|$ for $|z| = 1$. Then for any complex number β with $|\beta| \le 1$ and $R > r \ge 1$,*

$$|f(Rz) - \beta f(rz)| \le |F(Rz) - \beta F(rz)| \qquad (|z| \ge 1). \tag{13}$$

Equality holds for the polynomial $f(z) = \gamma F(x)$, where γ is a complex number such that $|\gamma| = 1$.

To obtain Theorem 7 from the Theorem 8, simply take $\beta = 1$, $r = 1$, divide the two sides of (13) by $(R - 1)$ and make $R \to 1$.

Szegö [86] proved inequality (10) under a weaker condition. Precisely, he [86] proved that

Theorem 9 *If $f \in \mathcal{P}_n$, then*

$$\max_{|z|=1} |f'(z)| \le n \max_{|z|=1} |\mathrm{Re} f(z)|. \tag{14}$$

Equality holds for $f(z) = \lambda z^n$ with $\lambda \in \mathbb{C}$.

Alternate proof of the above Theorem 9 was provided by Mohapatra, O'Hara and Rodriguez [65], and their proof is by using Lagrange's Interpolation Formula.

Malik [60] has given a proof of the above theorem based on a result of de Bruijn [20]. In the same paper, he [60] also proved the following improvement of Bernstein's inequality (also see Rahman [74]):

Theorem 10 *If $f \in \mathcal{P}_n$ and $g(z) = z^n \overline{f(1/\bar{z})}$ be the conjugate polynomial associated with f, then on $|z| = 1$*

$$|f'(z)| + |g'(z)| \le n \max_{|z|=1} |f(z)|. \tag{15}$$

Further generalizations can be found in [25] and [26]. Frappier et al. [26, Theorem 8] also provided the following generalization:

Theorem 11 *If $f \in \mathcal{P}_n$ and $z_1, z_2, \ldots, z_{2n}$ are any $2n$ equally spaced points on $|z| = 1$, then*

$$\max_{|z|=1}|f'(z)| \leq n \max_{1\leq k\leq 2n} |f(z_k)|.$$

Let $f \in \mathcal{P}_n$ and $p > 0$ be any real number. It is well known [71] that

$$\lim_{p\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} |f\left(\mathrm{e}^{\mathrm{i}\theta}\right)|^p d\theta\right)^{\frac{1}{p}} = \max_{|z|=1}|f(z)|,$$

and thus $\left(\frac{1}{2\pi}\int_0^{2\pi} |f\left(\mathrm{e}^{\mathrm{i}\theta}\right)|^p d\theta\right)^{\frac{1}{p}}$ can be seen as a generalization of $\max_{|z|=1}|f(z)|$. In view of this observation, the following inequality by Zygmund [89] can be seen as a generalization of the Bernstein's inequality (10).

Theorem 12 *If $f \in \mathcal{P}_n$, then*

$$\left(\int_0^{2\pi} |f'(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leq n\left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \qquad (p \geq 1). \tag{16}$$

The above inequality is best possible with equality holds true for $f(z) = \lambda z^n$.

Obviously, it would be of interest to know what happens when $0 < p < 1$ in the above theorem. In his proof of Theorem 12, Zygmund used the convexity of the function $\phi : x \to x^p$ which is valid only if $p \geq 1$. Attempts to resolve this were made by Klein [58], Ivanov [56], and Storoženko et al. [85]. Surprisingly, it took almost 50 years to solve the problem completely, and almost 40 years to make some definite progress when Osval'd [69] proved

Theorem 13 *If $f \in \mathcal{P}_n$, then*

$$\left(\int_0^{2\pi} |f'\left(e^{i\theta}\right)|^p d\theta\right)^{\frac{1}{p}} \leq nC_p\left(\int_0^{2\pi} |f\left(e^{i\theta}\right)|^p d\theta\right)^{\frac{1}{p}} \qquad (0 < p < 1). \tag{17}$$

where C_p is a constant that depends only on p.

In 1979, Paul Nevai [66] proved that in the above inequality $C_p \leq (8/p)^{1/p}$. It is in fact less than or equal to $(11)^{1/p}$, see Maté and Nevai [61].

The problem was completely resolved by Arestov [2] who used subharmonic functions and Jensen's formula to derive the following sharp bound.

Theorem 14 *If $f \in \mathcal{P}_n$, then*

$$\left(\int_0^{2\pi} |f'(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leq n\left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \qquad (0 < p < \infty). \tag{18}$$

The above inequality is best possible. Equality holds true for $f(z) = \lambda z^n$.

Golitschek and Lorentz [34] gave a simpler proof of this inequality and also obtained its generalization. The sharp inequality analogous to (18) when f has no zeros in $|z| < 1$ was obtained by Rahman and Schmeisser [76].

For historical details on these theorems and related literatures on the development of approximation theory, we refer readers to [47, 64, 70, 78]. For some generalizations of Bernstein's inequality for rational functions, we refer readers to [46], where polynomial inequalities and their generalizations to rational functions have been studied.

The chapter is expository in nature. Having discussed some of the generalizations of Bernstein's and Markov's inequalities in this Sect. 1, we will discuss constrained Bernstein type inequalities, that is, inequalities for different classes of polynomials in Sect. 2. Then Sect. 3 deals with the extension of some of the results of Sect. 2 for entire functions of exponential type, and finally, Sect. 4 contains some of the open problems, discussed in the text of this chapter, that could be of interest to some of the readers.

2 Constrained Bernstein Type Inequalities for Polynomials

As mentioned in Sect. 1, in this section we will discuss Bernstein type inequalities for some classes of polynomials, along with some Bernstein type inequalities in the L^p norm.

2.1 Polynomials Having No Zeros Inside a Circle

Since the equality holds in the Bernstein inequality given in Eq. (10) if and only if $f(z) = \lambda z^n$ which has all its zeros at the origin, one would expect that there is a relationship between the bound n and the distance of the zeros of the polynomial from the origin. This fact was observed by Erdös [24] who conjectured that if the polynomial $f(z)$ has no zero in $|z| < 1$, then $\max_{|z|=1}|f'(z)| \le (n/2)\max_{|z|=1}|f(z)|$. This conjecture was proved in the special case when $f(z)$ has all its zeros on $|z| = 1$ independently by Polya and Szegö [59]. In the general case the conjecture was proved for the first time by Lax [59].

Theorem 15 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < 1$, then*

$$\max_{|z|=1}|f'(z)| \le \frac{n}{2}\max_{|z|=1}|f(z)|. \tag{19}$$

Equality in (19) holds for any polynomial which has all its zeros on $|z| = 1$.

Simpler proofs of this result were given by de Bruijn [20] and Aziz and Mohammad [6]. In this section we will be discussing some of them.

It was proposed by Professor R. P. Boas to obtain inequalities analogous to (19) for polynomials having no zeros in $|z| < K$. In this connection following partial result is due to Malik [60].

Theorem 16 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then*

$$\max_{|z|=1}|f'(z)| \leq \left(\frac{n}{1+K}\right)\max_{|z|=1}|f(z)|. \tag{20}$$

Equality holds for $f(z) = (z+K)^n$.

For quite some time it was believed that if $f(z) \neq 0$ in $|z| < K$ where $K \leq 1$, then the inequality analogous to (19) should be

$$\max_{|z|=1}|f'(z)| \leq \frac{n}{1+K^n}\max_{|z|=1}|f(z)|, \tag{21}$$

till E. B. Saff gave the example $f(z) = (z - \frac{1}{2})(z + \frac{1}{3})$ to counter this belief. For this polynomial, $\max_{|z|=1}|f'(z)| \approx 2 \cdot 1666$ while the right hand side of (21) is $(2/(1+(1/3)^2))\max_{|z|=1}|f(z)| \approx 2 \cdot 144 < 2 \cdot 166$ and so (21) does not hold for this polynomial. Govil [37], however proved that if f in $\mathcal{P}_n$ has no zero in $|z| < K$, $K \leq 1$, then

$$\max_{|z|=1}|f'(z)| \leq \frac{n}{K^n + K^{n-1}}\max_{|z|=1}|f(z)|. \tag{22}$$

It is clear that the above bound is of interest only if $K^n + K^{n-1} > 1$. For another result in this direction, see Govil [38].

Govil and Rahman [48] found an extension of Theorem 16 for higher order derivatives. They proved that

Theorem 17 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then*

$$\max_{|z|=1}|f^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{1+K^s}\max_{|z|=1}|f(z)|. \tag{23}$$

For $s = 1$, (23) reduces to (20).

A polynomial of the form $f(z) = a_0 + \sum_{\nu=1}^{n} a_\nu z^{m_\nu}$, where $0 < m_1 < \ldots < m_n$ are given integers, is called Lacunary polynomial. Chan and Malik [17] proved the following extension of Theorem 16 for a special class of Lacunary polynomials.

Theorem 18 *If $f(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ is a polynomial in $\mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then*

$$\max_{|z|=1}|f'(z)| \leq \frac{n}{1+K^\mu}\max_{|z|=1}|f(z)|. \tag{24}$$

$f(z) = (z^\mu + K^\mu)^{n/\mu}$ shows that the inequality is sharp where n is a multiple of μ.

Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n$. It can be shown that if $f(z) \neq 0$ in $|z| < K$, $K \geq 1$ then the equality in (20) can hold if and only if $|a_1/a_0| = n/K$ and hence it should be possible to improve upon (20) if $|a_1/a_0| \leq cn/K$ where $0 \leq c \leq 1$. This fact was observed by Govil et al. [51] who obtained a bound in terms of the coefficients a_0, a_1, and a_2. They proved

Theorem 19 *If* $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial in* $\mathcal{P}_n$ *such that* $f(z) \neq 0$ *in* $|z| < K$ *where* $K \geq 1$*, then*

$$\max_{|z|=1}|f'(z)| \leq \frac{n|a_0| + K^2|a_1|}{(1+K^2)n|a_0| + 2K^2|a_1|} \max_{|z|=1}|f(z)|; \tag{25}$$

furthermore

$$\max_{|z|=1}|f'(z)| \leq \left(\frac{n}{1+K}\right) \frac{(1-|\lambda|)(1+K^2|\lambda|) + K(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-K+K^2+K|\lambda|) + K(n-1)|\mu-\lambda^2|} \max_{|z|=1}|f(z)|, \tag{26}$$

where

$$\lambda = \frac{Ka_1}{na_0}, \quad \mu = \frac{2K^2}{n(n-1)} \frac{a_2}{a_0}.$$

Both the above inequalities are best possible. For even n, the equality in (25) holds for

$$f(z) = \frac{a_0}{K^n}\left(ze^{i\gamma} + Ke^{i\alpha}\right)^{n/2}\left(ze^{i\gamma} + Ke^{-i\alpha}\right)^{n/2},$$

where γ and α are arbitrary real numbers. Whether n is even or odd, equality holds in (26) for

$$f(z) = \frac{a_0}{K^n}(z+K)^{n_1}\left(z^2 + 2Kz\frac{na - n_1}{n - n_1} + K^2\right)^{(n-n_1)/2},$$

and in fact for $f(ze^{i\gamma})$ for all real γ, if n_1 is an integer such that $n/3 \leq n_1 \leq n$, $(n - n_1)$ is even, and $(3n_1 - n)/(n + n_1) \leq a \leq 1$.

It is worth noting that the bound in inequality (20) due to Malik [60] depends only on the zero with the smallest modulus. To illustrate it, take $f_1(z) = (z+K)^n$ and $f_2(z) = (z+K)(z+K+\ell)^{n-1}$, $K \geq 1, \ell > 0$. One can see that (20) gives the same bound for these polynomials. So, it is of interest to look for a bound that depends upon the location of all the zeros rather than just on the location of the the zero of smallest modulus. In this direction, Govil and Labelle [44] proved the following

Theorem 20 *Let* $f(z) = a_n \Pi_{\nu=1}^{n}(z - z_\nu)$, $a_n \neq 0$*, be a polynomial of degree* n*. If* $|z_\nu| \geq K_\nu \geq 1, 1 \leq \nu \leq n$*, then*

$$\max_{|z|=1}|f'(z)| \leq n\left\{\left(\sum_{\nu=1}^{n}\frac{1}{K_\nu - 1}\right) \Big/ \left(\sum_{\nu=1}^{n}\frac{K_\nu + 1}{K_\nu - 1}\right)\right\}\max_{|z|=1}|f(z)|. \tag{27}$$

Equality holds for $f(z) = (z+K)^n$, $K \geq 1$.

Remark 1 It can be easily verified, that the right hand side of the inequality (27) is in fact equal to

$$\frac{n}{2}\left\{1 - \frac{1}{1 + \frac{2}{n}\sum_{\nu=1}^{n}\frac{1}{K_\nu - 1}}\right\}\max_{|z|=1}|f(z)|. \tag{28}$$

If $K_\nu \geq K, \quad K \geq 1$ for $1 \leq \nu \leq n$, then clearly

$$\sum_{\nu=1}^{n} \frac{1}{K_\nu - 1} \Big/ \sum_{\nu=1}^{n} \frac{K_\nu + 1}{K_\nu - 1} \leq \frac{1}{1+K},$$

so, the bound in (27) is in general at least as sharp as in Malik's bound (20). In fact, except for the case when the polynomial $f(z)$ has all its zeros on $|z| = K, \quad K > 1$, the bound obtained by (27) is always sharper than the bound obtainable from (20). If $K_\nu = 1$ for some $\nu, \quad 1 \leq \nu \leq n$, then the inequality (27) reduces to Lax's inequality (19).

The statement of the Theorem 20 might suggest that one needs to know all the zeros of the polynomial but it is not so. No doubt, the usefulness of the theorem will be heightened if the polynomial is given in terms of the zeros. If in particular, the polynomial $f(z)$ is the product of two or more polynomials having zeros in $|z| \geq K_1 > 1, \quad |z| \geq K_2 > 1$, etc., each of norm ≤ 1, then $f(z)$ would be of norm ≤ 1, and one would have a better estimate for $\max_{|z|=1}|f'(z)|$ by (27) than from (20).

Aziz and Dawood [5] considered the problem given in Theorem 15 under an additional condition that the $\min_{|z|=1}|f(z)|$ is also given. In this direction, they proved that

Theorem 21 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < 1$, then*

$$\max_{|z|=1}|f'(z)| \leq \frac{n}{2}\{\max_{|z|=1}|f(z)| - \min_{|z|=1}|f(z)|\}. \tag{29}$$

The result is best possible and equality holds for $f(z) = \alpha z^n + \beta$ where $|\beta| \geq |\alpha|$.

The above result of Aziz and Dawood [5] was generalized by Govil [40] who proved that

Theorem 22 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then*

$$\max_{|z|=1}|f^{(s)}(z)| \leq \frac{n(n-1)\cdots(n-s+1)}{1+K^s}\{\max_{|z|=1}|f(z)| - \min_{|z|=K}|f(z)|\} \tag{30}$$

which sharpens Theorem 17 due to Govil and Rahman [48].

Also, for $s = 1$, the Theorem 22 reduces to

Theorem 23 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < K$ where $K \geq 1$, then*

$$\max_{|z|=1}|f'(z)| \leq \left(\frac{n}{1+K}\right)\{\max_{|z|=1}|f(z)| - \min_{|z|=K}|f(z)|\}. \tag{31}$$

Equality is attained for $f(z) = (z+K)^n$.

For polynomials not vanishing in $|z| < 1$, de Bruijn [20] proved the following generalization of Theorem 15.

Theorem 24 *If $f \in \mathcal{P}_n$ such that $f(z) \neq 0$ in $|z| < 1$, then for $p \geq 1$,*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f'\left(e^{i\theta}\right)|^p d\theta\right)^{\frac{1}{p}} \leq n c_p^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_0^{\pi} |f\left(e^{i\theta}\right)|^p d\theta\right)^{\frac{1}{p}} \tag{32}$$

where $c_p = 2^{-p}\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)/\Gamma\left(\frac{1}{2}p+\frac{1}{2}\right)$. *The result is sharp and the equality holds for* $f(z) = (\alpha + \beta z^n)$, $|\alpha| = |\beta|$.

To obtain Lax's inequality (19) from (32), simply make $p \to \infty$ and note that $\lim_{p\to\infty} c_p^{1/p} = 1/2$. For an alternate proof of Theorem 24, see Rahman [74]. The inequality (32) in fact holds for $p > 0$ and this was proved by Rahman and Schmeisser [61]. A simpler proof and a generalization of Theorem 24 were given by Aziz [4].

For polynomials not vanishing in $|z| < K$, $K \geq 1$, Govil and Rahman [48] proved

Theorem 25 *If* $f \in \mathcal{P}_n$ *such that* $f(z) \neq 0$ *in* $|z| < K$ *where* $K \geq 1$, *then for* $p \geq 1$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leq n\, E_p^{\frac{1}{p}} \left(\frac{1}{2pi}\int_0^{\pi} |f(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \tag{33}$$

where $E_p = 2\pi/\int_0^{2\pi} |K + e^{i\alpha}|^p d\alpha$.

Since $\lim_{p\to\infty} E_p^{1/p} = 1/(1+K)$, we get (20) by taking $p \to \infty$ in (33). For $K = 1$, Theorem 25 reduces to Theorem 24 of de Bruijn [20].

Gardner and Govil [30] have generalized the above result of Govil and Rahman [48] by proving the following theorem.

Theorem 26 *Let* $f(z) = a_n\Pi_{\nu=1}^n(z - z_\nu), a_n \neq 0$, *be a polynomial of degree n. If* $|z_\nu| \geq K_\nu \geq 1$, $1 \leq \nu \leq n$, *then for* $p > 0$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leq n\, F_p^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_0^{\pi} |f(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \tag{34}$$

where $F_p = \{2\pi/\int_0^{2\pi} |t_0 + e^{i\theta}|^p d\theta\}$, *and* $t_0 = \{1 + n/\sum_{\nu=1}^n \frac{1}{K_\nu - 1}\}$, *if* $K_\nu > 1$ *for all* ν, $1 \leq \nu \leq n$, *and* $t_0 = 1$ *if* $K_\nu = 1$ *for some* ν, $1 \leq \nu \leq n$. *The result is best possible in the case* $K_\nu = 1$, $1 \leq \nu \leq n$, *and the equality holds for* $f(z) = (1+z)^n$.

The above result in the case $p \geq 1$ was also proved by Gardner and Govil [28]. If $K_\nu = 1$ for some ν, $1 \leq \nu \leq n$ then $t_0 = 1$ and (34) reduces to the inequality (32) due to de Bruijn [20]. If $K_\nu \geq K$ for some $K > 1$, $1 \leq \nu \leq n$, then as it is easy to verify that $F_p \leq \{2\pi/\int_0^{2\pi} |K + e^{i\alpha}|^p d\alpha\}^{\frac{1}{p}}$, and so the above inequality reduces to the inequality (33) due to Govil and Rahman [48]. Further, if in Theorem 26, we make $p \to \infty$, we get Theorem 20, due to Govil and Labelle [44].

2.2 *Polynomials Having All the Zeros in a Circle*

We again begin with Bernstein's inequality that if $f \in \mathcal{P}_n$, then

$$\max_{|z|=1}|f'(z)| \leq n \max_{|z|=1}|f(z)|\,. \tag{35}$$

Equality in (35) holds only for polynomials of the form λz^n, $\lambda \neq 0$ is a complex number.

As it is evident from λz^n (λ a complex number), it is not possible to improve upon the bound in (35), if $f(z)$ has all its zeros in $|z| \leq 1$. Hence it would be of interest to obtain an inequality in the reverse direction and this was done by Turán [88], who proved

Theorem 27 *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{2}\max_{|z|=1}|f(z)|. \tag{36}$$

The result is best possible and the equality holds for all polynomials of degree n which have all their zeros on $|z| = 1$.

It will obviously be of interest to obtain an inequality analogous to (36) for polynomials having all their zeros in $|z| \leq K$, $K > 0$. In this regard, Malik [60] considered the case when $K \leq 1$, and by using his theorem (see Theorem 16), he obtained

Theorem 28 *If $f(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq K \leq 1$, $K > 0$, then*

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{1+K}\max_{|z|=1}|f(z)|. \tag{37}$$

Equality holds for the polynomial $f(z) = (z + K)^n$.

A simple and direct proof of this result was given by Govil [35] which is as follows.

If $f(z) = a_n\Pi_{\nu=1}^n(z - z_\nu)$ is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, then

$$\left|\frac{f'(e^{i\theta})}{f(e^{i\theta})}\right| \geq \mathrm{Re}\left(e^{i\theta}\frac{f'(e^{i\theta})}{f(e^{i\theta})}\right) = \sum_{\nu=1}^{n}\mathrm{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-z_\nu}\right) \geq \sum_{\nu=1}^{n}\frac{1}{1+K},$$

that is,

$$|f(e^{i\theta})| \geq \frac{n}{1+K}|f(e^{i\theta})|,$$

where θ is real. Choosing θ_0 such that $|f(e^{i\theta_0})| = \max_{0\leq\theta<2\pi}|f(e^{i\theta})|$, we get

$$|f'(e^{i\theta_0})| \geq \frac{n}{1+K}\max_{0\leq\theta<2\pi}|f(e^{i\theta})|,$$

from which (37) follows.

The above argument does not hold for $K > 1$ because then $\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}/(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{z}_\nu)\right)$ may not be greater than or equal to $1/(1 + K)$. Govil [35] also settled the case when $K > 1$, by proving

Theorem 29 *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then*

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{1+K^n}\max_{|z|=1}|f(z)|. \tag{38}$$

The result is best possible and the equality holds for the polynomial $f(z) = z^n + K^n$.

A simpler proof of this result was later given by Datt [19]. Note that for $K > 1$, the extremal polynomial turns out to be of the form $(z^n + K^n)$ while for $K < 1$, it has the form $(z + K)^n$. Thus $K = 1$ is a critical value of the parameter under consideration and one should not expect the same kind of reasoning to work for both $K < 1$ and $K > 1$.

The following refinement of Theorem 29 was done by Giroux et al. [33].

Theorem 30 *Let* $f(z) = a_n \Pi_{\nu=1}^{n}(z - z_\nu)$ *be a polynomial of degree* n *such that* $|z_\nu| \leq 1$ *for* $1 \leq \nu \leq n$*, then*

$$\max_{|z|=1}|f'(z)| \geq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \max_{|z|=1}|f(z)|. \tag{39}$$

Equality holds in (39)*, if the zeros are all positive.*

A generalization of the above Theorem was obtained by Aziz [3].

Theorem 31 *Let* $f(z) = a_n \Pi_{\nu=1}^{n}(z - z_\nu)$ *be a polynomial of degree* n *such that* $|z_\nu| \leq K$ *for* $1 \leq \nu \leq n$*, then*

$$\max_{|z|=1}|f'(z)| \geq \frac{2}{1 + K^n} \sum_{\nu=1}^{n} \frac{K}{K + |z_\nu|} \max_{|z|=1}|f(z)|. \tag{40}$$

Equality holds again for $f(z) = z^n + K^n$.

Inequality (40) is also a refinement of the inequality (38) due to Govil [35].

Although, the Theorem 29 due to Govil [35] is sharp, but as is easy to see, it has two drawbacks. First, the bound in (38) depends only on the zero of largest modulus, and not on other zeros even if some of the zeros are very close to the origin. Second, since the extremal polynomial in (38) is $(z^n + K^n)$, it should be possible to improve upon the bound for polynomials $\sum_{\nu=0}^{n} a_\nu z^\nu$, where not all the coefficients $a_1, a_2, \ldots a_{n-1}$ are zero. This was observed by Govil [39] who proved the following refinement of Theorem 29.

Theorem 32 *Let* $f(z) = \sum_{\nu=0}^{\nu} a_\nu z^\nu = a_n \Pi_{\nu=1}^{n}(z - z_\nu), a_n \neq 0$ *be a polynomial of degree* $n \geq 2$*,* $|z_\nu| \leq K_\nu, \quad 1 \leq \nu \leq n$*, and let* $K = \max(K_1, K_2, \ldots, K_n) \geq 1$. *Then, for* $n > 2$

$$\begin{aligned}\max_{|z|=1}|f'(z)| \geq \frac{2}{1 + K^n} \left(\sum_{\nu=1}^{n} \frac{K}{K + K_\nu}\right) \max_{|z|=1}|f(z)| + |a_1| \left(1 - \frac{1}{K^2}\right) \\ + \frac{2|a_{n-1}|}{1 + K^n} \sum_{\nu=1}^{n} \frac{1}{K + K_\nu} \left(\frac{K^n - 1}{n} - \frac{K^{n-2} - 1}{n - 2}\right),\end{aligned} \tag{41}$$

and, if $n = 2$

$$\max_{|z|=1}|f'(z)| \geq \frac{2}{1+K^n}\left(\sum_{\nu=1}^{n}\frac{K}{K+K_\nu}\right)\max_{|z|=1}|f(z)| \tag{42}$$
$$+|a_1|\left(1-\frac{1}{K}\right)+\frac{(K-1)^n}{1+K^n}|a_1|\sum_{\nu=1}^{n}\frac{1}{K+K_\nu}.$$

In these estimates equality holds for $f(z)=z^n+K^n$.

The case $n=1$ in Theorem 32 is trivial because in that case $\max_{|z|=1}|f'(z)| = (1/(1+K))\max_{|z|=1}|f(z)|$, where K is the modulus of the zero of $f(z)$. Since $K/(K+K_\nu) \geq 1/2$ $(1 \leq \nu \leq n)$, from Theorem 32 follows trivially

Theorem 33 *If* $f(z) = a_n \Pi_{\nu=1}^{n}(z-z_\nu)$, $a_\nu \neq 0$, *is a polynomial of degree n having all its zeros in* $|z| \leq K$ *where* $K \geq 1$, *then for* $n > 2$

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{1+K^n}\max_{|z|=1}|f(z)|$$
$$+|a_1|\left(1-\frac{1}{K^2}\right)+\frac{n|a_{n-1}|}{K(1+K^n)}\left(\frac{K^n-1}{n}-\frac{K^{n-2}-1}{n-2}\right), \tag{43}$$

and, if $n=2$

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{1+K^n}\max_{|z|=1}|f(z)|+|a_1|\left(\frac{(K-1)^n}{K(1+K)^n}+\frac{K-1}{K}\right). \tag{44}$$

Inequalities (43) and (44) together provide a refinement of Theorem 29 because as can be easily verified that for $K > 1$ and $n > 2$, we have

$$\frac{K^n-1}{n}-\frac{K^{n-2}-1}{n-2} > 0.$$

A refinement of Theorem 27 was given by Aziz and Dawood [5] which is as follows.

Theorem 34 *If* $f(z)$ *is a polynomial of degree n having all its zeros in* $|z| \leq 1$, *then*

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{2}\{\max_{|z|=1}|f(z)|+\min_{|z|=1}|f(z)|\}. \tag{45}$$

Equality holds for $f(z)=\alpha z^n+\beta$, $|\beta| \leq |\alpha|$.

The above theorem has been generalized by Govil [40] who proved the following more general

Theorem 35 *If* $f(z)$ *is a polynomial of degree n, having all its zeros in* $|z| \leq K$, *then*

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{1+K}\max_{|z|=1}|f(z)|+\frac{n}{K^{n-1}(1+K)}\min_{|z|=K}|f(z)|, \tag{46}$$

if $K \leq 1$, *and for* $K \geq 1$

$$\max_{|z|=1}|f'(z)| \geq \frac{n}{1+K}\{\max_{|z|=1}|f(z)|+\min_{|z|=K}|f(z)|\}. \tag{47}$$

Both the above inequalities are best possible. In the first case the equality is attained for $f(z) = (z + K)^n$ and in the second for $f(z) = z^n + K^n$.

For generalization of the above inequalities for polar derivative, we refer to the papers of Aziz and Rather [7] and Govil and McTume [45].

2.3 Self-Inversive and Self-Reciprocal Polynomials

In this section we will discuss some inequalities concerning self-inversive and self-reciprocal polynomials. We begin with the definition of self-inversive polynomials.

Definition 1 A polynomial f in $\mathcal{P}_n$ is called n-self inversive (self inversive), if it satisfies the condition $z^n \overline{f(\overline{1/z})} \equiv f(z)$.

We represent the class of self-inversive polynomials of degree at most n by $\mathcal{P}_n^{\sim}$.

If $f \in \mathcal{P}_n$ and $g(z) := z^n \overline{f(1/\overline{z})}$, then, from Theorem 10, one gets

$$|f'(z)| + |g'(z)| \le n \max_{|z|=1} |f(z)| \qquad (|z| = 1)\,.$$

In particular, if $f \in \mathcal{P}_n^{\sim}$ then $f(z) \equiv g(z)$ and hence $f'(z) \equiv g'(z)$. Thus, from the above inequality, we have

$$\max_{|z|=1} |f'(z)| \le \frac{n}{2} \max_{|z|=1} |f(z)|\,. \tag{48}$$

Let f be a polynomial of degree n, and z_0 a point on the unit circle such that $|f(z_0)| = \max_{|z|=1} |f(z)|$. Then, $|f'(z_0)| = |g'(z_0)| = |n f(z_0) - z_0 f'(z_0)| \ge n\,|f(z_0)| - |f'(z_0)|$. Hence,

$$\max_{|z|=1} |f'(z)| \ge |f'(z_0| \ge \frac{n}{2} |f(z_0)| = \frac{n}{2} \max_{|z|=1} |f(z)|. \tag{49}$$

Now, if we combine (48) and (49), we get the following result (see Govil [35, Lemma 4].

Theorem 36 *If $f(z)$ is a self-inversive polynomial of degree n, then*

$$\max_{|z|=1} |f'(z)| = \frac{n}{2} \max_{|z|=1} |f(z)|. \tag{50}$$

The above result also appears in a paper of O'Hara and Rodriguez [68] and Saff and Shiel-Small [83].

The L^p inequality for self-inversive polynomials was obtained by Dewan and Govil [21], and in this regard they proved the following.

Theorem 37 *If $f(z)$ is a self-inversive polynomial of degree n, then for $p \ge 1$*

$$\left(\int_0^{2\pi} |f'(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \le n c_p^{\frac{1}{p}} \left(\int_0^{2\pi} |f(e^{i\theta)}|^p d\theta \right)^{\frac{1}{p}}, \tag{51}$$

where $c_p = 2^{-p}\frac{\sqrt{\pi}\,\Gamma(p/2+1)}{\Gamma(p/2+1/2)}$. *The above inequality is best possible and it reduces to equality for* $f(z) = (z^n + 1)$.

Later on Govil and Jain [43] proved the following more complete result.

Theorem 38 *If* $f(z)$ *is a self-inversive polynomial of degree n, then for* $p \geq 1$

$$\frac{n}{2}\left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leq \left(\int_0^{2\pi} |f'(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \leq n c_p^{\frac{1}{p}} \left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \tag{52}$$

where $c_p = 2^{-p}\frac{\sqrt{\pi}\,\Gamma(p/2+1)}{\Gamma(p/2+1/2)}$. *Both the inequalities are best possible and they both reduce to equality for* $f(z) = (z^n + 1)$.

The above result of Govil and Jain [43] has been extended for $p > 0$ by Govil [41].

It can be seen that $\lim_{p\to\infty} c_p^{1/p} = 1/2$, and $\lim_{p\to\infty} ((1/2\pi)\int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^p d\theta)^{(1/p)} = \max_{|z|=1}|f(z)|$, and thus from (52), we get once again the conclusion of Theorem 36 that if $f(z)$ is a self-inversive polynomial of degree n, then

$$\max_{|z|=1}|f'(z)| = \frac{n}{2}\max_{|z|=1}|f(z)|.$$

Now we will study the class of self-reciprocal polynomials.

Definition 2 A polynomial f in $\mathcal{P}_n$ is called n self reciprocal (self reciprocal), if it satisfies the condition $z^n f(1/z) \equiv f(z)$.

Following Rahman and Schmeisser [78], the class of self-reciprocal polynomials of degree at most n will be denoted by $\mathcal{P}_n^{\vee}$.

Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a self-reciprocal polynomial. Then the following observations are evident from its definition.

- $a_\nu = a_{n-\nu}$, for $0 \leq \nu \leq n$.
- If $\zeta \neq 0$ is a zero of f then so is $1/\zeta$.Thus self-reciprocal polynomials have at least half of their zeros outside the open unit disk. It is assumed that a polynomial f belonging to $\mathcal{P}_n$ but of degree $m < n$ has $n - m$ of its zeros at ∞.
- If the degree of f is odd then it has a zero at -1.

We will now discuss the Bernstein's inequality given in (10) for this class. Let us start with a polynomial f in $\mathcal{P}_1^{\vee}$. From the Bullet 3 given above, $f(z) = c(z + 1)$, where $c \in \mathbb{C}$. We have $\max_{|z|=1}|f'(z)| = |c|$ and $\max_{|z|=1}|f(z)| = 2|c|$. Thus

$$\max_{|z|=1}|f'(z)| = \frac{1}{2}\max_{|z|=1}|f(z)| \qquad \left(f \in \mathcal{P}_1^{\vee}\right) \tag{53}$$

which is consistent with Theorem 36, as $\mathcal{P}_1^{\vee}$ and $\mathcal{P}_1^{\sim}$ are the same.

Next, let f belong to $\mathcal{P}_2^{\vee}$. From Bullet 1, we can write $f(z) = a(z^2 + 1) + bz$, and without loss of generality, we can take $a = 1$.

$$\max_{|z|=1}|f'(z)| = \max_{|z|=1}|2z + b| = 2 + |b|. \tag{54}$$

Then

$$\max_{|z|=1}|f(z)| = \max_{|z|=1}|(z^2+1)+bz|$$
$$= \max_{0\le\theta\le 2\pi}|(\mathrm{e}^{2\mathrm{i}\theta}+1)+b\mathrm{e}^{\mathrm{i}\theta}| = 2\max_{0\le\theta\le 2\pi}|\cos\theta+\frac{b}{2}| \ge \sqrt{4+|b|^2}. \tag{55}$$

So, from (54) and (55), we have

$$\frac{\max_{|z|=1}|f'(z)|}{\max_{|z|=1}|f'(z)|} \le \frac{2+|b|}{\sqrt{4+|b|^2}}. \tag{56}$$

It can be easily verified that for $x \ge 0$,

$$\frac{2+x}{\sqrt{4+x^2}} \le \sqrt{2}. \tag{57}$$

Therefore from (56) and (57), we conclude that

$$\max_{|z|=1}|f'(z)| \le \sqrt{2}\max_{|z|=1}|f(z)| \qquad (f \in \mathcal{P}_2^{\vee}), \tag{58}$$

and equality holds in (58) for $f(z) = z^2+2iz+1$.
Thus, we have a sharp estimate in the Bernstein inequality for $\mathcal{P}_2^{\vee}$. For $n \ge 3$, the sharp estimate in Bernstein equality remains unknown even though the class is under investigation for well over 40 years.

Frappier et al. [26, p. 97] constructed a polynomial $f(z) := \{(1-\mathrm{i}z)^2 + z^{n-2}(z-\mathrm{i})^2\}/4$ of degree n for which $f(z) = z^n f(1/z)$ holds and

$$\max_{|z|=1}|f(z)| = 1 = |f(\mathrm{i})| \ \text{ whereas } \ |f'(-\mathrm{i})| = n-1\,.$$

This example exhibits the existence of a polynomial f in $\mathcal{P}_n^{\vee}$ for which

$$\max_{|z|=1}|f'(z)| \ge |f'(-\mathrm{i})| = (n-1)\max_{|z|=1}|f(z)|. \tag{59}$$

Thus the bound in the Bernstein inequality for $\mathcal{P}_n^{\vee}$ is atleast $n-1$.
Frappier et al. [27, Theorem 2] studied another class of polynomials $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ whose constant term a_0 is equal to the coefficient of the leading term a_n. For such polynomials they proved that

$$\max_{|z|=1}|f'(z)| \le \left(n-\frac{1}{2}+\frac{1}{2(n+1)}\right)\max_{|z|=1}|f(z)|. \tag{60}$$

As noted above in bullet 1, f belongs to $\mathcal{P}_n^{\vee}$ if and only if $a_\nu = a_{n-\nu}$ for each $0 \le \nu \le n$. Hence, in particular, $\nu = 0$, which gives $a_0 = a_n$. Thus the inequality (60) holds for polynomials in $\mathcal{P}_n^{\vee}$ as well. Combining inequalities (59) and (60), one gets

$$(n-1)\max_{|z|=1}|f(z)| \le \max_{|z|=1}|f'(z)| \le \left(n-\frac{1}{2}+\frac{1}{2(n+1)}\right)\max_{|z|=1}|f(z)| \tag{61}$$

which shows that in general there will be no meaningful improvement in (10) for $\mathcal{P}_n^\vee$. This is quite surprising as the class $\mathcal{P}_n^\vee$ is quite restrictive in some sense. For example it has as many zeros inside the unit disk as it has outside.

We can, however obtain improvements in (10) if we impose some additional restrictions on $\mathcal{P}_n^\vee$. We will consider two types of conditions here; restrictions on the location of zeros and restrictions on the coefficients of polynomials. We will start with the following theorem of Govil et al. [50].

Theorem 39 *Let f belong to $\mathcal{P}_n^\vee$ such that all its zeros are either in the left-half plane or in the right-half plane. Then*

$$\max_{|z|=1}|f'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1}|f(z)|. \tag{62}$$

The result is sharp. Equality holds for polynomial $f(z) = c(1+z)^n$ if all the zeros are in the left-half plane and for polynomial $f(z) = c(1-z)^n$ if all the zeros are in the right-half plane.

Recently, Tariq [87] has noted a property of polynomials in $\mathcal{P}_n^\vee$ whose zeros lie in the left-half plane. His observation is given in the next theorem.

Theorem 40 *Let f belong to $\mathcal{P}_n^\vee$ such that all its zeros are in the left-half plane. In addition, suppose that its zeros in the second quadrant are of modulus at most* 1. *Then*

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})| \le |f'(\mathrm{e}^{\mathrm{i}\theta})| \qquad (0 \le \theta \le \pi). \tag{63}$$

As an application of above theorem, he proved the following:

Theorem 41 *Let f belong to $\mathcal{P}_n^\vee$ such that all its zeros are in the left-half plane. In addition, suppose that its zeros in the second quadrant are of modulus at most* 1. *Further assume that $|f(\mathrm{e}^{-\mathrm{i}\theta})| \le M$ for $0 \le \theta \le \pi$. Then*

$$|f'(\mathrm{e}^{-\mathrm{i}\theta})| \le M\,\frac{n}{2} \qquad (0 \le \theta \le \pi). \tag{64}$$

The example $f(z) = (z^2+1)^{n/2}$, shows that the estimate is sharp when n is even. For odd n, the equality holds for $f(z) = (z+1)^n$.

Now, we will discuss some Bernstein type inequalities for $\mathcal{P}_n^\vee$ that are obtained by considering restrictions on the coefficients of polynomials. Aziz [3] investigated the polynomials in $\mathcal{P}_n^\vee$ whose coefficients lie in the first quadrant. For such polynomials, he proved the following

Theorem 42 *Let $f(z) = \sum_{\nu=0}^{n} (\alpha_\nu + i\beta_\nu)z^\nu$, $\alpha_\nu \ge 0$, $\beta_\nu \ge 0$, $\nu = 0, 1, 2, \ldots n$ be a polynomial in $\mathcal{P}_n^\vee$. Then*

$$\max_{|z|=1}|f'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1}|f(z)|. \tag{65}$$

The result is sharp when n is even. Equality holds for the polynomial $f(z) = z^n + 2iz^{n/2} + 1$.

Proof Let us write $f(z) = f_1(z) + \mathrm{i} f_2(z)$, where $f_1(z) = \sum_{\nu=0}^{n} \alpha_\nu z^\nu$, and $f_2(z) = \sum_{\nu=0}^{n} \beta_\nu z^\nu$. Since $\alpha_\nu \geq 0$ and $\beta_\nu \geq 0$, we have $\max_{|z|=1}|f_1(z)| = |f_1(1)| = f_1(1)$ and $\max_{|z|=1}|f_2(z)| = |f_2(1)| = f_2(1)$. Also, note that f_1 and f_2 are self-inversive polynomials. Thus, we have $\max_{|z|=1}|f_1'(z)| = (n/2)\max_{|z|=1}|f_1(z)| = (n/2)f_1(1)$ and $\max_{|z|=1}|f_2'(z)| = (n/2)\max_{|z|=1}|f_2(z)| = (n/2)f_2(1)$. Let θ_0 be the number such that $\max_{|z|=1}|f'(z)| = |f'(\mathrm{e}^{\mathrm{i}\theta_0})|$. Then

$$\max_{|z|=1}|f'(z)| = |f'(\mathrm{e}^{\mathrm{i}\theta_0})| \leq |f_1'(\mathrm{e}^{\mathrm{i}\theta_0})| + |f_2'(\mathrm{e}^{\mathrm{i}\theta_0})| = \frac{n}{2}\{f_1(1) + f_2(1)\} \tag{66}$$

Since, $\{f_1(1) + f_1(1)\} \leq \sqrt{2\{f_1^2(1) + f_2^2(1)\}} = \sqrt{2}|f(1)| \leq \sqrt{2}\max_{|z|=1}|f(z)|$, we get the desired result from (66). □

If f is a polynomial in $\mathcal{P}_n^\vee$ whose zeros lie in a sector of opening $\pi/2$, say in, $\psi \leq arg\ z \leq \psi + \pi/2$, for some real ψ, then the polynomial $g(z) = e^{-i\psi} f(z)$ belongs to $\mathcal{P}_n^\vee$ such that its coefficients lie in the first quadrant of the complex plane. Moreover $\max_{|z|=1}|g(z)| = \max_{|z|=1}|f(z)|$ and $\max_{|z|=1}|g'(z)| = \max_{|z|=1}|f'(z)|$. So applying Theorem 42 on $g(z)$ one can get the following result in Jain [57].

Theorem 43 *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ where $a_\nu = \alpha_\nu e^{i\phi} + \beta_\nu e^{i\psi}$, $\alpha_\nu \geq 0$, $\beta_\nu \geq 0$, where $0 \leq \nu \leq n$, $0 \leq |\phi - \psi| \leq \pi/2$, be a polynomial in $\mathcal{P}_n^\vee$. Then*

$$\max_{|z|=1}|f'(z)| \leq \frac{n}{\sqrt{2}}\max_{|z|=1}|f(z)|. \tag{67}$$

The result is best possible. Equality holds for the polynomial $p(z) = z^n + 2iz^{n/2} + 1$, n being an even integer.

Govil and Vetterlein [49] considered the class of self-reciprocal polynomials whose coefficients lie in a sector of an angle γ centered at origin. Their estimate for $\max_{|z|=1}|f'(z)|$ depends on the angle γ and contains Theorem 42 and Theorem 43 as special cases. More precisely, their result is

Theorem 44 *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, whose coefficients lie in a sector of opening γ with vertex at the origin, belong to $\mathcal{P}_n^\vee$. Then*

$$\max_{|z|=1}|f'(z)| \leq \frac{n}{2\cos(\gamma/2)}\max_{|z|=1}|f(z)| \qquad \left(0 \leq \gamma \leq \frac{2\pi}{3}\right). \tag{68}$$

Equality holds for the polynomial $f(z) = z^n + 2\mathrm{e}^{\mathrm{i}\gamma} z^{n/2} + 1$, where n is an even integer.

It is important to note that the above theorem produces better estimate than (10) only for $|\gamma| \leq 2\pi/3$.

For a polynomial f in $\mathcal{P}_n^\vee$, in general $\max_{|z|=1}|f(z)|$ can occur at any point on the unit circle; not necessarily at $z = 1$. Rahman and Tariq [79] observed that, under the condition of Theorem 44, a sharp estimate for $\max_{|z|=1}|f'(z)|$ in (68) can be obtained in terms of $|f(1)|$ rather than $\max_{|z|=1}|f(z)|$. Then it makes sense to take γ in $[0, \pi)$ instead of $[0, 2\pi/3]$. They used the theory of entire functions of exponential type to prove the following theorem.

Theorem 45 *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, whose coefficients lie in a sector of opening γ with vertex at the origin, belong to $\mathcal{P}_n^\vee$. Then*

$$\max_{|z|=1} |f'(z)| \leq \frac{n}{2\cos(\gamma/2)} |f(1)|. \tag{69}$$

In the case where n is even, the polynomial $p(z) := z^n + 2\,\mathrm{e}^{\mathrm{i}\gamma}\, z^{n/2} + 1$ shows that the above inequality is sharp for any $\gamma \in [0, \pi)$.

Next, we will discuss few integral inequalities of Bernstein type associated with self-reciprocal polynomials. We will start with a result of Aziz and Zerger [8] who considered the L^2 analogue of (10) for polynomials in $\mathcal{P}_n^\vee$ and proved that

Theorem 46 *If $f(z)$ belongs to $\mathcal{P}_n^\vee$, then*

$$\frac{n^2}{4} \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta \leq \int_0^{2\pi} |f'(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta \leq \frac{n^2}{2} \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta. \tag{70}$$

Both estimates are sharp. Equality holds on the right side of the inequality for $f(z) = c(z+1)^n$ for all $n \geq 1$. On the left side, equality holds for $f(z) = cz^{n/2}$ when n is even.

Alzer [1] extended the above result for higher order derivatives. He used ideas from discrete mathematics and obtained the following generalization.

Theorem 47 *Let $f(z)$ be a polynomial in $\mathcal{P}_n^\vee$ and k an integer such that $1 \leq k \leq n$. Then*

$$\alpha_n(k) \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta \leq \int_0^{2\pi} |f^{(k)}(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta \leq \beta_n(k) \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2 d\theta \tag{71}$$

where

$$\alpha_n(k) = \begin{cases} \Pi_{j=0}^{k-1} \left(\frac{n}{2} - j\right)^2 & n \text{ is even} \\ \frac{1}{2}\left\{\Pi_{j=0}^{k-1}\left(\frac{n+1}{2} - j\right)^2 + \Pi_{j=0}^{k-1}\left(\frac{n-1}{2} - j\right)^2\right\} & n \text{ is odd} \end{cases}$$

$$\beta_n(k) = \frac{1}{2} \Pi_{j=0}^{k-1} (n-j)^2.$$

The inequalities are best possible. Equality holds on the right side of inequality for $w(z) = z^n + 1$. On the left-hand side, the equality holds for $u(z) = z^{n/2}$ if n is even and for $v(z) = z^{n-1/2}(1+z)$ if n is odd.

Using Theorem 40 discussed earlier, Tariq [87] has found an L^p inequality for $\mathcal{P}_n^\vee$ that is valid for $p \geq 1$. More precisely he proved the following

Theorem 48 *Let f, which has all its zeros in the left-half plane, belong to $\mathcal{P}_n^\vee$. Furthermore, the zeros in the second quadrant are in the unit disk $\{z : |z| \leq 1\}$. Then, for $p \geq 1$*

$$\int_{-\pi}^{0} |f'(\mathrm{e}^{\mathrm{i}\theta})|^p \, d\theta \leq n^p \, C_p \int_{-\pi}^{0} |f(\mathrm{e}^{\mathrm{i}\theta})|^p \, d\theta, \tag{72}$$

where C_p is as given by

$$C_p = \frac{2\pi}{\int_{-\pi}^{\pi} |1 + e^{i\alpha}|^p d\alpha} = 2^{-p}\, \frac{\sqrt{\pi}\ \Gamma(p/2+1)}{\Gamma(p/2+1/2)}. \tag{73}$$

Recall that for a polynomial f in $\mathcal{P}_n$, $\|f\|_p = (1/2\pi \int_0^{2\pi} |f(e^{i\theta})|^p d\theta)^{1/p}$ and $\|f\|_\infty = \max_{|z|=1}|f(z)|$ denote the L^p and uniform norms respectively. Qazi [72] investigated a Bernstein type inequality in which he considered the L^p norm of the derivative f' and L^∞ norm of f and asked the question:

What is the best value for A_n in the following $\|f'\|_p \le A_n\|f'\|_\infty$, where $f \in \mathcal{P}_n^\vee$? In this direction, he proved the following

Theorem 49 *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n^\vee$ and $0 \le p \le 2$? Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(e^{i\theta})|^p\, d\theta \le \frac{n^p}{2^{p/2}}\{\|f\|_\infty^2 - 2|a_0|\}^{p/2}. \tag{74}$$

The example $f(z) = 1 + z^n$ shows that the above inequality is sharp.

We will close this section with an inequality in the opposite direction. Let f be a self-reciprocal polynomial. From the definition, we have $z^n f(1/z) \equiv f(z)$. If we differentiate both sides with respect to z, we get $f'(z) = nz^{n-1}f(1/z) - z^{n-2}f'(1/z)$. Choose the complex number z_0 on the unit circle $\{z : |z| = 1\}$, such that $|f(1/z_0)| = \max_{|z|=1}|f(z)|$. Then, we have

$$n\max_{|z|=1}|f(z)| = n|f(1/z_0)| = |f'(z_0) + z^{n-2}f'(1/z_0)| \le 2\max_{|z|=1}|f'(z)|.$$

Thus we have the following theorem of Dewan and Govil [22].

Theorem 50 *If $f \in \mathcal{P}_n^\vee$, then*

$$\max_{|z|=1}|f'(z)| \ge \frac{n}{2}\max_{|z|=1}|f(z)|. \tag{75}$$

The result is sharp. Equality holds for polynomial $f(z) = c(1+z)^n$.

Although the class $\mathcal{P}_n^\vee$ has been extensively studied among others by Frappier and Rahman [25] and Frappier et al. [27], Govil et al. [50], a sharp Bernstein's type inequality for this class is still unknown for $n \ge 3$.

3 Entire Functions of Exponential Type

In this section, we will study the extensions of results about polynomials, discussed in Sect. 2, to the entire functions of exponential type. We will start with the following definition of entire functions of exponential type.

Definition 3 An entire function f is said to be an entire function of exponential type τ if for every $\varepsilon > 0$ there is a constant $k(\varepsilon)$ depending only on ε but not on z such that $|f(z)| < k(\varepsilon)\,e^{(\tau+\varepsilon)|z|}$ for all $z \in \mathbb{C}$.

Let f be an entire function and r be any positive real number. Denote the maximum modulus of the function f on the circle of radius r by $M_f(r)$. That is $M_f(r) := \max_{|z|=r} |f(z)|$. If there is no ambiguity, we write $M_f(r) = M(r)$.
The order of an entire function f, denoted by ρ, is defined by

$$\rho := \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}. \tag{76}$$

It is a convention to take the order of a constant function of modulus less than or equal to one as 0.
An entire function of finite order ρ is said to have type T, where T is given by

$$T := \limsup_{r \to \infty} \frac{\log M(r)}{r^{\rho}}. \tag{77}$$

It is clear that entire functions of order less than 1 are of exponential type τ, where τ can be taken to be any number greater than or equal to 0. Also entire functions of order 1 and type $T \leq \tau$ are of exponential type τ.

Examples of entire functions of exponential type include polynomials with complex coefficients, $t(z) = \sum_{\nu=0}^{n} a_\nu \cos \nu z + b_\nu \sin \nu z$, where coefficients belong to $\mathbb{C}$, etc.

Definition 4 Let f be an entire function of exponential type. The function

$$h_f(\theta) := \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r}, \qquad (0 \leq \theta < 2\pi) \tag{78}$$

is called the indicator function of f. It describes the growth of the function along a ray $\{z : argz = \theta\}$. It is finite or $-\infty$. Unless $h_f(\theta) \equiv -\infty$, it is a continuous function of θ. If f is an entire function of exponential type τ, then $h_f(\theta) \leq \tau$, for $0 \leq \theta \leq 2\pi$.

Bernstein (see [9, p. 102]) himself found the extension of inequality (10) for the entire functions of exponential type. He proved that

Theorem 51 *If f is an entire function of exponential type τ, then*

$$\sup_{-\infty < x < \infty} |f'(x)| \leq \tau \sup_{-\infty < x < \infty} |f(x)|. \tag{79}$$

Equality holds if and only if $f(z) = a\,e^{i\tau z} + b\,e^{-i\tau z}$, where a, $b \in \mathbb{C}$ and $|a| + |b| > 0$.

Genčev [31] observed that the conclusion of above theorem is still valid even if one considers the supremum of $|\mathrm{Re}\, f(x)|$ instead of $|f(x)|$ over $\mathbb{R}$ in (79). Using Levitan polynomials [55], he proved the following extension of Bernstein's inequality.

Theorem 52 *If f is an entire function of exponential type τ such that $h_f(\pi/2) \leq 0$, then*

$$\sup_{-\infty < x < \infty} |f'(x)| \leq \tau \sup_{-\infty < x < \infty} |\mathrm{Re}\, f(x)|. \tag{80}$$

Equality holds for $f(z) = a\,\mathrm{e}^{\mathrm{i}\tau z}$.

This result may be seen as a generalization of the result of Szegö (Theorem 9). For various other refinements of Theorem 51, we refer readers to [13, Chap. 11].

Let $p > 0$ be a real number. We say that a function f belongs to L^p on the real line if, $\int_{-\infty}^{\infty} |f(x)|^p\, dx < \infty$. It can be verified that $\lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p} = \sup_{-\infty<x<\infty} |f(x)|$. In view of this, the following generalization of Theorem 51 is given in the next theorem [13, p. 211].

Theorem 53 *Let f be an entire function of exponential type τ that belongs to L^p on the real line, where $p \geq 1$ is a real number. Then*

$$\int_{-\infty}^{\infty} |f'(x)|^p\, dx \leq \tau^p \int_{-\infty}^{\infty} |f(x)|^p\, dx. \tag{81}$$

For various refinements and extensions of above result, we refer readers to the paper of Rahman and Schmeisser [77].
As an L^p analogue of Theorem 52, Dostanić [23] has recently proved the following

Theorem 54 *If f is an entire function of exponential type τ such that $h_f(\pi/2) \leq 0$, then*

$$\int_{-\infty}^{\infty} |f'(x)|^p\, dx \leq C_p \tau^p \int_{-\infty}^{\infty} |\mathrm{Re} f(x)|^p\, dx \qquad (p \geq 1) \tag{82}$$

where C_p is given by (73).

3.1 Bernstein Type Inequalities for Entire Functions Having No Zero in the Upper-Half Plane

In this section, we will discuss few inequalities about entire functions of exponential type when the function has no zero in the open upper-half plane $\{z : Im(z) > 0\}$. The theorems discussed here may be seen as extension of results in Sect. 2.1 for entire functions of exponential type.

To motivate ourself, let us take a polynomial g in $\mathcal{P}_n$ such that $g(z) \neq 0$ in $|z| < 1$. From Theorem 15, $\max_{|z|=1}|g'(z)| \leq (n/2)\max_{|z|=1}|g(z)|$. Define a function $f(z) = g(\mathrm{e}^{\mathrm{i}z})$. It is obvious that f is an entire function of exponential type n. Since g has no zero in $|z| < 1$, f has no zero in the open-half plane $\{z : Im(z) > 0\}$ and $h_f(\pi/2) = 0$. Thus, if we can obtain a bound for $\sup_{-\infty<x<\infty} |f'(x)|$ in terms of $\sup_{-\infty<x<\infty} |f(x)|$, then it will give us a generalization of Lax's result, Theorem 15.

Perhaps, in view of these observations, Boas [14] (see also [75]) formulated and proved the following general result for entire functions of exponential type.

Theorem 55 *Let f be an entire function of exponential type τ such that $h_f(\pi/2) = 0$ and $f(x + \mathrm{i}y) \neq 0$ for $y > 0$. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \frac{\tau}{2} \sup_{-\infty<x<\infty} |f(x)|. \tag{83}$$

Equality is attained for $f(x) = (1 + \mathrm{e}^{\mathrm{i}\tau z})/2$.

For generalizations of the above inequality of Boas to polar derivatives of entire functions, see Gardner and Govil [29].

It is worth pointing out that the condition $h_f(\pi/2) = 0$ in the theorem is indeed necessary. To see it, let $f(z) = \cos\tau z, \tau > 0$. The function f is an entire function of exponential type τ with only real zeros. Furthermore, $\sup_{-\infty<x<\infty} |f(x)| = 1$ and

$$h_f\left(\frac{\pi}{2}\right) = \limsup_{y\to\infty} \frac{\log\left(\frac{\mathrm{e}^{-\tau y}+\mathrm{e}^{\tau y}}{2}\right)}{y} = \tau > 0$$

and

$$\sup_{-\infty<x<\infty} |f'(x)| = \tau = \tau \sup_{-\infty<x<\infty} |f(x)|$$

which contradicts the conclusion of the theorem.

In 1959, Professor R.P. Boas asked the following question concerning the generalization of Theorem 55, a partial answer to which was given by Govil and Rahman [48].

Let f be an entire function of exponential type τ such that $|f(x)| \le 1$ for real x, $h_f(\pi/2) = 0$ and $f(x + \mathrm{i}y) \ne 0$ for $y > k$, where $-\infty < k < \infty$. Then what can be said about the bound for $|f'(x)|$?

The hypothesis $f(x + \mathrm{i}y) \ne 0$ for $y > k$ is a more general than $f(x + \mathrm{i}y) \ne 0$ for $y > 0$, if $k < 0$. So one might expect an improved estimate in (83) under this restriction. However, the following example of Govil and Rahman [48] shows that it is not the case.

Example

Let n_1, n_2, and n_3 be positive integers, $\tau = n_1/n_2$, $a = 1/n_3n_2$, and $k \le 0$. Define a function f_a as follows

$$f_a(z) = \left\{\frac{e^{\mathrm{i}az} - e^{-ak}}{1 + e^{-ak}}\right\}^{\tau/a}.$$

It is clear that $f_a(z)$ is an entire function of exponential type τ with $h_{f_a}(\pi/2) = 0$, $\sup_{-\infty<x<\infty} |f_a(x)| = 1$, and $f_a(z) \equiv f_a(x + \mathrm{i}y)$ has all its zeros on $y = k$. Also

$$\sup_{-\infty<x<\infty} |f_a'(x)| = \sup_{-\infty<x<\infty} \left\{\frac{e^{\mathrm{i}az} - e^{-ak}}{1 + e^{-ak}}\right\}^{\tau/a-1} \frac{\tau}{1 + e^{-ak}} = \frac{\tau}{1 + e^{-ak}} > \frac{\tau}{2} - \varepsilon$$

by making a sufficiently small.

Thus the bound in (83) cannot be improved in general by simply taking $f(z) \ne 0$ in a larger half plane unless some more conditions are imposed on the function. In addition to the already given conditions, Govil and Rahman [48] added a restriction on the indicator function of f' and found the following extension of Theorem 55.

Theorem 56 *Let f be an entire function of exponential type τ such that $f(x+\mathrm{i}y) = 0$ for $y = k$ where $k \le 0$. If $h_f(\pi/2) = 0,\ h_{f'}(\pi/2) = -c < 0$, then*

$$\sup_{-\infty<x<\infty} |f'(x)| \le \frac{\tau}{1 + e^{c|k|}} \sup_{-\infty<x<\infty} |f(x)|. \tag{84}$$

The inequality is sharp. Equality holds for

$$f_c(z) = \left\{ \frac{e^{icz} - e^{-ak}}{1 + e^{-ck}} \right\}^{\tau/c} \tag{85}$$

if τ/c is a positive integer.

Even though the zero free-half plane $\{z : Im(z) > k\}, k \leq 0$ in Theorem 56 is larger than $\{z : Im(z) > 0\}$ but the condition that requires all the zeros of f to lie on a horizontal line $y = k$ is still too restrictive. Govil and Rahman [48] were able to relax this restriction also by imposing one more condition on the conjugate of the function f. The conjugate of an entire function f of exponential type τ is a function g defined by $g(z) = \mathrm{e}^{i\tau z}\overline{f(\overline{z})}$. They proved the following theorem.

Theorem 57 *Let f be an entire function of exponential type τ such that $f(x + \mathrm{i}y) \neq 0$ for $y > k$ where $k \leq 0$. If $h_f(\pi/2) = 0$, $h_{f'}(\pi/2) = -c < 0$, and $h_{g'}(\pi/2) = -c < 0$ where $g(z) = \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\overline{z})}$. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \frac{\tau}{1 + e^{c|k|}} \sup_{-\infty<x<\infty} |f(x)|. \tag{86}$$

If τ/c is a positive integer, then the function

$$f_c(z) = \left\{ \frac{e^{icz} - e^{-ak}}{1 + e^{-ck}} \right\}^{\tau/c}$$

satisfies the condition of the above theorem and

$$\sup_{-\infty<x<\infty} |f_c'(x)| = \frac{\tau}{1 + e^{c|k|}} \sup_{-\infty<x<\infty} |f_c(x)|.$$

It may be remarked that although Theorem 57 answers question raised by Professor R. P. Boas, Jr. in the case when $f(x + \mathrm{i}y) \neq 0$ for $y > k$ where $k \leq 0$, the case when $f(x + \mathrm{i}y) \neq 0$ for $y > k$ where $k \geq 0$ is still completely unsolved.
Theorem 57 generalizes the result of Malik [60] discussed in Sect. 2.1. To see this, let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial such that $p(z) \neq 0$ in $|z| \leq K$ where $K \geq 1$. The function $f(z) = \sum_{\nu=0}^{n} a_\nu \mathrm{e}^{i\nu z}$ is an entire function of exponential type n. Since $p(z) \neq 0$ in the disk $|z| < K$, $f(z)$ has all its zeros in $y > k$ and $h_f(\pi/2) = 0$. For $z = iy$, we have $f'(iy) = \sum_{\nu=1}^{n} \mathrm{i}\ \nu\ a_\nu \mathrm{e}^{-\nu y}$. Thus $|f'(\mathrm{i}y)| \leq \mathrm{e}^{-y}(1 + \phi(y))$, where $\phi(y) \to 0$ as $y \to 0$. Thus we have $h_{f'}(\pi/2) \leq -1$. The conjugate of f is given by $g(z) = \sum_{\nu=0}^{n} \overline{a_{n-\nu}} \mathrm{e}^{i\nu z}$. Similar reasoning as used in the case of $h_{f'}(\pi/2)$, gives $h_{g'}(\pi/2) \leq -1$ as well. So all the conditions of Theorem 57 are satisfied. Thus we have

$$|f'(x)| \leq \frac{n}{1 + e^{|k|}} \sup_{-\infty<x<\infty} |f(x)|.$$

If p is a nonconstant polynomial such that $|p(z)| \geq m > 0$ on the unit circle, then $|p(z)| > m$ in the open unit disk U provided that $p(z) \neq 0$ in U. Therefore $p(z) - \lambda m \neq 0$ in U for any λ such that $|\lambda| \leq 1$. This fact plays an important

role in obtaining a generalization of a Theorem of Aziz and Dawood [5], which is a refinement of Erdös conjecture proved by Lax [59]. However, if f is a nonconstant transcendental entire function of exponential type such that $|f(x)| \geq m > 0$ on the real axis, then it may be that $|f(z)|$ is not greater than m at any point of the upper-half plane $H := \{z : Im(z) > 0\}$, even if $f(z) \neq 0$ in H, as the example $f(z) := e^{iz}$ shows.

Note that for the function $f(z) = e^{iz}$, which is an entire function of exponential type, we have $h_f(\pi/2) = -1$, but if we assume that $h_f(\pi/2) \geq 0$, and that f has no zeros in H, it turns out that $|f(z)| > m$ everywhere in H. Keeping this in view, Govil et al. [52] have proved the following more general result.

Theorem 58 *Let f be an entire function of exponential type having no zeros in the closed upper-half plane $\overline{H}$, and suppose that $|f(x)| \geq m > 0$ on the real axis. Furthermore, let $h_f(\pi/2) = a$. Then*

$$|f(x + \mathrm{i}y)| > m\mathrm{e}^{a\,y} \qquad (y > 0\,,\ x \in \mathbb{R}) \tag{87}$$

except for $f(z) := c\,\mathrm{e}^{-\mathrm{i}az}$, $c \in \mathbb{C}$, $|c| = m$.

Making use of the above Theorem 58, Govil et al. in [52] have proved the following sharpening of Theorem 55.

Theorem 59 *Let f be an entire function of exponential type τ such that $h_f(\pi/2) = 0$ and $f(x + \mathrm{i}y) \neq 0$ for $y > 0$. Assume that for $x \in \mathbb{R}$, $0 \leq m \leq |f(x)| \leq M$ where $M = \sup_{-\infty<x<\infty} |f(x)|$. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \leq \frac{M - m}{2}\tau. \tag{88}$$

Equality is attained for $f(x) = (M + m)/2\mathrm{e}^{\mathrm{i}\alpha} + (M - m)/2\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{\mathrm{i}\tau z}, \alpha, \beta \in \mathbb{R}$.

The theorem reduces to Theorem 55, when $m = 0$, and includes the theorem of Aziz and Dawood (29) discussed in Sect. 2.1.

Next, we will discuss the L^p analogues of Theorem 55 and Theorem 57. Rahman [75] found the following result which provides the L^p analogue of Theorem 55 of Boas [14].

Theorem 60 *If f is an entire function of exponential type τ in L^p, $p \geq 1$ such that $f(x + \mathrm{i}y) \neq 0$ for $y > 0$, $h_f(\pi/2) = 0$, then*

$$\int_{-\infty}^{\infty} |f'(x)|^p\, dx \leq \tau^p\, C_p \int_{-\infty}^{\infty} |\, f(x)|^p dx\,,$$

where C_p is as given in (73).

The L^p inequality corresponding to Theorem 57 has been given by Govil and Rahman [48]. They in fact proved the following.

Theorem 61 *Let f be an entire function of exponential type τ such that $f(x + \mathrm{i}y) \neq 0$ for $y > k$ where $k \leq 0$. If $h_f(\pi/2) = 0$, $h'_f(\pi/2) = -c < 0$, and*

$h'_g(\pi/2) = -c < 0$ *where* $g(z) = \mathrm{e}^{\mathrm{i}\tau z}\overline{f(\overline{z})}$. *Then for* $p \geq 1$

$$\int_{-\infty}^{\infty} |f'(x)|^p \, dx \leq \tau^p \; D_p \int_{-\infty}^{\infty} |\, f(x)|^p dx \,,$$

where D_p *is as given by*

$$D_p = \frac{2\pi}{\int_{-\pi}^{\pi} |e^{c|k|} + e^{i\alpha}|^p d\alpha}. \tag{89}$$

3.2 Bernstein Type Inequalities for Entire Functions Having No Zero in the Lower-Half Plane

In this section, we will study the Bernstein's type inequalities for entire functions of exponential type which has no zero in the lower open-half plane $\{z : Im(z) < 0\}$. The theorems discussed here may be seen as extensions of results in Sect. 2.2 for entire functions of exponential type.

Let us take a polynomial g in $\mathcal{P}_n$ such that $g(z) \neq 0$ for $|z| > 1$. From Theorem 27, $\max_{|z|=1}|g'(z)| \geq (n/2)\max_{|z|=1}|g(z)|$. Define a function $f(z) = g(\mathrm{e}^{\mathrm{i}z})$. Clearly, f is an entire function of exponential type n and has no zero in the open-half plane $\{z : Im(z) < 0\}$. Also $h_f(\pi/2) \leq 0$, and therefore to obtain generalization of Theorem 27 of Turan [88] for entire functions of exponential type, Rahman [73] proved the following.

Theorem 62 *Let* f *be an entire function of exponential type* τ *such that* $f(z) \equiv f(x + \mathrm{i}y) \neq 0$ *for* $y < 0$, $h_f(\pi/2) \leq 0$ *and* $h_f(-\pi/2) \leq \tau$. *Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \;\geq\; \frac{\tau}{2} \sup_{-\infty<x<\infty} |f(x)|. \tag{90}$$

Equality is attained for $f(x) = (1 + \mathrm{e}^{\mathrm{i}\tau z})/2$.

Govil [36] considered the following generalization of the above theorem. *Let* f *be an entire function of exponential type* τ *such that* $|f(x)| \leq 1$ *for* $x \in \mathbb{R}$, $h_f(\pi/2) \leq 0$, $h_f(-\pi/2) = \tau$ *and* $f(x + \mathrm{i}y) \neq 0$ *for* $y < k \leq 0$. *What is the best bound for* $|f'(x)|$?

In this direction, he proved that

Theorem 63 *If* f *is an entire function of exponential type of order 1, type* τ *such that* $f(x + \mathrm{i}y) \neq 0$ *for* $y < k \leq 0$, $h_f(\pi/2) \leq 0$, $h_f(-\pi/2) = \tau$, *then*

$$\sup_{-\infty<x<\infty} |f'(x)| \;\geq\; \frac{\tau}{1 + e^{\tau|k|}} \sup_{-\infty<x<\infty} |f(x)|. \tag{91}$$

The result is best possible. Equality holds for the function

$$f(z) = \frac{e^{i\tau z} - e^{-\tau k}}{1 + e^{-\tau k}}.$$

Let us study another generalization of Theorem 62. One of the hypotheses in the theorem states that the function f has no zero in the open-lower-half plane $\{z : Im(z) < 0\}$. Govil et al. [52] looked for the improvement in the conclusion under the assumption that $|f(x)| \geq m \geq 0$ for $x \in \mathbb{R}$. If $m > 0$, then f will have no zero in the closed-half plane $\{z : Im(z) \leq 0\}$ and one should expect a better estimate in Theorem 62. Using Theorem 59, they obtained the following result which can be seen as yet another generalization of Theorem 62.

Theorem 64 *Let f be an entire function of order-one type τ such that $f(x+\mathrm{i}y) \neq 0$ for $y < 0$. For $x \in R$, $0 \leq m \leq |f(x)| \leq M$ where $M = \sup_{-\infty<x<\infty} |f(x)| < \infty$, and $h_f(\pi/2) \leq 0$. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \geq \frac{M+m}{2}\tau. \tag{92}$$

Theorem 64 reduces to Theorem 62, when $m = 0$ and includes the theorem of Aziz and Dawood (45) discussed in Sect. 2.2.

3.3 Bernstein Type Inequalities for Subclass of Entire Functions Satisfying $f(z) = e^{i\tau z} f(-z)$

In this section, we will discuss some results for the class of entire functions of exponential type that can be seen as an extension of class of self-reciprocal polynomials.

Note that if $g(z)$ is a polynomial of degree n then the function $f(z) = g(\mathrm{e}^{\mathrm{i}z})$ is an entire function of exponential type n. Further, if the polynomial $g(z)$ is self reciprocal then, as is easy to see, the function $f(z)$ will satisfy the condition

$$f(z) \equiv \mathrm{e}^{\mathrm{i}nz} f(-z). \tag{93}$$

Therefore, the class of entire functions of exponential type τ whose elements satisfy the condition $f(z) \equiv \mathrm{e}^{\mathrm{i}\tau z} f(-z)$ is a natural extension of the class of self-reciprocal polynomials. Let us denote the class of such entire functions of exponential type by $\mathcal{F}_\tau^\vee$.

Govil [42] considered this class and proved several results. For example, he proved the following theorem which is a generalization of (75) for entire functions of exponential type. He deduced the conclusion as a consequence of another inequality he proved for this class. In this chapter we will give a direct proof.

Theorem 65 *If $f \in \mathcal{F}_\tau^\vee$, then*

$$\sup_{-\infty<x<\infty} |f'(x)| \geq \frac{\tau}{2} \sup_{-\infty<x<\infty} |f(x)|. \tag{94}$$

The result is best possible and the equality holds for $f(z) = (1 + e^{i\tau z})$.

Proof Let f be a function in $\mathcal{F}_\tau^\vee$ and x, an arbitrary real number. On the real line, the function f satisfies the condition $f(x) \equiv e^{i\tau x} f(-x)$. Differentiating both sides with respect to x, we get $f'(x) + e^{i\tau x} f'(-x) = i\tau e^{i\tau x} f(-x)$. Using triangle inequality, we get $|\tau f(-x)| \le |f'(x)| + |f'(-x)| \le 2|\sup_{-\infty<x<\infty} |f'(x)|$. Since x is an arbitrary real number, we get $\sup_{-\infty<x<\infty} |f(x)| \le 2\sup_{-\infty<x<\infty} |f'(x)|$ and the result follows. □

We know from Theorem 51 that equality in (79) holds if and only if the function is of the form $a\,e^{i\tau z} + b\,e^{-i\tau z}$, where $a, b \in \mathbb{C}$ and $|a| + |b| > 0$. It is obvious that the functions in $\mathcal{F}_\tau^\vee$ cannot be of the form $a\,e^{i\tau z} + b\,e^{-i\tau z}$ and hence equality cannot hold in (79) for functions in $\mathcal{F}_\tau^\vee$. Thus, for any f in $\mathcal{F}_\tau^\vee$

$$\frac{\sup_{-\infty<x<\infty} |f'(x)|}{\sup_{-\infty<x<\infty} |f(x)|} < \tau.$$

So the question is: *what is the best estimate in Theorem 51, if* $f \in \mathcal{F}_\tau^\vee$*?*

Rahman and Tariq [80] have shown that it could be as close to τ as one wish. In fact, the following result holds true.

Theorem 66 *Given any number* $\varepsilon \in (0, \tau)$*, we can find an entire function* $f_\varepsilon \in \mathcal{F}_\tau^\vee$ *such that*

$$\sup_{-\infty<x<\infty} |f_\varepsilon'(x)| \ge (\tau - \varepsilon) \sup_{-\infty<x<\infty} |f_\varepsilon(x)|.$$

This theorem may be seen as an extension of (59) for entire functions of exponential type.

Recently, Tariq [87] has investigated the functions in $\mathcal{F}_\tau^\vee$ whose zeros satisfy certain conditions. Let f belong to $\mathcal{F}_\tau^\vee$ and ζ, a zero of f. From the definition of $\mathcal{F}_\tau^\vee$, $-\zeta$ is also a zero of f. Thus f has half of its zero in the upper-half plane. Also if ζ lies in the first quadrant then $-\zeta$ will lie in the third quadrant. Tariq [87] has recently observed the following property of functions in $\mathcal{F}_\tau^\vee$ whose zeros lie in the first and the third quadrants.

Theorem 67 *Let* f*, which has all its zeros in the first and the third quadrants, belong to* $\mathcal{F}_\tau^\vee$*. Then*

$$|f'(-x)| \le |f'(x)| \qquad (x > 0). \tag{95}$$

Using above observation, he has obtained few new inequalities for functions in $\mathcal{F}_\tau^\vee$. We will state one of them here [87].

Theorem 68 *Let* f*, which has all its zeros in the first and the third quadrants, belong to* $\mathcal{F}_\tau^\vee$*. Further assume that* $|f(x)| \le M$ *on* $(-\infty, 0)$*. Then*

$$|f'(x)| \le \frac{M\tau}{2} \qquad (x \le 0). \tag{96}$$

The estimate is sharp as the example $M(1 + e^{i\tau z})/2$ *shows.*

Rahman and Tariq [80] formulated and proved a theorem that can be seen as a generalization of Theorem 44, which is due to Govil and Vetterlien [49]. The main issue they encountered while deciding about the extension of Theorem 44 to the entire function of exponential type was:

What class of entire functions of exponential type would admit an extension of Theorem 44?

If one simply takes functions of the form $f(z) = p(\mathrm{e}^{\mathrm{i}z}) = \sum_{\nu=0}^{n} a_\nu \, \mathrm{e}^{\mathrm{i}\nu z}$ and required coefficients to lie in a sector, then it is indeed an entire function of exponential type but too restrictive as an arbitrary entire function of exponential type, in general, cannot be expressed as a finite or infinite sum of the form $\sum a_\nu \, \mathrm{e}^{\mathrm{i}\nu z}$. According to Rahman and Tariq [80] an appropriate class of entire functions of exponential type for which Theorem 45 would admit an extension is the one whose elements are uniformly almost periodic on the real line. For the definition and the related materials on uniformly almost periodic functions, we refer readers to [11, 16, 80].

Under certain conditions, functions that are uniformly almost periodic on the real line will have a Fourier series expansion of the form $\sum a_\nu \, \mathrm{e}^{\mathrm{i}\lambda_\nu z}$. The a_ν's are called Fourier coefficients and λ_ν's are called Fourier exponents. By putting certain restrictions on the Fourier coefficients, Rahman and Tariq [80] formulated and proved the following theorem for entire functions of exponential type which can be seen as an extension of Theorem 45.

Theorem 69 *Let $f \in \mathcal{F}_\tau^\vee$ be uniformly almost periodic on the real axis, with Fourier series $f(x) \sim \sum_{n=1}^{\infty} A_n \, \mathrm{e}^{\mathrm{i}\Lambda_n x}$, where the coefficients $A_1, A_2, \ldots$ lie in a sector of opening $\gamma \in [0\,,\,\pi)$ with vertex at the origin. Then*

$$\sup_{-\infty<x<\infty} |f'(x)| \;\le\; \frac{\tau}{2\,\cos(\gamma/2)}\,|f(0)|\,. \tag{97}$$

The example $f(z) := \mathrm{e}^{\mathrm{i}\tau z} + 2\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}\tau z/2} + 1$ shows that the estimate is sharp.

Let us now turn our attention to some integral inequalities associated with $\mathcal{F}_\tau^\vee$. It is well known, see for example [11, p. 15], that if a function f is uniformly almost periodic on the real line, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)dx = \lim_{T\to\infty}\frac{1}{T}\int_{-T}^0 f(x)dx = \lim_{T\to\infty}\frac{1}{2\,T}\int_{-T}^T f(x)dx \tag{98}$$

exists. These integrals are called the mean value of the function. We will denote the mean value of f by $\mathcal{M}(f)$. It is also known that the absolute value $|f|$ and the derivative f' of a uniformly almost periodic function f are also uniformly almost periodic [11, pp. 3–6]. These two results ensure that $\mathcal{M}(|f|)$ and $\mathcal{M}(|f'|)$ exist. Thus the following theorem is a generalization of (70) for entire functions of exponential type.

Theorem 70 *Let $f \in \mathcal{F}_\tau^\vee$ be a uniformly almost periodic function on the real line. Then*

$$\frac{\tau^2}{4}\mathcal{M}(|f|^2) \le \mathcal{M}(|f'|^2) \le \frac{\tau^2}{2}\mathcal{M}(|f|^2). \tag{99}$$

The right side of the above inequality is sharp as equality holds for $f(z) := (1 + e^{i\tau z})/2$. *By taking* $f(z) := e^{i\tau z/2}$, *we see that the left-hand side of the inequality is also sharp.*

The right side of the inequality was proved earlier by Rahman and Tariq [81]. However for the sake of completeness, we will outline the proof of both sides of the inequality.

Proof Let f, a uniformly periodic function on $\mathbb{R}$, belong to $\mathcal{F}_\tau^\vee$ and λ be an arbitrary real number. It is well known that $\mathcal{M}\{e^{-i\lambda x} f(x)\}$, the mean value of $e^{-i\lambda x} f(x)$, is 0 except for at the most countably many λ's where [11, p. 18]

$$\mathcal{M}\{e^{-i\lambda x} f(x)\} = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\lambda x} f(x) dx. \tag{100}$$

Let $\Lambda = \{\Lambda_1, \Lambda_2, \cdots\}$ be the collections of λ's for which $\mathcal{M}\left\{e^{-i\lambda x} f(x)\right\} \neq 0$. The elements of Λ are called Fourier exponents of the function f. Let Λ_ν be a Fourier exponent. The mean value $\mathcal{M}\{e^{-i\Lambda_\nu x} f(x)\}$ is called the Fourier coefficient corresponding to Fourier exponent Λ_ν and is denoted by A_ν .

One can associate a series (see [11, p. 18]) called Fourier series $\sum_{\nu=0}^\infty A_\nu e^{i\Lambda_\nu x}$ with a uniformly almost periodic function f . We denote it by $f(x) \sim \sum_{\nu=0}^\infty A_\nu e^{i\Lambda_\nu x}$. From Bohr's fundamental theorem ([16], p. 17), we have

$$\mathcal{M}(|f|^2) = \sum_{\nu=0}^\infty |A_\nu|^2. \tag{101}$$

Since, $e^{i\tau x}$ is periodic and hence uniformly almost periodic, and $f(x)$ is given to be uniformly almost periodic, the product $g(x) = e^{i\tau x} f(-x)$ is also uniformly almost periodic [11, p. 6]. Thus the Fourier series of $g(x)$ can be obtained by multiplying the Fourier series of $f(-x)$ by $e^{i\tau x}$. So $g(x) \sim \sum_{\nu=0}^\infty A_\nu e^{i(\tau-\Lambda_\nu)x}$. Since $f(x) \equiv g(x)$ for $x \in \mathbb{R}$, $f(x) \sim \sum_{\nu=0}^\infty A_\nu e^{i\Lambda_\nu x}$ and $g(x) \sim \sum_{\nu=0}^\infty A_\nu e^{i(\tau-\Lambda_\nu)x}$ have to be the same. We conclude that $\tau - \Lambda_\nu$ is a Fourier exponent of f if Λ_ν is.

From a result of Boas [12] (also see [80, Lemma 3]), one has $|\Lambda_\nu| \leq \tau$ and $|\tau - \Lambda_\nu| \leq \tau$ for each ν, which actually implies that $0 \leq \Lambda_\nu \leq \tau$.

f' and g' are also uniformly almost periodic (see [11, Chap. 3]) with $f'(x) \sim \sum_{\nu=0}^\infty A_\nu i\Lambda_\nu e^{i\Lambda_\nu x}$ and $g'(x) \sim \sum_{\nu=0}^\infty A_\nu i(\tau - \Lambda_\nu) e^{i(\tau-\Lambda_\nu)x}$ respectively. Once again from Bohr's Theorem, $\mathcal{M}(|f'|^2) = \sum_{\nu=0}^\infty |A_\nu|^2 \Lambda_\nu^2$ and $\mathcal{M}(|g'|^2) = \sum_{\nu=0}^\infty |A_\nu|^2 (\tau - \Lambda_\nu)^2$. Since $f(x) \equiv g(x)$, we have $f'(x) = g'(x)$ as well and hence $\mathcal{M}(|f'|^2) = \mathcal{M}(|g'|^2)$. Thus

$$\mathcal{M}(|f'|^2) = \frac{\mathcal{M}(|f'|^2) + \mathcal{M}(|g'|^2)}{2} = \sum_{\nu=0}^\infty \frac{(\tau - \Lambda_\nu)^2 + \Lambda_\nu^2}{2} |A_\nu|^2. \tag{102}$$

It can be easily checked that for $0 \leq \Lambda_\nu \leq \tau$,

$$\frac{\tau^2}{4} \leq \frac{(\tau - \Lambda_\nu)^2 + \Lambda_\nu^2}{2} \leq \frac{\tau^2}{2}.$$

So, we have

$$\frac{\tau^2}{4}\sum_{\nu=0}^{\infty}|A_\nu|^2 \leq \sum_{\nu=0}^{\infty}\frac{(\tau-\Lambda_\nu)^2+\Lambda_\nu^2}{2}|A_\nu|^2 \leq \frac{\tau^2}{2}\sum_{\nu=0}^{\infty}|A_\nu|^2. \tag{103}$$

From (101), (102), and (103) we get

$$\frac{\tau^2}{4}\mathcal{M}(|f|^2) \leq \mathcal{M}(|f'|^2) \leq \frac{\tau^2}{2}\mathcal{M}(|f|^2)$$

and the proof is complete.
$f(z) := e^{i\tau z/2}$ shows that the left-hand inequality is sharp, because $\mathcal{M}(|f|^2) = 1$, $f'(x) = \tau/2e^{i\tau x/2}$, and $\mathcal{M}(|f'|^2) = \tau^2/4$. So $\mathcal{M}(|f'|^2) = \tau^2/4\mathcal{M}(|f|^2)$. To see that the right-hand inequality is sharp, take $f(z) := (1 + e^{i\tau z})/2$ and note that $\mathcal{M}(|f'|^2) = \tau^2/4$, and $\mathcal{M}(|f|^2)$ is $\lim_{T\to\infty}(1/T)\int_0^T(1 + e^{i\tau x}/2)dx = 1/2$. So $\mathcal{M}(|f'|^2) = \tau^2/2\,\mathcal{M}(|f|^2)$. □

For functions in $\mathcal{F}_\tau^\vee$ which belong to L^2 on the real line, Rahman and Tariq [81] have proved the following

Theorem 71 *Let f belong to $\mathcal{F}_\tau^\vee$ such that$\int_\infty^\infty|f(x)|^2dx < \infty$. Then*

$$\int_{-\infty}^{\infty}|f'(x)|^2dx \leq \frac{\tau^2}{2}\int_{-\infty}^{\infty}|f(x)|^2dx \tag{104}$$

The coefficient $(\tau^2/2)$ in (104) can not be replaced by a smaller number.

We observe that under the condition given in Theorem 71, one can even prove that

$$\frac{\tau^2}{4}\int_{-\infty}^{\infty}|f(x)|^2dx \leq \int_{-\infty}^{\infty}|f'(x)|^2dx \leq \frac{\tau^2}{2}\int_{-\infty}^{\infty}|f(x)|^2dx. \tag{105}$$

Then (105) can be seen as an extension of Theorem 46 for entire functions of exponential type.

We end this section by stating an inequality recently obtained by Tariq [87]. It is an L^p analogue of Theorem 48 on the half line $(-\infty, 0)$.

Theorem 72 *Let f, which has all its zeros in the first and the third quadrants, belong to$\mathcal{F}_\tau^\vee$. Further suppose that $f \in L^p$ on $(-\infty, 0)$. Then, for $p \geq 1$*

$$\int_{-\infty}^{0}|f'(x)|^p\,dx \leq \tau^p\,C_p\int_{-\infty}^{0}|f(x)|^p\,dx \tag{106}$$

where C_p is as given in (73).

4 Some Open Problems

In this section, we present some of the problems discussed in Sects. 1–3 of this chapter, which we believe are still open. Since some of these problems have been open for quite some time, there is a possibility that some of them might have already been solved or a significant progress made toward their solution, of which we may not be aware.

Also, it may be remarked that none of these problems are due to authors of this chapter. In fact, these problems were told to the authors of this chapter by other mathematicians and the authors only worked to solve these problems, and in some cases, made some progress.

Problem 1 The problem of finding a sharp inequality analogous to the inequality (20) due to Malik [60] when $K < 1$ is still open. The sharp inequality is not known even for $n = 2$ except in the case where both the zeros lie on $|z| = K$. This problem was told to us by Professor Q. I. Rahman.

Problem 2 We believe that the inequality (23) in Theorem 17, which is due to Govil and Rahman [48] is not sharp, and thus the problem of finding a sharp inequality would be of interest and is open.

Problem 3 The problem of obtaining sharp bound in Theorem 25, which is due to Govil and Rahman [48], is open. The inequality obtained in Theorem 25 is not best possible and the best possible inequality is not available even in the case when $p = 2$. Similarly, the problem of obtaining a sharp inequality in Theorem 26 is also open.

Problem 4 It was proposed by late Professor R. P. Boas, Jr. to obtain an inequality corresponding to Bernstein's inequality when the polynomial f has k $(0 < k < n)$ zeros inside the unit circle. In this connection, it was shown by Giroux and Rahman [32] that for every positive integer n, there exists a polynomial $f(z)$ of degree n having a zero on $|z| = 1$, such that

$$\max_{|z|=1}|f'(z)| \geq (n - c/n)\max_{|z|=1}|f(z)|.$$

On the other hand for an arbitrary polynomial $f(z)$ of degree n having a zero on $|z| = 1$, they showed that

$$\max_{|z|=1}|f'(z)| \leq \left(n - \frac{1 - \sin 1}{4\pi n}\right)\max_{|z|=1}|f(z)|.$$

Also, S. Ruscheweyh, in 1986 has shown that there exist polynomials $f(z)$ of degree n having all but one zero on $|z| = 1$, such that

$$\max_{|z|=1}|f'(z)| = [An + o(n)]\max_{|z|=1}|f(z)|,$$

where $A \simeq 0.884$, thus showing that even if we assume that all but one zeros lie on $|z| = 1$, bound in the Bernstein's inequality cannot really be very significantly improved.

Problem 5 The Bernstein's inequality for the class of self-reciprocal polynomials discussed in Sect. 2.3 is unknown for $n \geq 3$. If f is a self-reciprocal polynomial, we only know that

$$(n-1)\max_{|z|=1}|f(z)| \leq \max_{|z|=1}|f'(z)| \leq \left(n - \frac{1}{2} + \frac{1}{2(n+1)}\right) \max_{|z|=1}|f(z)|$$

which itself is quite remarkable, as half of its zeros are in the unit disk. The problem of obtaining Bernstein type inequality for the the class of self-reciprocal polynomials was proposed to us by Professor Q. I. Rahman.

Problem 6 Although, Theorem 57 answers question raised by late Professor R. P. Boas, Jr. in the case when $f(x + \mathrm{i}y) \neq 0$ for $y > k$ where $k \leq 0$, but the case when $f(x + \mathrm{i}y) \neq 0$ for $y > k$ where $k > 0$ is still completely unsolved.

Problem 7 For entire functions of exponential type satisfying the condition $f(z) \equiv \mathrm{e}^{\mathrm{i}\tau z} f(-z)$, the result of Rahman and Tariq (Theorem 71) gives an L^p analogue of Theorem 53 for $p = 2$. Recently, Tariq [87] has found an L^p inequality on the half line under certain restrictions on the zeros of f. However an L^p inequality for this class in full generality is still an open problem.

Problem 8 Let $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, $c_n \neq 0$ be a polynomial of a degree n having all its zeros in the open-unit disk. We define

$$M_p(f\,;\,R) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p \,\mathrm{d}\theta\right)^{1/p} \qquad (p \neq 0\,;\; R \geq 1)\,. \tag{107}$$

This is the usual definition of the mean $M_p(f\,;\,R)$, $p > 0$ when the zeros of f are not restricted to lie in the open unit disk; the integral in (107) may not exist for $p \leq -1/n$ if f has zeros on $|z| = R$.

It is known (see [5, Theorem 1] or [64, p. 686, Theorem 3.1.21]) that if $f(z)$ is a polynomial of degree n having all its zeros in the open-unit disk $\{z : |z| < 1\}$ such that $|f(z)| \geq \mu$ for $|z| = 1$, then

$$\max_{|z|=1}|f'(z)| \geq \mu n. \tag{108}$$

In view of the fact (for example, see [54, p. 143 in § 193] that $M_p(f\,;\,1) \to \min_{|z|=1}|f(z)|$ and $M_p(f'\,;\,1) \to \min_{|z|=1}|f'(z)|$ as $p \to -\infty$, for any given $p < 0$ the problem of obtaining the best possible bound for $\frac{M_p(f'\,;\,1)}{M_p(f\,;\,1)}$, where f is a polynomial of degree n having all its zeros in $|z| < 1$ will obviously be of interest, because it will, in particular, generalize the above inequality (108). This problem was also proposed to us by Professor Q. I. Rahman.

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On Approximation Properties of Szász–Mirakyan Operators

Vijay Gupta

Abstract In the present chapter, we present approximation properties of the well-known Szász-Mirakyan operators. These operators were introduced in the middle of last century and because of their important properties, researchers continued to work on such operators and their different modifications. Although there are several modifications of the Szász-Mirakyan operators available in the literature viz. integral modifications due to Kantorovich, Durrmeyer and mixed operators, but here we discuss only the discrete modifications of these operators which were proposed by several researchers in last 60 years. In the recent years, overconvergence properties were studied by considering the complex version of Szász-Mirakyan operators. In the last section, we consider complex Szász-Stancu operators and establish upper bound and a Voronovskaja type result with quantitative estimates for these operators attached to analytic functions of exponential growth on compact disks.

Keywords Bernstein polynomials · Divided differences · Linear combinations · Asymptotic expansion · Rate of convergence · q integer ·

1 Introduction

In the middle of last century, O. Szász [28], J. Favard [8] and G. M. Mirakyan [22] (also spelled Mirakian or Mirakjan) generalized the Bernstein polynomials to an infinite interval and proposed an important operators for $f \in C[0,\infty)$, $x \in [0,\infty)$ and $n \in \mathbb{N}$ as

$$S_n(f,x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1}$$

Szász [28] showed that the operator (1) converges uniformly to $f(x)$, if $f(t)$ is bounded on every finite subinterval of $[0,\infty)$, equal to $O(t^k)$ for some $k > 0$ as $t \to \infty$ and is continuous at a point $t = x$.

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Mirakyan [22] considered the partial sum of the operators $S_n(f,x)$ as

$$S_{n,m}(f,x)=\sum_{k=0}^{m}e^{-nx}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)$$

and he proved that $\lim_{n\to\infty}S_{n,m}(f,x)=f(x)$ uniformly in $[0,r']$, if $\lim_{n\to\infty}\frac{m}{n}=r<r'>0$.

In the year 1977, Hermann [15] proved the following result and showed that the operators $S_n(f,x)$ does not converge if $f(t)\geq t^{\phi(t).t}$, where $\phi(t)$ is any monotonically increasing function such that $\lim_{t\to\infty}\phi(t)=\infty$.

Theorem 1 [15] *If f is continuous on $[0,\infty)$ and is equal to $O(e^{\alpha x})$ for some $\alpha>0$ as $t\to\infty$, then for all $A>0$, we have*

$$S_n(f,x)-f(x)=O(\omega_{2A}(f,n^{-1/2})),\;\; x\in[0,A],$$

where

$$\omega_A(f,\delta)=\sup_{x\in[0,A]}\{|f(x+t)-f(x)|:|t|\leq\delta\}.$$

Totik [29] represented the Szász operators in the form of difference function as

$$S_n(f,x):=\sum_{k=0}^{\infty}\frac{(-nx)^k}{k!}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}=\sum_{k=0}^{\infty}\Delta_{1/n}^k(f;0)\frac{(nx)^k}{k!},$$

where

$$\Delta_h^k(f;x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih).$$

If f is of exponential growth on $[0,\infty)$, then Lupas [19] observed that the Szász-Mirakyan operators can be written in terms of divided differences, i.e.

$$S_n(f,x)=\sum_{k=0}^{\infty}[0,1/n,\ldots,k/n;f]x^k,$$

where $[0,1/n,\cdots,j/n;f]$ denotes the divided difference of f on the knots $0,1/n,\cdots,j/n$. The quantitative estimates in approximation of Szász-Mirakyan operators were also established by several researchers. We mention below some of the important results on these operators: Stancu [25] obtained the following result on uniform norm, using probabilistic methods:

Theorem 2 [25] *Let $f\in C^1[0,a], a>0$, then for $n\in\mathbb{N}$, we have*

$$||S_n(f,.)-f||\leq(a+\sqrt{a}).\frac{1}{\sqrt{n}}\omega(f',1/\sqrt{n}).$$

Singh [24] obtained the following sharp estimate in simultaneous approximation:

Theorem 3 [24] *Let* $f \in C^{r+1}[0,a], a > 0$*, then for* $n \in \mathbb{N}$*, we have*

$$||S_n^{(r)}(f,.) - f^{(r)}|| \leq \frac{r}{n}||f^{(r+1)}|| + K_{n,r}.\frac{1}{\sqrt{n}}\omega(f^{(r+1)}, 1/\sqrt{n}),$$

where $K_{n,r} = [(a/2) + (r/2\sqrt{n}) + (r^2/4n)((r^2/4n) + a)^{1/2}.(1 + (r/2\sqrt{n}))]$.

By $C_B[0,\infty)$, we mean the space of all real valued continuous bounded functions f defined on $[0,\infty)$. Totik [29] obtained the following equivalence results for the Szász operators. The modified modulus of smoothness considered in [29] is defined as

$$\omega(\delta) = \sup_{\substack{0\leq x<\infty \\ 0<h\leq\delta}} |\Delta^2_{h\sqrt{x}}(f;x)|, \delta > 0,$$

for an absolute constant K, $\omega(\lambda\delta) \leq K\lambda^2\omega(\delta), \lambda \geq 1$.

Theorem 4 [29] *Let* $f \in C_B[0,\infty)$*, the following are equivalent:*
(i) $S_n(f,x) - f(x) = o(1), n \to \infty$.
(ii) $\omega(\delta) = o(1), \delta \to 0$.
(iii) $f(x + h\sqrt{x}) - f(x) = o(1)$*, as* $h \to 0$ *uniformly in* x.
(iv) the function $f(x^2)$ *is uniformly continuous.*

Totik showed that equivalence of $(ii) \Leftrightarrow (i)$ holds even if $f \in C_B[0,\infty)$ is replaced by weaker assumption on $f \in C[0,\infty), \omega(1) < \infty$.

Theorem 5 [29] *Let* $0 < \alpha \leq 1$. *For* $f \in C_B[0,\infty)$*, the following are equivalent*
(i) $S_n(f,x) - f(x) = o(n^{-\alpha})$.
(ii) $\omega(\delta) = O(\delta^{2\alpha})$.

Kasana and Arawal [17] extended the studies and estimated a result for linear combinations of Szász operators. The kth order linear combinations $S_n(f,k,x)$ of the operators $S_{d_jn}(f,x)$, discussed in [21] are given by

$$S_n(f,k,x) = \sum_{j=0}^{k} C(j,k)S_{d_jn}(f,x),$$

where

$$C(j,k) = \prod_{i=0 i\neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0; C(0,0) = 1$$

and $d_0, d_1, ...,d_k$ are arbitrary but fixed distinct positive integers. In an alternate form, the linear combinations $S_n(f,k,x)$ can be represented in the following form.

$$S_n(f,k,x) = \frac{1}{\triangle}\begin{vmatrix} S_{d_0n}(f,x) & d_0^{-1} & d_0^{-2} & .... & d_0^{-k} \\ S_{d_1n}(f,x) & d_1^{-1} & d_1^{-2} & .... & d_1^{-k} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ S_{d_kn}(f,x) & d_k^{-1} & d_k^{-2} & .... & d_k^{-k} \end{vmatrix}$$

where $\triangle$ is the Vandermonde determent obtained by replacing the operator column of the above determent by the entries 1. The following error estimation was done in [17].

Theorem 6 [17] *Let f be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(t^{\alpha t})$ as $t \to \infty$, for some $\alpha > 0$. If $f^{(r+1)} \in C < a, b >$, then for n sufficiently large*

$$\left|\left|S_n^{(r)}(f,k,.) - f^{(r)}\right|\right| \leq C_1 n^{-1/2}\omega(f^{(r+1)}, n^{-1/2}) + C_2 n^{-(k+1)},$$

where $C_1 = C_1(k,r), C_2 = C_2(k,r,f)$ and $\omega(f^{(r+1)},\delta)$ is the modulus of continuity of $f^{(r)}$ on $< a, b >$, which denotes an open interval in $[0,\infty)$ continuing the closed interval $[a,b]$.

2 Asymptotic Expansion for Szász-Mirakyan Operators

In the year 2007, Abel et al. [1] established an asymptotic expansion of the Szász-Mirakyan operators. They took advantage of Stirling numbers to obtain the asymptotic expansion. Usually the Stirling numbers of first $s(n,k)$ are defined by the equation

$$x^{\overline{n}} = \sum_{k=0}^{n} s(n,k)x^k, \;\; n = 0, 1, 2, \ldots$$

where $x^{\overline{k}} = x(x-1)\ldots(x-k+1), \;\; x^{\overline{0}} = 1$, is the falling factorial.

Also the Stirling numbers of second kind $S(n,k)$ can be computed from the generating relation

$$x^n = \sum_{k=0}^{n} S(n,k)x^{\overline{k}}, \;\; n = 0, 1, 2, \ldots$$

It was given in [7] that Stirling numbers of second kind possess the representations

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

and

$$S(n,n-k) = \sum_{j=k}^{2k}\binom{n}{j}S_2(j,j-k),$$

where S_2 are the associated Stirling numbers of second kind defined by the double generating functions

$$\sum_{n,k\geq 0} S_2(n,k)t^n u^k/n! = \exp\left(u(e^t - 1 - t)\right)).$$

It was observed that after simple computation

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\sum_{l=0}^{j}\binom{j}{l}\frac{n!}{(n-l)!}(k-j)^{n-l},\ \ (n \geq k)$$

otherwise $S_2(n,k) = 0$. Abel et al. [1] derived the following asymptotic expansion:

Theorem 7 [1] *Let $q \in \mathbb{N}_0$. Assume that $f \in C^q[0,\infty)$ and $f^{(q)}$ is uniformly continuous. Then, the Szász-Mirakyan operators possess the representation*

$$S_n(f,x) = \sum_{k=0}^{q} n^{-k}\sum_{s=k}^{\max\{q,2k\}}\frac{f^{(s)}(x)}{s!}x^{s-k}S_2(s,s-k) + R_n^{[q]}(x).$$

The remainder satisfies the estimate

$$\left|R_n^{[q]}(x)\right| \leq M_q.\frac{1+x^{q+1}}{n^{q/2}}\omega(f^{(q)},n^{-1/2}),$$

with a constant M_q independent of f.

3 Jain Modification of Szász-Mirakyan Operators

In the year 1972, Jain [16] proposed a new operator with the help of a Poisson distribution. He considered its convergence properties and gave its degree of approximation. The special case of the operators of Jain turns out to be Szász-Mirakyan operators. The operators are defined as

$$S_n^{\beta}(f,x) = \sum_{k=0}^{\infty} w_{\beta}(k,nx)f(k/n),$$

where $w_{\beta}(k,\alpha) = \alpha(\alpha+k\beta)^{k-1}e^{-(\alpha+k\beta)}/k!$ with $0 < \alpha < \infty$ and $0 \leq \beta < 1$. The parameter β may depend on the natural number n. It is easy to see that for $\beta = 0$, these operators reduce to the Szász-Mirakjan operators.

Theorem 8 [16] *If $f \in C[0,\infty)$ and $\beta \to 0$ as $n \to \infty$, then the sequence $\{S_n^{\beta}(f,x)\}$ converges uniformly to $f(x)$ in $[a,b]$, where $0 \leq a < b < \infty$.*

Theorem 9 [16] *If $f \in C[0,\lambda]$ and $1 > \beta'/n \geq \beta \geq 0$ then*

$$\left|S_n^{\beta}(f,x) - f(x)\right| \leq [1+\lambda^{1/2}(1+\lambda\beta\beta')^{1/2}]\omega(n^{-1/2}),$$

where $\omega(\delta) = \sup|f(x_2) - f(x_1)|; x_1, x_2 \in [0,\lambda]$, δ being a positive number such that $|x_2 - x_1| < \delta$.

Theorem 10 [16] *If $f \in C'[0,\lambda]$ and $1 > \beta'/n \geq \beta \geq 0$ then*

$$\left|S_n^{\beta}(f,x) - f(x)\right| \leq \lambda^{1/2}(1+\lambda\beta\beta')^{1/2}[1+\lambda^{1/2}(1+\lambda\beta\beta')^{1/2}]\omega_1^{-1/2})/\sqrt{n},$$

where $\omega_1(\delta)$ is the modulus of continuity of f'.

We may observe here that not much work on these operators has been done as of its complicated behavior.

4 Szász-Chlodovsky Operators

The Szász-Chlodovsky operators considered in 1974 by Stypińki [26] are defined as

$$S_n(f,x,h_n) = \sum_{v=0}^{\infty} s_{n,v}\left(\frac{x}{h_n}\right) f\left(\frac{vh_n}{n}\right),$$

where f denotes a function defined on $\langle 0,\infty)$ and bounded on every segment $\langle 0,h\rangle \subset \langle 0,\infty)$ and $s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}, v = 0,1,2,\ldots\ n \in \mathbb{N}\ \{h_n\}, n = 1,2,\ldots$ denotes a sequence of positive numbers increasing to infinity. It was observed in [26] that the inequality $0 \leq z \leq \frac{3}{2}\sqrt{nt}, t \in \langle 0,h\rangle, h > 0$ implies that

$$\sum_{|v-nt|\geq 2z\sqrt{nt}} s_{n,v}(t) \leq 2ze^{-z^2}.$$

Also, if $L_{n,4}(t) = \sum_{v=0}^{\infty}(v-nt)^4 s_{n,v}(t)$, then $L_{n,4}(t) = 3(nt)^2 + nt$.

The following Voronovskaja type asymptotic formula was proved by Stypińki [26].

Theorem 11 [26] If

1. $h_n > 0, \lim\limits_{n\to\infty} h_n = \infty, \lim\limits_{n\to\infty}\dfrac{h_n}{n} = 0.$
2. $\lim\limits_{n\to\infty} M(h_n)\dfrac{n}{h_n}e^{-\alpha}\dfrac{n}{h_n} = 0$ for every $\alpha > 0$.
3. $f''(x)$ exists at a fixed point $x \geq 0$, then

$$\lim_{n\to\infty}\frac{n}{h_n}[\,|S_n(f,x,h_n) - f(x)|\,] = \frac{1}{2}xf''(x).$$

5 Rate of Convergence

The important topic in the last thirty years is to obtain the rate of convergence for function of bounded variation. In this direction, Cheng [6] first estimated the rate of convergence for Szász-Mirakyan Operators and proved the following result.

Theorem 12 [6] *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $f(t) = O(t^{\alpha t})$ for some $\alpha > 0$ as $t \to \infty$. If $x \in (0,\infty)$ is irrational, then for n sufficiently large, we have*

$$\left|S_n(f,x) - \frac{(f(x+)+f(x-))}{2}\right| \leq \frac{(3+x)x^{-1}}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{O(x^{-1/2})}{n^{1/2}}|f(x+)-f(x-)| + O(1)(4x)^{4\alpha x}(nx)^{-1/2}\left(\frac{e}{4}\right)^n x,$$

where $V_a^b(g)$ is the total variation of g on $[a,b]$ and the auxiliary function is defined by

$$g_x(t) = \begin{cases} f(t)-f(x-), & 0 \leq t < x \\ 0, & t = x, \\ f(t)-f(x+), & x < t < \infty \end{cases}$$

Sun [27] gave an estimate in simultaneous approximation on functions of bounded variation with growth of order $O(t^{\alpha t})$. He considered the following class of functions of generalized bounded variation as

$$\begin{aligned} B_r^{(\alpha)} = \{f : f^{(r-1)} \in C[0,\infty),\ & f_{\pm}^{(r)}(x) \text{ exists everywhere and are bounded} \\ & \text{on every finite subinterval of } [0,\infty) \text{ and } f_{\pm}^{(r)}(t) = O(t^{\alpha t}), (t \to \infty) \\ & \text{for some } \alpha > 0\}, \end{aligned}$$

where $f_{\pm}^{(0)}(x)$ means $f(x\pm)$. Sun [27] obtained the following estimates for the rate of convergence:

Theorem 13 [27] *If $f \in B_r^{(\alpha)}, r \in \mathbb{N}\bigcup\{0\}$, then for $n \geq 3+4r^2$, we have*

$$\left|S_n^{(r)}(f,x) - \frac{(f_+^{(r)}(x)+f_-^{(r)}(x))}{2}\right| \leq (73\Delta(x)/n)\sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k}) + 73\sqrt{\Delta(x)/n}w_x(x+3) + O(e^{-cn}) + |f_+^{(r)}(x) - f_-^{(r)}(x)|/(1+\sqrt{nx}),$$

where the sign O is independent of f and n but depends on x and α and $w_x(t) = w_x(h_r,t)$ is the point-wise modulus of continuity of h_r at x, $\Delta(x) = \max\{1,x\}$ and h_r is defined as

$$h_r(x) = \begin{cases} f^{(r)}(t) - f_-^{(r)}(x), & x \leq t < 0 \\ 0, & t = x \\ f^{(r)}(t) - f_+^{(r)}(x), & 0 \leq t < \infty \end{cases}$$

Theorem 14 [27] *If $f \in B_{r+1}^{(\alpha)}$, then for $x \in [0,A](A>0)$ and $n \geq 4r^2$, we have*

$$\begin{aligned}\left|S_n^{(r)}(f,x)-f^{(r)}(x)\right| &\leq (21\Delta(x)/n)\sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k}\\ &+(3/2)|f_+^{(r+1)}(x)-f_-^{(r+1)}(x)|\\ &+\sqrt{\Delta(x)/n}+O(1/n),\end{aligned}$$

where the sign O is independent of x, n and f but depends on α and A.

He remarked that for continuous derivatives his estimate does not include the case $f' \in Lip\ 1$ on every finite subinterval of $[0,\infty)$. He obtained in such case $S_n^{(r)}(f,x)-f^{(r)}(x)=O(\log n/n), r=0,1,2,\ldots$ which is worse than the usual order of approximation $O(1/n)$. Sun also put up a question of whether a unified approach can be developed which may improve the estimate for the class $f' \in Lip\ 1$ on every finite subinterval of $[0,\infty)$.

Zeng and Piriou [33] improved the estimate of Theorem 12 by considering a more general class of functions than $BV_{loc}[0,\infty)$, namely

$$I_{loc,B}=\{f : f \text{ is bounded in every finite subinterval of } [0,\infty)\}.$$

Set

$$\Omega(x,f,\lambda)=\sup_{t\in[x-\lambda,x+\lambda]}|f(t)-f(x)|,$$

where $f \in I_{loc,B}, x \in [0,\infty)$ is fixed and $\lambda \geq 0$.

Theorem 15 [33] *Assume that $I_{loc,B}$ and $f(t)=O(t^{\alpha t})$ for some $\alpha>0$ as $t \to \infty$. If $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0,\infty)$, then for n sufficiently large, we have*

$$\begin{aligned}&\left|S_n(f,x)-\frac{f(x+)+f(x-)}{2}-\frac{v(f,n,x)}{\sqrt{2\pi nx}}\right|\\ &\leq \frac{5+x}{nx+1}\sum_{k=1}^{n}\Omega(x,g_x,x/\sqrt{k})+O(n^{-1}),\end{aligned}$$

where $g_x(t)$ is defined as in Theorem 12, $O(n^{-1})$ depends on x and

$$v(f,n,x)=(f(x+)-f(x-))(nx-[nx]-2/3)+(f(x)-f(x-))\delta_{[nx]}(nx),$$

$[nx]$ denotes the greatest integer not exceeding nx.

6 Rate of Convergence For Szász-Bézier Operators

For $\alpha > 0$, Zeng [32] proposed the Bézier variant of Szász-Mirakyan operators as

$$S_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x)\left(\frac{k}{n}\right),$$

where $Q_{n,k}^{\alpha}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,k}(x) = \sum_{j=k}^{\infty} s_{n,j}(x)$ with the Szász basis function given by $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$. It was observed in [32] that

1. $J_{n,k}(x) - J_{n,k+1}(x) = s_{n,k}(x), k = 0, 1, 2, \ldots$
2. $J'_{n,k}(x) = ns_{n,k-1}(x), k = 1, 2, 3, \ldots$
3. $J_{n,k}(x) = n\int_0^x s_{n,k-1}(u)du, k = 1, 2, 3, \ldots$
4. $\sum_{k=1}^{\infty} J_{n,k}(x) = n\int_0^x \sum_{k=1}^{\infty} s_{n,k-1}(u)du = nx$
5. $J_{n,0}(x) > J_{n,1}(x) > \ldots > J_{n,k}(x) > J_{n,k+1}(x) > \ldots$

and for every natural number k, $0 \leq J_{n,k}(x) < 1$ and $J_{n,k}(x)$ increase strictly on $[0, \infty)$. The following convergence theorems were studied.

Theorem 16 [32] *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$ and let $f(t) = O(e^{\beta . t})$ for some $\beta > 0$ as $t \to \infty$. Then, for $x \in [0, \infty)$ and n sufficiently large, we have*

$$\begin{aligned}
&\left|S_{n,\alpha}(f,x) - \frac{1}{2^{\alpha}}f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right)f(x-)\right| \\
&\leq \frac{(3+x)\alpha}{nx+1/2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) \\
&+ \frac{(0.8\sqrt{1+3x}+1/2)\alpha}{\sqrt{nx}+1}|f(x+) - f(x-)| \\
&+ \frac{\alpha/\sqrt{\alpha}2\pi + 1}{\sqrt{nx}+1}|f(x) - f(x-)| \\
&+ O(1)\frac{\alpha(2x+1)^{(2x+1)\beta}}{\sqrt{nx}+1}\left(\frac{e}{4}\right)^{nx},
\end{aligned}$$

where

$$\varepsilon_n(x) = \begin{cases} 1, & \text{if } x = k'/n \text{ for some } k' \in \mathbb{N} \\ 0, & \text{if } x \neq k/n \text{ for all } k \in \mathbb{N} \end{cases}$$

when $x = 0$, we set $1/2^{\alpha} f(x+) + (1 - 1/2^{\alpha})f(x-) = f(0)$. Also $V_a^b(g_x)$ is the total variation of g_x on $[a, b]$.

Theorem 17 [32] *Let* $f \in BV[0,\infty)$, $x \in [0,\infty)$ *and* f *be normalized at* x. *Then, for* $n \geq 1$, *we have*

$$\left| S_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right|$$
$$\leq \frac{(2x+1)\alpha}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{\alpha.\min\{2x+2, 2\sqrt{2}+2\}}{\sqrt{nx}+1} |f(x+) - f(x-)|,$$

where $I_0 = [0,\infty), I_k = [x - 1/\sqrt{k}, x + 1/\sqrt{k}] \cap [0,\infty), k = 1, 2, \ldots, n.$

7 *q* Szász-Mirakyan Operators

The applications of q calculus has been a new area for last 25 years. Several new operators were introduced and their convergence behaviors were discussed. We refer the readers to the recent book by Aral-Gupta-Agarwal [5] in which a collection of some of the papers is presented. We first mention here some basic definitions. Given the value of $q > 0$, we define the q-integer $[n]_q$ by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases},$$

for $n \in \mathbb{N}$. The q factorial is defined as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \ldots [1]_q, & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases},$$

for $n \in \mathbb{N}$.

We define the q-binomial coefficients by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, 0 \leq k \leq n,$$

for $n, k \in \mathbb{N}$. A q-analogue of classical exponential function e^x is defined as

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{((1-q)x;q)_\infty}, |x| < \frac{1}{1-q}, |q| < 1$$

Another q-analogue of classical exponential function is given by

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]_q!} = (-(1-q)x;q)_\infty \, x \in \mathbb{R}, |q| < 1$$

where $(x;q)_\infty = \prod_{k=1}^{\infty}(1 - xq^{k-1})$. It is observed that

$$e_q(x)\, E_q(-x) = E_q(x)\, e_q(-x) = 1.$$

For $0 < q < 1$, Aral [3] defined new operators that we call the q-Szász-Mirakyan operators as

$$S_n^q(f,x) := \sum_{k=0}^{\infty} s_{n,k}^q(x) f\left(\frac{[k]_q b_n}{[n]_q}\right) = E_q\left(-[n]_q \frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right)\frac{\left([n]_q x\right)^k}{[k]_q!\,(b_n)^k},$$

where $0 \le x < \alpha_q(n)$, $\alpha_q(n) := \frac{b_n}{(1-q)[n]_q}$, $f \in C(\mathbb{R}_0)$ and (b_n) is a sequence of positive numbers such that $\lim\limits_{n\to\infty} b_n = \infty$. We observe that these operators are positive and linear. Furthermore, as a special case if $q = 1$, we recapture the classical Szász-Mirakyan operators. Depending on the selection of q, the q-Szász-Mirakyan operators are more flexible than the classical Szász-Mirakyan operators while retaining their approximation properties. A Voronovskaya-type relation for these operators is as follows:

Theorem 18 [3] *Let $f \in C(\mathbb{R}_0)$ be a bounded function and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the second derivative $D_{q_n}^2 f(x)$ exists at a point $x \in [0, \alpha_{q_n}(n))$ for n large enough. If $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then*

$$\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}(f,x) - f(x)\right) = \frac{1}{2} x \lim_{q_n\to 1} D_{q_n}^2 f(x).$$

Let $B_\rho(\mathbb{R}_0)$ be the set of all functions f satisfying the condition $|f(x)| \le M_f \rho(x), x \in \mathbb{R}_0$ with some constant M_f depending only on f. We denote by $C_\rho(\mathbb{R}_0)$ the space of all continuous functions belonging to $B_\rho(\mathbb{R}_0)$. Also

$$C_\rho^0(\mathbb{R}_0) = \left\{ f \in C_\rho(\mathbb{R}_0) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} < \infty \right\}.$$

The following result is in the weighted spaces.

Theorem 19 [3] *Let (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. For any function $f \in C_{2m}^0(\mathbb{R}_0)$, if $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then*

$$\lim_{n\to\infty} \sup_{0\le x\le \alpha_{q_n}(n)} \frac{\left|S_n^{q_n}(f,x) - f(x)\right|}{1 + x^{2m}} = 0.$$

Moreover, for n large enough

$$\sup_{0\le x\le \alpha_{q_n}(n)} \frac{\left|S_n^{q_n}(f,x) - f(x)\right|}{1 + x^{2m}} \le \left(2 + \sqrt{2}\right) \omega\left(f; \sqrt{\frac{b_n}{[n]_{q_n}}}\right),$$

where $\omega(f;\cdot)$ is the classical modulus of continuity.

Aral [3] also gave two representations of rth q-derivative of the q-Szász-Mirakyan operators in terms of the q-differences and the divided differences. In this continuation, Aral and Gupta [4] extended the studies and obtained some important properties for the q-Szász-Mirakyan operators. We mention some of the results below.

Theorem 20 [4] *Let $D_q^r f \in C[0,\infty)$ for some r and q, $r \geq 0$ and $0, q < 1$. If $m \leq D_q^r(f)(x) \leq M$ for $x \in [0,\infty)$ then there exist, $\widehat{q} \in (0,1)$ such that, for all $q \in (\widehat{q}, 1)$ and for $x \in [0, b_n/(1-q^n))$, the inequality*

$$\frac{mq^{r(r-1)/2}}{2^r} < D_q^r(S_n^q(f,x)) \leq q^{r(r-1)/2}M$$

holds for sufficiently large n.

Theorem 21 [4] *Let $b_n = o([n]_q)$ as $n \to \infty$ and $q \to 1$. If $D_q^r(f) \in C_{x^2}[0,\infty)$ for some integer r, then for all $x \in [0, A]$, we have*

$$\lim_{\substack{n\to\infty\\ q\to 1}} D_q^r(S_n^q(f,x)) = \lim_{q\to 1} D_q^r(f)(x)$$

Theorem 22 [4] *Let $r \geq 0$ and $s \geq 1$ be natural numbers. Suppose $q_n \to 1$ as $n \to \infty$. If $b_n = o([n]_{q_n})$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n}\left[D_{q_n}^r(S_n^{q_n}(t^{r+s},x)) - \frac{[r+s]_{q_n}}{[s]_{q_n}!}q_n^{(r+s)(r+s-1)/2}x^s\right]$$
$$\leq \frac{(r+s-1)!}{(r-1)!}\frac{(r+s-1)(r+s)}{2}x^{s-1}.$$

The following theorem gives a Stancu-type remainder of the q- Szász-Mirakyan operators, which reduces to the formula for remainder of classical Szász-Mirakjan operators.

Theorem 23 [4] *If $x \in (0, b_n/(1-q^n)) \setminus \left\{\frac{[j]_q b_n}{[n]_q} : j = 0, 1, 2, \ldots\right\}$, then*

$$S_n^q(f,x) - f(x) = \frac{x\left(1 + (1-q^n)\frac{x}{b_n}\right)b_n}{[n]_q}$$
$$\times \sum_{j=0}^{\infty} f[x, \frac{[j]_q b_n}{[n]_q}, \frac{[j+1]_q b_n}{[n]_q}]s_{n,j}^q(qx).$$

Theorem 24 [4] *If $f(x)$ is convex on $[0,\infty)$, then*

$$S_n^q(f,x) > S_{n\oplus b_n}^q(f,x),$$

for all $n \geq 0$ *and* $x \in [0, b_n/(1-q^n))$ *such that* $0 < q < 1$. *If* f *is linear, then* $S_n^q(f,x) = S^q_{n \bigoplus b_n}(f,x)$, *where*

$$\left(x \bigoplus y\right)^j = \sum_{n=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q x^n y^{j-n}, j = 0, 1, 2, \ldots$$

It was pointed out in [4] that above theorem states whether or not $S_n^q(f,x)$ decreases when f is convex. This is still an open problem.

8 Modified Szász-Mirakjan Operators

In the last decade, Walczak [30] defined Szász-Mirakjan Operators as

$$S_n(f;m,x) := \frac{1}{g((nx+1)^2;m)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+m)!} f\left(\frac{k+m}{n(nx+1)}\right), \tag{2}$$

where $x \in [0,\infty)$ and $g(t;m) = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{(k+m)!}, t \in [0,\infty)$.

Walczak [30] considered the space C_p, $p \in \mathbb{N}_0$, associated with the weight function $w_0(x) := 1, w_p(x) := (1+x^p)^{-1}, p \geq 1$ and composed of all real-valued functions on $[0,\infty)$, for which $w_p(x)f(x)$ is uniformly continuous and bounded on $[0,\infty)$. The norm on C_p is defined as $||f||_p := \sup_{x\in[0,\infty)} w_p(x)|f(x)|$. In [30], it was proved that if $f \in C_p$, then for the operators (2), one has the following estimate:

$$||S_n(f;m,.) - f||_p \leq M_0 \omega\left(f; C_p; \frac{1}{n}\right), m, n \in \mathbb{N},$$

where M_0 is an absolute constant and the modulus of continuity $\omega(f;C_p;t) := \sup_{0\leq h\leq t} ||f(x+h) - f(x)||_p, t \in [0,\infty)$. In particular, if $f \in C_p^1 := \{f \in C_p : f' \in C_p\}$, $p \in \mathbb{N}_0$, then

$$||S_n(f;m,.) - f||_p \leq \frac{M_1}{n},$$

where M_1 is an absolute constant.

It was observed in [31] that the Szász-Mirakjan operators are defined in terms of a sample of the given function f on the points k/n, called knots. For the operators $S_n(f;m,x)$, the knots are the numbers $(k+m)/(n(nx+1))$ for fixed m. Thus, the question arises, whether the knots $(k+m)/(n(nx+1))$ cannot be replaced by a given subset of points, which are independent of x, provided this will not change the degree of convergence. In connection with this question, Walczak and Gupta [31] introduced the operators $L_n(f;p;r;s,x)$ for $f \in B_p$, $p \in \mathbb{N}$, which is a class of all real valued

continuous functions $f(x)$, on $[0,\infty)$ for which $w_p(x)x^k f^{(k)}(x), k = 0,1,2,\dots,p$ is continuous and bounded on $[0,\infty)$ and $f^{(p)}(x)$ is uniformly continuous on $[0,\infty)$.

$$L_n(f;p;r;s,x) := \begin{cases} \frac{1}{I_r(n^s x)} \sum_{k=0}^{\infty} \frac{(n^s x)^{2k+r}}{2^{2k+r} k! \Gamma(r+k+1)} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{2k}{n^s}\right)\left(x-\frac{2k}{n^s}\right)^j}{j!}, & x > 0 \\ f(0), & x = 0 \end{cases} \tag{3}$$

where I_r is the modified Bessel's function

$$I_r := \sum_{k=0}^{\infty} \frac{t^{2k+r}}{2^{2k+r} k! \Gamma(r+k+1)}.$$

Walczak and Gupta [31] estimated the rate of convergence of the operators $L_n(f;p;r;s,x)$.

Theorem 25 *Fix $p \in \mathbb{N}_0, r \in [0,\infty)$ and $s > 0$. Then, there exists a positive constant $M \equiv M(p,r,s)$ such that for $f \in B_{2\,p+1}$, we have*

$$||L_n(f;2\,p+1;r;s,.) - f||_{2\,p+1} \le M\omega\left(f^{(2\,p+1)}; C_0; n^{-s}\right).$$

Theorem 26 *Fix $p \in \mathbb{N}_0, r \in [0,\infty)$ and $s > 0$. Then, there exists a positive constant $M \equiv M(p,r,s)$ such that for $f \in B_{2\,p+2}$, we have*

$$||L_n(f;2\,p+2;r;s,.) - f||_{2\,p+2} \le \frac{M(p,r,s)}{n^s}||f^{(2\,p+2)}||_0.$$

9 Complex Szász-Mirakjan-Type Operator

The convergence of the Bernstein polynomials in the complex plane was initiated in [18]. In the recent book [10], S. G. Gal collected and presented the Voronovskaja-type results with quantitative estimates for several operators like the complex Bernstein, complex q-Bernstein, complex Baskakov, complex Favard-Szász-Mirakjan, complex Bernstein–Kantorovich, complex Balázs-Szabados and complex Stancu-Kantorovich operators attached to analytic functions on compact disks and the exact order of simultaneous approximation for such complex operators.

Very recently Gal and Gupta (see [11–13]) and Mahmudov-Gupta [20] established quantitative results for different versions of well-known Bernstein–Durrmeyer operators in complex domain. Agarwal and Gupta [2] extended the studies and obtained results for the q analogue of certain Bernstein–Durrmeyer operators in complex domain. In order to make the convergence faster to a function being approximated, very recently Ren and Zeng [23] introduced a kind of complex modified q-Durrmeyer type operators which can reproduce constant and linear functions. They obtained the order of simultaneous approximation and a Voronovskaja-type result with a quantitative estimate for the modified complex q-Durrmeyer type operators attached to analytic functions on compact disks.

Recently Gal [9] obtained quantitative estimate in the Vornonvskaja's theorem and the exact bounds in the approximation of analytic functions without exponential growth by complex Favard-Szász-Mirakjan operators. He considered the function $f : [0, \infty) \to \mathbb{C}$ bounded on $[0, \infty)$. In [9], the class $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ was considered.

Theorem 27 [9] *Let* $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ *be with* $2 < R < +\infty$ *and suppose that* $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *is bounded on* $[0, +\infty)$ *and analytic in* $\mathbb{D}_R$ *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

(i) *Let* $1 \le r < \frac{R}{2}$ *be arbitrary fixed. Then, for all* $|z| \le r$ *and* $n \in N$, *we have*

$$|S_n(f, z) - f(z)| \le \frac{C_{r,f}}{n},$$

where $C_{r,f} = 6 \sum_{k=2}^{\infty} |c_k|(k-1)(2r)^{k-1} < \infty$.

(ii) *For the simultaneous approximation by complex Favard-Szász-Mirakjan operators, we have: if* $1 \le r < r_1 < \frac{R}{2}$ *are arbitrary fixed, then for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$,

$$|S_n^{(p)}(f, z) - f^{(p)}(z)| \le \frac{p! r_1 C_{r_1,f}}{n(r_1 - r)^{p+1}},$$

where $C_{r_1,f}$ *is as given in (i) above.*

Theorem 28 [9] *Let* $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ *be with* $2 < R < +\infty$ *and suppose that* $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *is bounded on* $[0, +\infty)$ *and analytic in* $\mathbb{D}_R$ *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

If $1 \le r < \frac{R}{2}$ *be arbitrary fixed. Then, for all* $|z| \le r$ *and* $n \in N$, *we have*

$$\left| S_n(f, z) - f(z) - \frac{z}{2n} f''(z) \right| \le \frac{M_{r,f}|z|}{n^2},$$

where $M_{r,f} = 26 \sum_{k=3}^{\infty} |c_k|(k-1)^2(k-2)(2r)^{k-3} < \infty$.

Theorem 29 [9] *Let* $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ *be with* $2 < R < +\infty$ *and suppose that* $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *is bounded on* $[0, +\infty)$ *and analytic in* $\mathbb{D}_R$ *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$.

If $1 \le r < \frac{R}{2}$ *be arbitrary fixed and if* f *is not a polynomial of degree* ≤ 1, *then we have*

$$||S_n(f) - f||_r \ge \frac{C_r(f)}{n}, n \in \mathbb{N},$$

where the constant $C_r(f)$ *depends only on* f *and* r *and* $||f||_r = \max\{|f(z)| : |z| \le r\}$.

Also, with exponential growth, Gal [10] estimated the quantitative estimates for overconvergence of Favard-Szász-Mirakjan operators.

Theorem 30 [10] *Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ be with $1 < R < +\infty$ and suppose that $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ is continuous in $(R, +\infty) \cup \overline{\mathbb{D}}_R$, analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, and suppose that there exist $M, C, B > 0$ and $A \in (\frac{1}{R}, 1)$, with the property that $|c_k| \le M \frac{A^k}{k!}$, for all $k = 0, 1, \ldots$, (which implies $|f(z)| \le M e^{A|z|}$ for all $z \in \mathbb{D}_R$) and $|f(x)| \le C e^{Bx}$, for all $x \in [R, +\infty)$.*

(i) Let $1 \le r < \frac{1}{A}$. Then, for all $|z| \le r$ and $n \in N$, we have

$$|S_n(f, z) - f(z)| \le \frac{C_{r,A}}{n},$$

where $C_{r,A} = \frac{M}{2r} \sum_{k=2}^{\infty} (k+1)(rA)^k < \infty$;

(ii) If $1 \le r < r_1 < \frac{1}{A}$ are arbitrary fixed, then for all $|z| \le r$ and $n, p \in \mathbb{N}$,

$$|S_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{p! r_1 C_{r_1,A}}{n(r_1 - r)^{p+1}},$$

where $C_{r_1,A}$ is given as at the above point (i).

Theorem 31 [10] *Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ be with $1 < R < +\infty$ and suppose that $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ is continuous in $(R, +\infty) \cup \overline{\mathbb{D}}_R$, analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, and suppose that there exist $M, C, B > 0$ and $A \in (\frac{1}{R}, 1)$, with the property that $|c_k| \le M \frac{A^k}{(2k)!}$, for all $k = 0, 1, \ldots$, (which implies $|f(z)| \le M e^{A|z|}$ for all $z \in \mathbb{D}_R$) and $|f(x)| \le C e^{Bx}$, for all $x \in [R, +\infty)$. Suppose that $1 \le r < \frac{1}{A}$.*

(i) Then, following upper estimate in the Voronovskaja-type formula holds

$$\left| S_n(f, z) - f(z) - \frac{z}{2n} f''(z) \right| \le \frac{3MA|z|}{r^2 n^2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1},$$

for all $n \in \mathbb{N}$, $|z| \le r$.

(ii) We have the following equivalence in the Voronovskaja's formula

$$\left\| S_n(f) - f - \frac{e_1}{2n} f'' \right\|_r \sim \frac{1}{n^2},$$

where the constants in the equivalence depend on f and r but independent of n.

10 Complex Szász-Stancu Operator

The Szász-Stancu operator of real variable $x \in [0, \infty)$ is defined by

$$S_n^{\alpha,\beta}(f, x) = \sum_{\nu=0}^{\infty} s_{n,\nu}(x) f\left(\frac{\nu + \alpha}{n + \beta}\right),$$

where $s_{n,\nu}(x) = e^{-nx}\frac{(nx)^\nu}{\nu!}$ and α, β are two given parameters satisfying the conditions $0 \le \alpha \le \beta$. For $\alpha = \beta = 0$, we recapture the classical Szász operator.

$$S_n^{\alpha,\beta}(f, z) = \sum_{\nu=0}^{\infty} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \ldots, \frac{\alpha+\nu}{n+\beta}; f\right] z^\nu,$$

which was studied in the book by Gal [10], pp. 104–114. Here, $[x_0, x_1, \ldots, x_m; f]$ denotes the divided difference of the function f on the distinct points $x_0, x_1, \ldots, x_m$.

As suggested by the above mentioned Lupas' representation, we deal with the following complex form for the Szász-Stancu operators, these operators were recently studied by Gupta and Verma [14], who obtained some results for bounded functions in complex domain. The first main result is the upper estimate.

Theorem 32 [14] *For $2 < R < +\infty$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(a) Suppose that $0 \le \alpha \le \beta$ and $1 \le r < \frac{R}{2}$ are arbitrary fixed. Then, for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|S_n^{\alpha,\beta}(f, z) - f(z)| \le \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| r^{k-1} + \frac{A_r(f)}{n+\beta} + \frac{\alpha B_r(f)}{n+\beta} + \frac{\beta C_r(f)}{n+\beta},$$

$$\text{where} \sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty, B_r(f) = \sum_{k=1}^{\infty} |c_k| k r^{k-1} < +\infty, C_r(f) = \sum_{k=1}^{\infty} |c_k| k r^k < +\infty \text{ and } A_r(f) = 2\sum_{k=1}^{\infty} |c_k|(k-1)(2r)^{k-1} < +\infty.$$

(b) Suppose that $0 \le \alpha \le \beta$ and $1 \le r < r_1 < \frac{R}{2}$, then for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|[S_n^{\alpha,\beta}(f, z)]^{(p)} - f^{(p)}(z)| \le \frac{p! r_1}{(r_1 - r)^{p+1}} \cdot \frac{M_{r_1}(f)}{n+\beta},$$

where $M_{r_1}(f) = (\alpha + \beta r_1) \sum_{k=1}^{\infty} |c_k| \cdot r_1^{k-1} + A_{r_1}(f) + B_{r_1}(f) + C_{r_1}(f)$.

The next main result is a Voronovskaja-type asymptotic formula.

Theorem 33 [14] *For $2 < R < +\infty$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \le r < \frac{R}{2}$ and $0 \le \alpha \le \beta$. Then, for all $|z| \le r$ and $n \in \mathbb{N}$, we have the following Voronovskaja-type result*

$$\left| S_n^{\alpha,\beta}(f, z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z) \right| \le \frac{M_{1,r}(f)}{n^2} + \frac{\sum_{j=2}^{6} M_{j,r}(f)}{(n+\beta)^2},$$

where

$$M_{1,r}(f) = 26 \sum_{k=3}^{\infty} |c_k|(k-1)^2(k-2)(2r)^{k-2} < +\infty,$$

$$M_{2,r}(f)=\left(\frac{\alpha^2}{2}+2\alpha\right)\cdot\sum_{k=2}^{\infty}|c_k|\cdot k(k-1)(2r)^{k-2}<+\infty,$$

$$M_{3,r}(f)=\frac{\beta^2}{2}\sum_{k=2}^{\infty}|c_k|k(k-1)(2r)^{k}<+\infty,$$

$$M_{4,r}(f)=\beta\sum_{k=2}^{\infty}|c_k|k(k-1)(2r)^{k-1}<+\infty,$$

$$M_{5,r}(f)=\alpha\beta\sum_{k=0}^{\infty}|c_k|k(k-1)r^{k-1}<+\infty,$$

$$M_{6,r}(f)=\beta^2\sum_{k=0}^{\infty}|c_k|k(k-1)r^{k}<+\infty.$$

Following exactly the lines from the p. 104, in the book by Gal [10], we get that if f is of exponential growth on $[0,\infty)$, then the operator $S_n^{\alpha,\beta}(f,z)$ is also well defined for all $z\in\mathbb{C}$. In this section below, we present the over convergence of the Szász-Stancu operators having exponential growth. To prove the main results for growth, we need the following two lemmas:

Lemma 1 [14] *For all $n,k\in N\cup\{0\}$, $0\le\alpha\le\beta$, $z\in\mathbb{C}$, let us define*

$$S_n^{\alpha,\beta}(e_k,z)=\sum_{\nu=0}^{\infty}[\frac{\alpha}{n+\beta},\frac{\alpha+1}{n+\beta},\ldots,\frac{\alpha+\nu}{n+\beta};e_k]z^{\nu},$$

where $e_k(z)=z^k$. Then, $S_n^{\alpha,\beta}(e_0,z)=1$ and we have the following recurrence relation:

$$S_n^{\alpha,\beta}(e_{k+1},z)=\frac{z}{n+\beta}(S_n^{\alpha,\beta}(e_k,z))'+\frac{nz+\alpha}{n+\beta}S_n^{\alpha,\beta}(e_k,z).$$

Consequently

$$S_n^{\alpha,\beta}(e_1,z)=\frac{nz+\alpha}{n+\beta},\quad S_n^{\alpha,\beta}(e_2,z)=\frac{nz}{(n+\beta)^2}+\frac{(nz+\alpha)^2}{(n+\beta)^2}.$$

Lemma 2 [14] *Let α, β be satisfying $0\le\alpha\le\beta$. Denoting $S_n^{0,0}(e_j)$ by $S_n(e_j)$, for all $n,k\in N\cup\{0\}$, we have the following recursive relation:*

$$S_n^{\alpha,\beta}(e_k,z)=\sum_{j=0}^{k}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}S_n(e_j,z).$$

The results for unbounded functions have different approximation properties and analysis is different. Here, we deal with unbounded functions of exponential growth

on compact disks. We study the rate of approximation of analytic functions of exponential growth and the Voronovskaja type result for the Szász-Stancu operator $S_n^{\alpha,\beta}(f,z)$. Also, the exact order of approximation by this operator is obtained.

Our first main result is the following theorem for upper bound.

Theorem 34 *Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ be with $1 < R < +\infty$ and suppose that $f : [R,+\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ is continuous in $(R,+\infty) \cup \overline{\mathbb{D}}_R$, analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, and suppose that there exist $M > 0$ and $A \in (\frac{1}{R}, 1)$, with the property that $|c_k| \leq M\frac{A^k}{k!}$, for all $k = 0,1,\ldots,$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{D}_R$) and $|f(x)| \leq Ce^{Bx}$, for all $x \in [R,+\infty)$. Suppose that $0 \leq \alpha \leq \beta$ and $1 \leq r < \frac{1}{A}$. Then, for all $|z| \leq r$ and $n \in N$, we have*

$$|S_n^{\alpha,\beta}(f,z) - f(z)| \leq \frac{(\alpha+\beta r)}{(n+\beta)r}\sum_{k=1}^{\infty} M\frac{(rA)^k}{k!} + \sum_{k=1}^{\infty} M(rA)^k\frac{(k+1)}{2nr}.$$

Proof By using the recurrence relation of Lemma 1, we have

$$S_n^{\alpha,\beta}(e_{k+1},z) = \frac{z}{n+\beta}(S_n^{\alpha,\beta}(e_k,z))' + \frac{nz+\alpha}{n+\beta}S_n^{\alpha,\beta}(e_k,z),$$

for all $z \in \mathbb{C}, k \in \{0,1,2,\ldots.\}, n \in N$. From this, we immediately get the recurrence formula

$$\begin{aligned} S_n^{\alpha,\beta}(e_k,z) - z^k =& \frac{z}{n+\beta}\left[(S_n^{\alpha,\beta}(e_{k-1},z)) - z^{k-1}\right]' + \frac{nz+\alpha}{n+\beta}\left[S_n^{\alpha,\beta}(e_{k-1},z) - z^{k-1}\right] \\ &+ \frac{(k-1)+\alpha-\beta z}{n+\beta}z^{k-1}, \end{aligned}$$

for all $z \in \mathbb{C}, k,n \in N$. Clearly, $S_n^{\alpha,\beta}(e_0,z) - e_0 = 0$ and from the above relation, we have

$$|S_n^{\alpha,\beta}(e_1,z) - e_1(z)| = \left|\frac{\alpha-\beta z}{n+\beta}\right| \leq \frac{\alpha+\beta r}{n+\beta}.$$

Now let $1 \leq r < R$ if we denote the norm-$||.||_r$ in $C(\overline{\mathbb{D}}_r)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \leq r\}$, then by a linear transformation, the Bernstein's inequality in the closed unit disk becomes $|P_k'(z)| \leq \frac{k}{r}||P_k||_r$, for all $|z| \leq r$, where $P_k(z)$ is a polynomial of degree $\leq k$. Thus, from the above recurrence relation, we get

$$\begin{aligned} ||S_n^{\alpha,\beta}(e_k,.) - e_k||_r \leq& \frac{r}{n+\beta}||(S_n^{\alpha,\beta}(e_{k-1},.)) - e_{k-1}||_r\frac{(k-1)}{r} \\ &+ r||S_n^{\alpha,\beta}(e_{k-1},.) - e_{k-1}||_r \\ &+ \frac{(k-1)+\alpha+\beta r}{n+\beta}r^{k-1}, \end{aligned}$$

implying

$$||S_n^{\alpha,\beta}(e_k,.) - e_k||_r \leq \left(r + \frac{k-1}{n+\beta}\right)||(S_n^{\alpha,\beta}(e_{k-1},.)) - e_{k-1}||_r$$

$$+\frac{(k-1)+\alpha+\beta r}{n+\beta}r^{k-1},$$

Proceeding along the lines of p. 106 of [10], we see by mathematical induction with respect to k that the above recurrence implies

$$||S_n^{\alpha,\beta}(e_k,.)-e_k||_r \leq \frac{(k+1)!}{2n}r^{k-1}+\frac{\alpha+\beta r}{n+\beta}r^{k-1}.$$

Thus, we have

$$S_n^{\alpha,\beta}(f,z)=\sum_{k=0}^{\infty}c_k S_n^{\alpha,\beta}(e_k,z),$$

which implies

$$\begin{aligned}|S_n^{\alpha,\beta}(f,z)-f(z)| &\leq \sum_{k=1}^{\infty}|c_k|\cdot|S_n^{\alpha,\beta}(e_k,z)-z^k|\\ &\leq \frac{\alpha+\beta r}{n+\beta}\sum_{k=1}^{\infty}|c_k|\cdot r^{k-1}+\sum_{k=1}^{\infty}|c_k|\frac{(k+1)!}{2n}r^{k-1}\\ &\leq \frac{(\alpha+\beta r)}{(n+\beta)r}\sum_{k=1}^{\infty}M\frac{(rA)^k}{k!}+\sum_{k=1}^{\infty}M\frac{A^k}{k!}\frac{(k+1)!}{2n}r^{k-1}.\end{aligned}$$

This proves the theorem.

The next main result is a Voronovskaja-type asymptotic formula.

Theorem 35 *Let $0 \leq \alpha \leq \beta$. Suppose that the hypothesis on the function f and on the constants R, M, C, B, A in the statement of Theorem 32 hold and let $1 \leq r < \frac{1}{A}$ be fixed. We have the following Voronovskaja-type result*

$$\left|S_n^{\alpha,\beta}(f,z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{nz}{2(n+\beta)^2}f''(z)\right|$$
$$\leq \frac{\beta MA^2}{2(n+\beta)^3}M_{1,r}(f)+\frac{|z|MA[3+4(\alpha+\beta r)]}{n(n+\beta)r^2}M_{2,r}(f)+\frac{MA^2}{(n+\beta)^2}M_{3,r}(f),$$

where by $|c_k| \leq M\frac{A^k}{k!}$,

$$M_{1,r}(f)=\sum_{k=2}^{\infty}(k-1)(k-2)[(2k-3)+(\alpha+\beta r)](rA)^{k-2},$$

$$M_{2,r}(f)=\sum_{k=2}^{\infty}(k+1)(rA)^{k-1},$$

and

$$M_{3,r}(f) = \sum_{k=2}^{\infty} \Big[(\alpha+\beta r)(k-1)(k-2) + (\alpha+\beta r)^2 + \frac{(k-1)(k-2)^2}{2} + \frac{(k-1)(k-2)(\alpha+\beta r)}{2}\Big](rA)^{k-2}.$$

Proof Denoting $e_k(z) = z^k$ and $\pi_{n,k}(z) = S_n^{\alpha,\beta}(f,z)$, we obtain

$$\left|S_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta} f'(z) - \frac{z}{2(n+\beta)^2} f''(z)\right|$$

$$\leq \sum_{k=1}^{\infty} |c_k| \left|\pi_{n,k}(z) - e_k(z) + \frac{kz^{k-1}(\beta z-\alpha)}{n+\beta} - \frac{nz^{k-1}k(k-1)}{2(n+\beta)^2}\right|.$$

By Lemma 1, we have

$$\pi_{n,k+1}(z) = \frac{z}{n+\beta}\pi'_{n,k}(z) + \frac{nz+\alpha}{n+\beta}\pi_{n,k}(z), z \in \mathbb{C}.$$

If we denote

$$E_{n,k}(z) = \pi_{n,k}(z) - e_k(z) + \frac{kz^{k-1}(\beta z-\alpha)}{n+\beta} - \frac{nz^{k-1}k(k-1)}{2(n+\beta)^2},$$

then it is clear that $E_{n,k}(z)$ is a polynomial of degree $\leq k$ and by above recurrence relation, we have

$$E_{n,k}(z) = \frac{z}{n+\beta}E'_{n,k-1}(z) + \frac{nz+\alpha}{n+\beta}E_{n,k-1}(z) + X_{n,k}(z),$$

where

$$\begin{aligned} X_{n,k}(z) = {} & \frac{(k-1)}{n+\beta}z^{k-1} + \frac{\alpha(k-1)(k-2)}{(n+\beta)^2}z^{k-2} - \frac{\beta(k-1)^2}{(n+\beta)^2}z^{k-1} \\ & + \frac{n(k-1)(k-2)^2}{2(n+\beta)^3}z^{k-2} + \frac{\alpha}{n+\beta}z^{k-1} + \frac{n}{n+\beta}z^k \\ & + \frac{n\alpha(k-1)}{(n+\beta)^2}z^{k-1} + \frac{\alpha^2(k-1)}{(n+\beta)^2}z^{k-2} - \frac{n\beta(k-1)}{(n+\beta)^2}z^k \\ & - \frac{\alpha\beta(k-1)}{(n+\beta)^2}z^{k-1} + \frac{n^2(k-1)(k-2)}{2(n+\beta)^3}z^{k-1} + \frac{\alpha n(k-1)(k-2)}{2(n+\beta)^3}z^{k-2} \\ & - z^k - \frac{\alpha k}{n+\beta}z^{k-1} + \frac{\beta k}{n+\beta}z^k - \frac{k(k-1)n}{2(n+\beta)^2}z^{k-1} \end{aligned}$$

$$= \frac{z^{k-1}}{n+\beta}[(k-1)+\alpha+nz-\alpha k+\beta kz]-z^k$$
$$+\frac{(k-1)z^{k-2}}{(n+\beta)^2}\left[\alpha(k-2)-\beta z(k-1)+\alpha nz+\alpha^2-n\beta z^2-\alpha\beta z-\frac{knz}{2}\right]$$
$$+\frac{(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[n(k-2)+n^2z+\alpha n].$$

Using $\alpha+nz=(\alpha-\beta z)+(n+\beta)z$, we have

$$X_{n,k}(z)=\frac{z^{k-1}}{n+\beta}[(k-1)+(\alpha-\beta z)+(n+\beta)z-\alpha k+\beta kz]-z^k$$
$$+\frac{(k-1)z^{k-2}}{(n+\beta)^2}\Big[\alpha(k-2)-\beta z(k-1)+(\alpha-\beta z)\alpha+(n+\beta)\alpha z-\beta z(\alpha-\beta z)$$
$$-(n+\beta)\beta z^2-\frac{knz}{2}\Big]+\frac{(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[n(k-2)+n(\alpha-\beta z)+(n+\beta)nz]$$
$$=\frac{z^{k-1}(k-1)}{n+\beta}[1-\alpha+\beta z]+\frac{(k-1)z^{k-2}}{(n+\beta)^2}\left[\alpha(k-2)-\beta z(k-1)+(\alpha-\beta z)\alpha-\beta z(\alpha-\beta z)-\frac{knz}{2}\right]$$
$$+\frac{(k-1)\alpha z^{k-1}}{(n+\beta)}-\frac{(k-1)\beta z^k}{(n+\beta)}+\frac{(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[n(k-2)+n(\alpha-\beta z)]+\frac{n(k-1)(k-2)z^{k-1}}{2(n+\beta)^2}$$
$$=\frac{z^{k-1}(k-1)}{n+\beta}+\frac{(k-1)z^{k-2}}{(n+\beta)^2}[\alpha(k-2)-\beta z(k-1)+(\alpha-\beta z)\alpha-\beta z(\alpha-\beta z)]$$
$$-\frac{k(k-1)nz^{k-1}}{2(n+\beta)^2}+\frac{(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[n(k-2)+n(\alpha-\beta z)]+\frac{n(k-1)(k-2)z^{k-1}}{2(n+\beta)^2}$$
$$=\frac{z^{k-1}(k-1)}{n+\beta}+\frac{(k-1)z^{k-2}}{(n+\beta)^2}[\alpha(k-2)-\beta z(k-1)+(\alpha-\beta z)^2]$$
$$+\frac{(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[n(k-2)+n(\alpha-\beta z)]-\frac{n(k-1)z^{k-1}}{(n+\beta)^2}$$
$$=\frac{(k-1)z^{k-2}}{(n+\beta)^2}[\alpha(k-2)-\beta z(k-1)+(\alpha-\beta z)^2]$$
$$+\frac{(k-1)(k-2)z^{k-2}}{2(n+\beta)^2}[(k-2)+(\alpha-\beta z)]$$
$$-\frac{\beta(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[(k-2)+(\alpha-\beta z)]+\frac{\beta(k-1)z^{k-1}}{(n+\beta)^2}$$
$$=\frac{(k-1)z^{k-2}}{(n+\beta)^2}\left[(\alpha-\beta z)(k-2)+(\alpha-\beta z)^2+\frac{(k-2)^2}{2}+\frac{(k-2)(\alpha-\beta z)}{2}\right]$$
$$-\frac{\beta(k-1)(k-2)z^{k-2}}{2(n+\beta)^3}[(k-2)+(\alpha-\beta z)].$$

Thus, for all $|z|\le r, k\ge 2$, we have

$$|E_{n,k}(z)|\le\frac{|z|}{2(n+\beta)}[2||E'_{n,k-1}(z)||_r]+\frac{n|z|+\alpha}{n+\beta}|E_{n,k-1}(z)|+|X_{n,k}(z)|$$

$$\leq r|E_{n,k-1}(z)| + \frac{|z|}{2(n+\beta)} \frac{2(k-1)}{r} ||E_{n,k-1}(z)||_r + |X_{n,k}(z)|$$

$$\leq r|E_{n,k-1}(z)| + \frac{|z|}{2(n+\beta)} \frac{2(k-1)}{r} \Big[||\pi_{n,k-1}(z) - e_{k-1}(z)||_r$$

$$+ \frac{(k-1)z^{k-2}(\beta z - \alpha)}{n+\beta} - \frac{nz^{k-2}(k-1)(k-2)}{2(n+\beta)^2} \Big] + |X_{n,k}(z)|$$

$$\leq r|E_{n,k-1}(z)| + \frac{|z|}{2(n+\beta)} \frac{2(k-1)}{r} \Big[\frac{k!}{2n} r^{k-2} + \frac{\alpha+\beta r}{n+\beta} r^{k-2}$$

$$+ \frac{(k-1)z^{k-2}(\beta z - \alpha)}{n+\beta} - \frac{nz^{k-2}(k-1)(k-2)}{2(n+\beta)^2} \Big] + |X_{n,k}(z)|$$

$$\leq r|E_{n,k-1}(z)| + \frac{|z|}{2(n+\beta)} \Big[\frac{2(k-1)}{r} \frac{k!}{2n} r^{k-2}$$

$$+ \frac{2(k-1)}{r} \frac{\alpha+\beta r}{n+\beta} r^{k-2} + \frac{2(k-1)}{r} \frac{(k-1)r^{k-2}(\beta z + \alpha)}{n+\beta}$$

$$+ \frac{2(k-1)}{r} \frac{r^{k-2}(k-1)(k-2)}{2(n+\beta)} + \frac{2(k-1)}{r} \frac{\beta(k-1)(k-2)r^{k-2}}{2(n+\beta)^2} \Big] + |X_{n,k}(z)|$$

$$\leq r|E_{n,k-1}(z)| + \frac{|z|}{2(n+\beta)} \Big[\frac{(k+1)!}{n} r^{k-3} [3 + 4(\alpha+\beta r)] + \frac{\beta(k-1)^2(k-2)}{(n+\beta)^2} r^{k-3} \Big]$$

$$+ \frac{(k-1)r^{k-2}}{(n+\beta)^2} \left[(\alpha+\beta r)(k-2) + (\alpha+\beta r)^2 + \frac{(k-2)^2}{2} + \frac{(k-2)(\alpha+\beta r)}{2} \right]$$

$$+ \frac{\beta(k-1)(k-2)r^{k-2}}{2(n+\beta)^3} [(k-2) + (\alpha+\beta r)]$$

$$\leq r|E_{n,k-1}(z)| + \frac{|z|(k+1)!}{2n(n+\beta)} r^{k-3} [3 + 4(\alpha+\beta r)]$$

$$+ \frac{(k-1)r^{k-2}}{(n+\beta)^2} \left[(\alpha+\beta r)(k-2) + (\alpha+\beta r)^2 + \frac{(k-2)^2}{2} + \frac{(k-2)(\alpha+\beta r)}{2} \right]$$

$$+ \frac{\beta(k-1)(k-2)r^{k-2}}{2(n+\beta)^3} [(2k-3) + (\alpha+\beta r)].$$

Taking $k = 2, 3, \ldots$ in the last inequality step by step, we obtain

$$\left| S_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{nz}{2(n+\beta)^2} f''(z) \right| \leq \sum_{k=2}^{\infty} |c_k||E_{n,k}(z)|$$

$$\leq \frac{|z|MA}{n(n+\beta)r^2} [3 + 4(\alpha+\beta r)] \sum_{k=2}^{\infty} (k+1)(rA)^{k-1}$$

$$+ \frac{MA^2}{(n+\beta)^2} \sum_{k=2}^{\infty} \Big[(\alpha+\beta r)(k-1)(k-2) + (\alpha+\beta r)^2$$

$$+ \frac{(k-1)(k-2)^2}{2} + \frac{(k-1)(k-2)(\alpha+\beta r)}{2} \Big] (rA)^{k-2}$$

$$+\frac{\beta MA^2}{2(n+\beta)^3}\sum_{k=2}^{\infty}(k-1)(k-2)[(2k-3)+(\alpha+\beta r)](rA)^{k-2},$$

which immediately proves the theorem.

Remark 1 For $\alpha=\beta=0$, the Theorems 32 and 33 become some of the results in the book Gal [10, pp. 104–113].

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Generalized Hardy–Hilbert Type Inequalities on Multiple Weighted Orlicz Spaces

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Abstract In this paper, we introduce the multiple weighted Orlicz spaces. We also give a multiple generalized Hardy-Hilbert type integral inequality with the general kernel on these new spaces. It includes many famous results as the special cases.

Keywords Hardy-Hilbert inequality · Weighted Orlicz space · Norm inequality

1 Introduction

Throughout this paper, we write

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p \omega(x) dx\right)^{1/p}, \quad \mathbb{R}^n_+$$
$$= \{x = (x_1, x_2, \ldots, x_n) : x_k \geq 0, 1 \leq k \leq n\},$$

$$L^p(\omega) = \{f : f \textit{ is measurable and } \|f\|_{p,\omega} < \infty\}; \quad \|x\| = \left(\sum_{k=1}^n |x_k|^2\right)^{1/2}.$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{sin\left(\frac{\pi}{p}\right)} \|f\|_p \|g\|_q \tag{1}$$

is called the Hilbert's inequalities, where $1 < p < \infty, (\frac{1}{p}) + (\frac{1}{q}) = 1$, and the constant factors $\frac{\pi}{sin\left(\frac{\pi}{p}\right)}$ is the best value (see [1]). Further, the following inequality of the form

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dxdy \leq C(p,q) \|f\|_{p,\omega_1} \|g\|_{q,\omega_2} \tag{2}$$

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is called the Hardy-Hilbert's inequalities with the general kernel. In view of the mathematical importance and applications, Hilbert's and Hardy-Hilbert's inequalities are field of interest of numerous mathematicians and were generalized in many different ways (see, e.g. [2–8] and the references cited therein). However, much less attention has been given to inequalities on the Orlicz spaces. In 2007, Kuang and Debnath obtained in [9] the Hilbert's inequalities with the homogeneous kernel on the weighted Orlicz spaces. The aim of this paper is to introduce the new multiple weighted Orlicz spaces and establish a new multiple generalized Hardy-Hilbert type inequality with the general kernel on these new spaces. It includes many famous results as the special cases.

2 Definitions and Statement of the Main Results

Definition 1 (see [9–12]) We call φ a Young's function if it is a non-negative increasing convex function on $(0, \infty)$ with $\varphi(0) = 0, \varphi(u) > 0, u > 0$, and

$$\lim_{u\to 0} \frac{\varphi(u)}{u} = 0, \lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty.$$

To Young's function φ, we can associate its convex conjugate function denoted by $\psi = \varphi^*$ and defined by

$$\psi(v) = \varphi^*(v) = \sup\{uv - \varphi(u) : u \geq 0\}, v \geq 0. \tag{3}$$

We note that $\psi = \varphi^*$ is also a Young's function and $\psi^* = (\varphi^*)^* = \varphi$. From the definition of $\psi = \varphi^*$, we get Young's inequality

$$uv \leq \varphi(u) + \psi(v), u, v > 0. \tag{4}$$

Let φ^{-1} be inverse function of φ, we have

$$v \leq \varphi^{-1}(v)\psi^{-1}(v) \leq 2v, v \geq 0. \tag{5}$$

The aim of this paper is to introduce the following new multiple weighted Orlicz spaces.

Definition 2 Let φ be a Young's function on $(0, \infty)$, for any measurable function f and non-negative weight function ω on $\mathbb{R}^n_+$, the multiple weighted Luxemburg norm is defined as follows:

$$\|f\|_{\varphi,\omega} = inf \left\{\lambda > 0 : \int_{\mathbb{R}^n_+} \varphi\left(\frac{|f(x)|}{\lambda}\right) \omega(x)dx \leq 1\right\}. \tag{6}$$

The multiple weighted Orlicz space is defined as follows:

$$L_\varphi(\omega) = \left\{f : \|f\|_{\varphi,\omega} < \infty\right\}. \tag{7}$$

In particular, if $\varphi(u) = u^p, 1 < p < \infty$, then $L_\varphi(\omega)$ is the weighted Lebesgue spaces $L^p(\omega)$; if $\varphi(u) = u(log(u + c))^q, q \geq 0, c > 0$, then $L_\varphi(\omega)$ is the weighted spaces $L(\omega)(log L(\omega))^q$.

Definition 3 (see [9, 10]) We call the Young's function φ on $(0, \infty)$ submultiplicative if

$$\varphi(uv) \leq \varphi(u)\varphi(v) \quad for \quad all \quad u, v \geq 0. \tag{8}$$

Remark 1 If φ satisfies (8), then φ also satisfies Orlicz ∇_2- condition, that is, there exists a constant $C > 1$ such that

$$\varphi(2u) \leq C\varphi(u) \quad for \quad all \quad u \geq 0.$$

Our main result is the following theorem:

Theorem 1 *Let the conjugate Young's functions φ, ψ on $(0, \infty)$ submultiplicative; $K(\|x\|, \|y\|)$ be a non-negative measurable function on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and satisfies:*

$$K(\|x\|, \|ty\|) = t^{-\lambda_2} K\left(t^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\|x\|, \|y\|\right), \quad t > 0, \tag{9}$$

where λ_1, λ_2 are real numbers and $\lambda_1\lambda_2 \neq 0$. Let $f \in L_\varphi(\omega_1), g \in L_\psi(\omega_2)$ and $\|f\|_{\varphi,\omega_1} > 0, \|g\|_{\psi,\omega_2} > 0$, where $\omega_1(x) = \|x\|^{-\lambda\lambda_1 + \frac{(n\lambda_1)}{\lambda_2}}, \quad \omega_2(y) = \|y\|^{-\lambda\lambda_2 + \frac{(n\lambda_2)}{\lambda_1}}$, $\lambda > 0$. If

$$C_1 = \frac{\pi^{n/2}\lambda_1}{2^{n-1}\Gamma(n/2)\lambda_2} \int_0^\infty K^\lambda(u, 1)\psi^{-1}(u)u^{\lambda\lambda_1 - \left(\frac{n\lambda_1}{\lambda_2}\right) - 1} du < \infty; \tag{10}$$

$$C_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty K^\lambda(u, 1)\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right) u^{n-1} du < \infty, \tag{11}$$

then

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K^\lambda(\|x\|, \|y\|)f(x)g(y)dxdy \leq C(\varphi, \psi)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \tag{12}$$

where $C(\varphi, \psi) = C_1 + C_2$ is defined by (10) and (11).

We obtain the following Corollary 1 by taking $\varphi(u) = u^p, \psi(v) = v^q, 1 < p, q < \infty, \left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$, in Theorem 1:

Corollary 1 *Let $K(x, y), \lambda_1, \lambda_2, \lambda, \omega_1$ and ω_2 satisfy the conditions of Theorem 1. If $f \in L^p(\omega_1)$, $g \in L^q(\omega_2)$, $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K^\lambda(\|x\|, \|y\|)f(x)g(y)dxdy \leq C(p, q)\|f\|_{p,\omega_1}\|g\|_{q,\omega_2}, \tag{13}$$

where

$$C(p,q)=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{\frac{\lambda_1}{\lambda_2}\int_0^{\infty}K^{\lambda}(u,1)u^{\frac{-1}{p}+\lambda\lambda_1-\left(\frac{n\lambda_1}{\lambda_2}\right)}du+\int_0^{\infty}K^{\lambda}(u,1)u^{\frac{-1}{p}+n-1}du\right\}. \quad (14)$$

In particular, if $\lambda_1=\lambda_2=\lambda=1$ in Corollary 1, then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}K(\|x\|,\|y\|)f(x)g(y)dxdy\le C(p,q)\|f\|_p\|g\|_q; \quad (15)$$

where

$$C(p,q)=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{\int_0^{\infty}K(u,1)u^{((1/q)-n)}du+\int_0^{\infty}K(u,1)u^{-(1/p)+n-1}du\right\}. \quad (16)$$

Remark 2 If $n=1,\lambda=1$, and $\lambda_1=\lambda_2=\lambda_0>0$, then

$$K(tx,ty)=t^{-\lambda_0}K(t^{-1}(tx),y)=t^{-\lambda_0}K(x,y),$$

that is, $K(x,y)$ is the homogeneous kernel of degree $(-\lambda_0)$, thus Theorem 1 reduces to the results of [9].

3 Proof of Theorem 1

We require the following lemmas to prove our result:

Lemma 1 (*see* [13]) *If* $a_k,b_k,p_k>0, 1\le k\le n, f$ *be a measurable function on* $[0,1]$. *Let* $D=\left\{(x_1,x_2,\dots,x_n):\sum_{k=1}^n\left(\frac{x_k}{a_k}\right)^{b_k}\le 1, x_k\ge 0\right\}$, *then*

$$\int_D f\left(\sum_{k=1}^n\left(\frac{x_k}{a_k}\right)^{b_k}\right)x_1^{p_1-1}\cdots x_n^{p_n-1}dx_1\cdots dx_n \quad (17)$$

$$=\frac{\prod_{k=1}^n(a_k)^{p_k}}{\prod_{k=1}^n b_k}\cdot\frac{\prod_{k=1}^n\Gamma\left(\frac{p_k}{b_k}\right)}{\Gamma\left(\sum_{k=1}^n\frac{p_k}{b_k}\right)}\cdot\int_0^1 f(t)t^{\left(\sum_{k=1}^n\frac{p_k}{b_k}-1\right)}dt.$$

From (17), we have the following lemma:

Lemma 2 *Let* f *be a measurable function on* $[0,\infty)$, *then*

$$\int_{\mathbb{R}_+^n}f\left(\|x\|^2\right)dx=\frac{\pi^{n/2}}{2^n\Gamma(n/2)}\int_0^{\infty}f(t)t^{(n/2)-1}dt. \quad (18)$$

Proof of Theorem 1

Proof Applying (5) and Young's inequality (4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} K^\lambda(\|x\|,\|y\|)f(x)g(y)dxdy\\
&\le \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left\{|f(x)|\varphi^{-1}\left(K^\lambda(\|x\|,\|y\|)\right)\right\}\left\{|g(y)|\psi^{-1}\left(K^\lambda(\|x\|,\|y\|)\right)\right\}dxdy\\
&= \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left\{|f(x)|\varphi^{-1}\left(K^\lambda(\|x\|,\|y\|)\right)\quad \varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)\right\}\\
&\quad\left\{|g(y)|\psi^{-1}\left(K^\lambda(\|x\|,\|y\|)\right)\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}dxdy\\
&\le \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\varphi\left\{|f(x)|\varphi^{-1}\left(K^\lambda(\|x\|,\|y\|)\right)\quad \varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)\right\}dxdy
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\quad+\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\psi\left\{|g(y)|\psi^{-1}\left(K^\lambda(\|x\|,\|y\|)\right)\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}dxdy\\
&= I_1+I_2.
\end{aligned}$$

Since φ on $(0,\infty)$ is submultiplicative, we have

$$\begin{aligned}
&\varphi\left\{|f(x)|\varphi^{-1}(K^\lambda(\|x\|,\|y\|))\varphi^{-1}(\psi^{-1}(\|x\|\cdot\|y\|^{-(\frac{\lambda_2}{\lambda_1})}))\right\}\\
&\quad\le \varphi(|f(x)|)\varphi\{\varphi^{-1}(K^\lambda(\|x\|,\|y\|))\varphi^{-1}(\psi^{-1}(\|x\|\cdot\|y\|^{-(\frac{\lambda_2}{\lambda_1})}))\}\\
&\quad\le \varphi(|f(x)|)K^\lambda(\|x\|,\|y\|)\psi^{-1}(\|x\|\cdot\|y\|^{-(\frac{\lambda_2}{\lambda_1})}).
\end{aligned} \tag{20}$$

Then, we have

$$\begin{aligned}
I_1 &\le \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\varphi(|f(x)|)K^\lambda(\|x\|,\|y\|)\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)dxdy\\
&= \int_{\mathbb{R}_+^n}\varphi(|f(x)|)\left\{\int_{\mathbb{R}_+^n}\|y\|^{-\lambda\lambda_2}K^\lambda\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)},1\right)\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)dy\right\}dx.
\end{aligned} \tag{21}$$

By (18), we have

$$\int_{\mathbb{R}_+^n}\|y\|^{-\lambda\lambda_2}K^\lambda\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)},1\right)\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)dy$$

$$=\frac{\pi^{n/2}}{2^n\Gamma(n/2)}\int_0^\infty t^{-\left(\frac{\lambda\lambda_2}{2}\right)}K^\lambda(\|x\|\cdot t^{-\left(\frac{\lambda_2}{2\lambda_1}\right)},1)\psi^{-1}\left(\|x\|\cdot t^{-\left(\frac{\lambda_2}{2\lambda_1}\right)}\right)t^{(n/2)-1}dt. \tag{22}$$

Let $u=\|x\|\cdot t^{-\frac{\lambda_2}{2\lambda_1}}$, and by (21), (22) and (10), we get

$$\begin{aligned}
I_1 \le& \frac{\pi^{n/2}\lambda_1}{2^{n-1}\Gamma(n/2)\lambda_2}\int_{\mathbb{R}_+^n}\int_0^\infty \varphi(|f(x)|)\|x\|^{-\lambda\lambda_1+\frac{n\lambda_1}{\lambda_2}}\cdot K^\lambda(u,1)\psi^{-1}(u)u^{\lambda\lambda_1-\frac{n\lambda_1}{\lambda_2}-1}dudx\\
&= \frac{\pi^{n/2}\lambda_1}{2^{n-1}\Gamma(n/2)\lambda_2}\left\{\int_0^\infty K^\lambda(u,1)\psi^{-1}(u)u^{\lambda\lambda_1-\frac{n\lambda_1}{\lambda_2}-1}du\right\}\cdot\left\{\int_{\mathbb{R}_+^n}\varphi(|f(x)|)\|x\|^{-\lambda\lambda_1+\frac{n\lambda_1}{\lambda_2}}dx\right\}\\
&= C_1\int_{\mathbb{R}_+^n}\varphi(|f(x)|)\omega_1(x)dx. \qquad (23)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\psi\left\{|g(y)|\psi^{-1}(K^\lambda(\|x\|,\|y\|))\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}\\
&\le \psi(|g(y)|)K^\lambda(\|x\|,\|y\|)\psi\left\{\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\} \qquad (24)\\
&\le \psi(|g(y)|)\|y\|^{-\lambda\lambda_2}K^\lambda\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)},1\right)\psi\left\{\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}.
\end{aligned}$$

By (18), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n}\|y\|^{-\lambda\lambda_2}K^\lambda\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)},1\right)\psi\left\{\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\times\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}dx\\
&=\|y\|^{-\lambda\lambda_2}\frac{\pi^{n/2}}{2^n\Gamma(n/2)}\int_0^\infty K^\lambda\left(t^{1/2}\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)},1\right)\psi\left\{\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(t^{1/2}\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}t^{(n/2)-1}dt. \qquad (25)
\end{aligned}$$

Let $u=t^{1/2}\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}$, and by (24), (25) and (11), we get

$$I_2=\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\psi\left\{|g(y)|\psi^{-1}(K^\lambda(\|x\|,\|y\|))\frac{1}{\varphi^{-1}\left(\psi^{-1}\left(\|x\|\cdot\|y\|^{-\left(\frac{\lambda_2}{\lambda_1}\right)}\right)\right)}\right\}dxdy$$

$$= \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty K^\lambda(u,1)\psi\left\{\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right\} u^{n-1}du$$
$$\times \int_{\mathbb{R}^n_+} \psi(|g(y)|)\|y\|^{-\lambda\lambda_2+\frac{n\lambda_2}{\lambda_1}}dy \tag{26}$$
$$= C_2 \int_{\mathbb{R}^n_+} \psi(|g(y)|)\omega_2(y)dy.$$

Thus, by (23) and (26), we obtain

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K^\lambda(\|x\|,\|y\|)f(x)g(y)dxdy$$
$$\leq C_1 \int_{\mathbb{R}^n_+} \varphi(|f(x)|)\omega_1(x)dx + C_2 \int_{\mathbb{R}^n_+} \psi(|g(y)|)\omega_2(y)dy. \tag{27}$$

It follows that

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K^\lambda(\|x\|,\|y\|)\left(\frac{f(x)}{\|f\|_{\varphi,\omega_1}}\right)\left(\frac{g(y)}{\|g\|_{\psi,\omega_2}}\right)dxdy$$
$$\leq C_1 \int_{\mathbb{R}^n_+} \varphi\left(\frac{|f(x)|}{\|f\|_{\varphi,\omega_1}}\right)\omega_1(x)dx + C_2 \int_{\mathbb{R}^n_+} \psi\left(\frac{|g(y)|}{\|g\|_{\psi,\omega_2}}\right)\omega_2(y)dy$$
$$\leq C_1 + C_2 = C(\varphi,\psi).$$

Hence,

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K^\lambda(\|x\|,\|y\|)f(x)g(y)dxdy \leq C(\varphi,\psi)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}.$$

The proof is complete.

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Inequalities for the Fisher's Information Measures

Christos P. Kitsos and Thomas L. Toulias

Abstract The objective of this chapter is to provide a thorough discussion on inequalities related to the entropy measures in connection to the γ-order generalized normal distribution (γ–GND). This three-term (position, scale and shape) family of distributions plays the role of the usual multivariate normal distribution in information theory. Moreover, the γ–GND is the appropriate family of distributions to support a generalized version of the entropy type Fisher's information measure. This generalized (entropy type) Fisher's information is also discussed as well as the generalized entropy power, while the γ-GND heavily contributes to these generalizations. The appropriate bounds and inequalities of these measures are also provided.

Keywords Fisher's entropy type information measure · Shannon entropy · Generalized normal distribution

1 Introduction

The well-known normal distribution, introduced by Gauss and, therefore, also known as Gaussian or normal distribution, plays an important role to all statistical problems. Interest is focused on the Information Theory and Statistics. An exponential power generalization of the normal distribution called the generalized γ-order normal distribution (γ-GND) has been discussed in [15], and studied in [19] and [21]. Moreover, new entropy measures were introduced in [16] and extensively discussed and proved in [15].

Entropy since the time of Clausius, 1865, plays an important role joining physical experimentation and statistical analysis. For the principle of maximum entropy, the normal distribution is essential and eventually is related with the energy and the variance involved. Moreover, the channel capacity is dependent on the entropy, since

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the time of Shannon, 1948. Therefore, we would like to know how entropy, energy and variance are related under the normal distribution, for practical problems. To proceed, we need a solid mathematical background to cover Statistics and Physics, despite the applicable form of this procedure. This is why these definitions are introduced in Sect. 2 under a mathematical analysis point of view, while they are so applicable (channel capacity etc.). Moreover, their relations through inequalities, either Poincaré or Sobolev, are briefly discussed.

There is also a connection with the optimal design theory. Fisher's parametric information measure is applied to the experimental design theory. The following example is presented. Let us consider two experiments $E_X \equiv (X, \xi)$ and $E_Y \equiv (Y, \delta)$ with X and Y being the design spaces while ξ and δ are the corresponding design measures from the design spaces $\varXi$ and $\varDelta$, respectively, see for details [9, 27]. In practice, the design space is where the experimenter performs the experiment and the design measure is, eventually, due to some mathematical insight, the proportion of the observations devoted for each design point. We shall say that the experiment E_X is sufficient for the experiment Y if there exist a transformation of X, say $t(X)$, such that $t(X)$ and Y have identical design measure, or coming from the same distribution. We shall write $E_X \geq E_Y$. In such a case, the Shannon information obtained from E_X, say H_X, is at least as that obtained in E_Y, say H_X, i.e. $\mathrm{H}_X \geq \mathrm{H}_Y$. Moreover, the same ordering occurs for the Fisher information in terms that $\mathrm{I}_\theta(X) - \mathrm{I}_\theta(Y)$ is non-negative definite, so $|\mathrm{I}_\theta(X)| \geq |\mathrm{I}_\theta(Y)|$, and therefore, one could say that D–optimal designs, see [9], between E_X and E_Y, the E_Y is more preferable. Consider two experiments, one coming from the Gaussian $\mathcal{N}(0, \sigma^2)$ and the other from the Gaussian $\mathcal{N}(0, \kappa^2\sigma^2)$. These experiments are equivalent in terms that the one is sufficient for the other. This is trivially true if all the observations of the first multiplied by κ, or divide all the observations of the second by κ. This is a brief explanation why there is an interest to have at least inequalities among various statistical–analytical measures concerning the Gaussian: to be able to compare the "information" obtained for an experiment. Usually it is assumed that the experimenter works with the Gaussian. Thus, it is of great importance to information theory.

Poincaré and Sobolev inequalities presented in Sect. 2 play an important role in the foundation of the generalized Fisher's entropy type information measure. Both these classes of inequalities offer a number of bounds for a number of physical applications, the most well known being the energy, among others. The Gaussian kernel or the error function (which produces the normal distribution), is certainly known, with two parameters—the mean and the variance. For the Gaussian kernel, an extra parameter was then introduced in [15], and therefore, a generalized form of the normal distribution was obtained. Specifically, the generalized Gaussian is obtained as an extremal for the logarithm Sobolev inequality (LSI) and is referred as the γ-order generalized distribution (γ-GND). In addition, the Poincaré inequality (PI) offers also the "best" constant for the Gaussian measure, and therefore is of interest to see how Poincaré and Sobolev inequalities are acting on the normal distribution.

That is, this chapter attempts to bridge the mathematical–analytical framework with statistical background as far as the Fisher's information measures (parametric and entropy type) is concerned. Emphasis is given to the entropy type Fisher's information and the generalization introduced.

2 Background

The PI is the most well-known result in the theory of Sobolev spaces, i.e. bounds can be obtained on a function f belonging to the Sobolev space $\mathbb{H}^1(\mathbb{R}^p, \mu) = \{f \in \mathcal{L}^2(\mathbb{R}^p, \mu) : \ \mathcal{E}_\mu(f) < \infty\}$ using the bounds on the derivatives, while the domain is still important. The energy $\mathcal{E}_\mu(f)$ of a local μ–integrable function f with $\nabla f \in \mathcal{L}^2(\mathbb{R}^p, \mu)$ is defined to be

$$\mathcal{E}_\mu(f) = \mathrm{Exp}_\mu(\|\nabla f\|^2).$$

The corresponding Poincaré constant, c_P, can easily be evaluated when the domain is convex. It holds that

$$\mathrm{Var}_\mu(f) \leq c_\mathrm{P}\mathcal{E}_\mu(f), \tag{1}$$

where $\mathrm{Exp}_\mu(f)$ and $\mathrm{Var}_\mu(f)$ are the expected value and the variance of f, respectively, corresponding to the probability measure μ, i.e. $\mathrm{Exp}_\mu(f) = \int f d\mu$ and $\mathrm{Var}_\mu(f) = \mathrm{Exp}_\mu([f - \mathrm{Exp}_\mu(f)]^2) = \mathrm{Exp}_\mu(f^2) - \mathrm{Exp}_\mu(f)^2$. Under some regularity conditions for the measure μ, there exists a constant $c_\mathrm{P} \in (0, +\infty)$ such that the PI as in (1), is

$$\mathrm{Var}_\mu(f) \leq c_\mathrm{P} \int_{\mathbb{R}}^{p} \|\nabla f\|^2 \, d\mu,$$

with f as a differentiable function having compact support. That is, bounds have to be evaluated for the variance and, therefore, for the information, either the parametric or the entropy type.

The entropy $\mathrm{Ent}_\mu f$ of a μ-integrable positive function f is defined to be

$$\mathrm{Ent}_\mu f := \mathrm{Exp}_\mu(f \log f) - \mathrm{Exp}_\mu f \log \mathrm{Exp}_\mu f, \tag{2}$$

where Exp is the expected value. Applying the inequality $uv \leq u \log u - u + e^v$, $u \in \mathbb{R}_+$, $v \in \mathbb{R}$, the so-called variational formula for the entropy is obtained,

$$\mathrm{Ent}_\mu f := \sup\left\{\mathrm{Exp}_\mu(fg) : \quad \mathrm{Exp}_\mu e^g = 1\right\}. \tag{3}$$

The quantity $\mathrm{Ent}_\mu f$ is finite if and only if $f \sup(0, \log f)$ is μ-integrable. Notice that when the expected value of f vanishes the definition (2) is simplified. Relation (3) is equivalent to the following inequality, known as entropy inequality,

$$\mathrm{Exp}_\mu(fg) \leq \tfrac{1}{t}\mathrm{Exp}_\mu f \log \mathrm{Exp}_\mu e^{tg} + \tfrac{1}{t}\mathrm{Ent}_\mu(f), \tag{4}$$

where f is every positive and square integrable function, g is a square integrable function and $t > 0$. The following Proposition 1 refers to the product probability space, as far as its variance and entropy concern.

Proposition 1 *Let* $(\mathbb{E}_i, F_i, \mu_i)$, $i = 1, 2, \ldots, p)$ *be* p *probability spaces and* $(\mathbb{E}^p, F^p, \mu^p)$ *the product probability space. Then,*

$$\mathrm{Var}_{\mu^p}(f) \leq \sum_{i=1}^{p} \mathrm{Exp}_{\mu^p}(\mathrm{Var}_{\mu_i}(f)),$$

$$\mathrm{Ent}_{\mu^p}(f) \leq \sum_{i=1}^{p} \mathrm{Exp}_{\mu^p}(\mathrm{Ent}_{\mu_i} f).$$

Proof Let a function g defined on $\mathbb{E}^p$ such that $\mathrm{Exp}_{\mu_p} e^g = 1$ and

$$g = \sum_{i=1}^{p} g_i \overset{\text{def.}}{=} \sum_{i=2}^{p} \log \frac{\int e^g d\mu_1(x_1) \cdots d\mu_{i-1}(x_{i-1})}{\int e^g d\mu_1(x_1) \cdots d\mu_i(x_i)},$$

with $g_i = g - \log \int e^g d\mu_i(x_i)$, $i = 1, 2, \ldots, p$. Hence, using the variational formula (3), for μ_i:

$$\sum_{i=1}^{p} \mathrm{Exp}_{\mu_i}(f g_i) \leq \sum_{i=1}^{p} \mathrm{Exp}_{\mu_i} \left(\mathrm{Ent}_{\mu_i}(f g_i)\right).$$

On the other hand

$$\begin{aligned} \mathrm{Exp}_{\mu^p}(fg) &= \sum_{i=1}^{p} \mathrm{Exp}_{\mu^p}(f g_i) \leq \sum_{i=1}^{p} \mathrm{Exp}_{\mu^p} \left(\mathrm{Exp}_{\mu_i}(f g_i)\right) \\ &\leq \sum_{i=1}^{p} \mathrm{Exp}_{\mu} \left(\mathrm{Ent}_{\mu_i} f\right). \end{aligned}$$

Combining the above two relationships, the result of the Proposition 1 is derived. Notice that, neither statistical properties, discussed for the above introduced functions of variance and entropy, nor physical properties are going to be discussed for the below defined energy.

In the case of $\mathbb{E} = \mathbb{R}^p$, the energy $\mathrm{Ener}_\mu f$ of a local integrable function f with $\nabla f \in \mathcal{L}^2(\mathbb{R}^p, \mu)$ is defined to be

$$\mathrm{Ener}_\mu f := \mathrm{Exp}_\mu \|\nabla f\|^2, \tag{5}$$

where ∇f is, as usual, the gradient of f. Hence, the energy is positive and invariant under the translations. Both variance and energy are crucial for the definition of the PI. Indeed the measure μ satisfies the PI for a certain function class $\mathcal{F}_P(\mathbb{E}, \mu)$ if there exists a constant $c \in \mathbb{R}_+$ such that

$$\mathrm{Var}_\mu(f) \leq c\mathrm{Ener}_\mu f. \tag{6}$$

for each function $f \in \mathcal{F}_P(\mathbb{E}, \mu)$.

Example 1 For example, one may consider $\mathcal{F}_P(\mathbb{E},\mu)$ to be the Sobolev space $\mathbb{S}^1(\mathbb{R}^p,\mu) = \{f \in \mathcal{L}^2(\mathbb{R}^p,\mu) : \ \mathrm{Ener}_\mu f < \infty\}$. The best constant $c_P(\mu)$ for the PI, for f not μ, a.e. constant, is defined to be

$$c_P(\mu) := \left(\inf\left\{\frac{\mathrm{Ener}_\mu f}{\mathrm{Var}_\mu(f)} : \ f \in \mathcal{F}_P(\mathbb{E},\mu)\right\}\right)^{-1}. \tag{7}$$

The measure μ satisfies the LSI for a certain function class $\mathcal{F}_{LS}(\mathbb{E},\mu)$, if there exists a constant $c \in \mathbb{R}_+^* = (0,+\infty)$ such that

$$\mathrm{Ent}_\mu f^2 \leq c\mathrm{Ener}_\mu f, \tag{8}$$

for each function $f \in \mathcal{F}_{LS}(\mathbb{E},\mu)$.

Example 2 One may consider $\mathcal{F}_{LS}(\mathbb{E},\mu)$ to be the Sobolev space $\mathbb{S}^1(\mathbb{R}^p,\mu)$. The best constant $c_{LS}(\mu)$ for the LSI, for f not μ, a.e. constant, is defined to be

$$c_{LS}(\mu) := \left(\inf\left\{\frac{\mathrm{Ener}_\mu f}{\mathrm{Var}_\mu(f)} : \ f \in \mathcal{F}_{LS}(\mathbb{E},\mu)\right\}\right)^{-1}. \tag{9}$$

Since

$$\mathrm{Ent}_\mu f^2 := \sup\left\{\mathrm{Exp}_\mu(f^2\, g) : \ \mathrm{Exp}_\mu e^g = 1\right\},$$

we have that the constant

$$c_{LS}(\mu) = \sup\left\{c(g) : \ \mathrm{Exp}_\mu e^g = 1\right\},$$

where (with $\mathrm{Ener}_\mu f > 1$)

$$c(g) := \left(\sup\left\{\frac{\mathrm{Exp}_\mu(f^2\, g)}{\mathrm{Ener}_\mu f} : \ f \in \mathcal{F}_{LS}(\mathbb{E},\mu)\right\}\right)^{-1}.$$

Under some regularity conditions for the measure μ, the PI as in (6) is

$$\mathrm{Var}_\mu(f) \leq c \int_{\mathbb{E}} \|\nabla f\|^2 d\mu,$$

with f a differentiable function having compact support, see [1] and the references there. The constant c is known as Poincaré constant and will be denoted as c_P in this chapter.

In principle, LSI attempts to estimate the lower order derivatives of a given function in terms of higher order derivatives. The well-known Sobolev inequalities were introduced in 1938, see [28] for details. The introductory and well-known LSI is

$$\left(\int_{\mathbb{E}^p} \|f(x)\|^{\frac{2p}{p-2}} d\mu(x)\right)^{\frac{p-2}{2p}} \leq c_S \left(\int_{\mathbb{E}}^{p} \|\nabla f(x)\|^2 d\mu(x)\right)^{1/2}, \tag{10}$$

or, in a compact form, through the norm

$$\|f\|_q \leq c_{\mathrm{S}} \|\nabla f\|_2, \quad q = \tfrac{2p}{p-2}.$$

The constant c_{S} is known as Sobolev constant. Since then, various attempts were tried to generalize (10). The first optimal Sobolev inequality was of the form

$$\left(\int_{\mathbb{E}^p} \|f(x)\|^{\frac{np}{p-n}} dx \right)^{\frac{p-n}{np}} \leq C_{p,n} \left(\int_{\mathbb{E}^p} \|\nabla f(x)\|^n dx \right)^{1/n}, \tag{11}$$

with $n \in [1, p)$.

Recall the inequalities (6), (8), (10) and (86). These inequalities are dependent on a constant c, which is being evaluated, in the optimal sense as in (7) and (9) for the PI and LSI, respectively. Therefore, in all these integral inequalities the crucial points are: the exponent, and the value of the critical constant, which is usually dependent on the gamma function. This is clear on the generalized form of normal distribution, introduced in [15] and discussed in [16] and [18].

The PI and the LSI are linked with the parametric Fisher's information measure, as it is briefly discussed in the next section.

3 PI and LSI for the Parametric Fisher's Information

One of the merits that normal distribution offers to the information theory is that for any random variable X and the estimator est$X$, the following inequality holds:

$$\mathrm{Exp}\,(X - \mathrm{est}X)^2 \geq (2\pi e)^{-1} \exp\{2\, h(X)\},$$

with $h(X)$ being the differential entropy. The equality holds if and only if X is normally distributed and $\mathrm{Exp}(X)$ is the mean of X. This very useful result can also be extended even when side information is given for the estimator [6].

Moreover, the normal distribution is adopted for the noise acting additively to the input variable when an input–output time discrete channel is formed. Therefore, the Gaussian distribution needs a special treatment evaluating Poincaré and Sobolev inequalities. Both the PI and LSI are applied to statistical distributions to evaluate the bounds between variance, entropy and energy. Moreover, the development of the PI and LSI for the normal distribution depends on the development on the Bernoulli measure due to a theoretical insight, which is not presented here. Therefore, a discussion of the Bernoulli case is first provided.

If $\mathbb{E} = \{0, 1\}$, the Bernoulli measure β_n of $\mathbb{E}$ with the parameter $n \in (0, 1)$ is the following probability measure:

$$\beta_n := n\delta_0 + m\delta_1, \tag{12}$$

where $m = 1-n$ and δ_a is the Dirac measure at a. It is $\mathrm{Exp}_{\beta_n} f = nf(0)+mf(1)$ and the energy is evaluated to be $\mathrm{Ener}_{\beta_n} f = nm\|f(0) - f(1)\|^2$. A simple calculation gives $\mathrm{Var}_{\beta_n}(f) = \mathrm{Ener}_{\beta_n} f$ that leads to the PI for the Bernoulli measure.

Theorem 1 (*PI for the Bernoulli Measure*)

$$\mathrm{Var}_{\beta_n}(f) \leq \mathrm{Ener}_{\beta_n} f, \quad \text{i.e. } c_P(\beta_n) = 1.$$

Next the sharp LSI for Bernoulli measure is given, so that to clear the application and the comparison between the continuous and the discrete case.

Theorem 2 (*LSI for Bernoulli Measure*) The best constant for the inequality

$$\mathrm{Ent}_{\beta_n} f^2 \leq c_{LS} \mathrm{Ener}_{\beta_n} f, \tag{13}$$

is

$$c_{LS} = \begin{cases} 2, & \text{if } n = \frac{1}{2} \\ \frac{\log m - \log n}{m-n}, & \text{otherwise.} \end{cases}$$

Proof By symmetry we are restricted to the case $0 < p \leq \frac{1}{2}$. The variational formula

$$c_{LS}(\beta_p) = \sup\{c(g) : \ \mathrm{Exp}_{\beta_p} e^g = 1\},$$

is used where

$$c(g) := \sup\left\{ \frac{\mathrm{Exp}_{\beta_p}(f^2 g)}{\mathcal{E}_{\beta_p} f} \ : \ f \in \mathcal{C}_{LS}(\mathbb{E}, \beta_p), \mathrm{Exp}_{\beta_p} f > 0 \right\}.$$

Let $\alpha = g(0)$ and $b = g(1)$. It is then

$$\mathrm{Exp}_{\beta_p}(e^g) = pe^{\alpha} + qe^{b} = 1 = e^0,$$

and hence, $\alpha b < 0$. Note that $\mathrm{Exp}_{\beta_p}|f| \leq \mathrm{Exp}_{\beta_p} f$. So, $f \geq 0$ is assumed. For $x = f(0)$ with $x > 0$, it is

$$pqc(g) = sup\left\{ \frac{p\alpha x^2 + qb}{(x-1)^2} \ : \ x > 0 \ \text{ and } \ x \neq 1 \right\}.$$

The supremum is attained for $x = -qb/p\alpha$ and it is $c(g) = (\frac{p}{b} + \frac{q}{\alpha})^{-1}$. Therefore,

$$c_{LS}(p\delta_0 + q\delta_1) = \left(\inf\left\{\tfrac{p}{b} + \tfrac{q}{\alpha} \ : \ pe^{\alpha} + qe^{b} = 1\right\}\right)^{-1}.$$

Let $t = e^{\alpha}$, $s = e^{b} = \frac{1-pt}{q}$ and define $\varphi(t) \stackrel{def}{=} \frac{p}{\log s} + \frac{q}{\log t}$. Since $\alpha b < 0$,

$$c_{LS}(p\delta_0 + q\delta_1) = \left(\inf\left\{\varphi(t) \ : \ t \in (0,1) \cup (1, 1/p)\right\}\right)^{-1}.$$

The definition domain of φ can be extended by setting $\varphi(0) = -\frac{p}{\log q}$, $\varphi(1) = \frac{1}{2}$ and $\varphi(p^-) = -\frac{q}{\log p}$. Remark that 1 is a local minimum if and only if $p = \frac{1}{2}$. Then

$$\varphi'(t) = \frac{p^2}{qs(\log s)^2} - \frac{q}{t(logt)^2}.$$

Notice that the constant c_{LS} is a concave function of the parameter n. It diverges to $+\infty$ as p tends to 0 and has minimum for $n = 1/2$, (as one could expect for the Bernoulli trials) and then the constant depends only on the parameter n. Therefore, considering $\mathbb{E} = \{a, b\}$ and $\beta_n := n\delta_a + m\delta_b$, we have the same constant for the inequality. In this case, the energy is evaluated as $\mathrm{Ener}_{\beta_n} f = nm\|f(b) - f(a)\|^2$.

Using the tensorisation property of variance and entropy, the PI as well as the LSI for Gaussian measure are obtained from the above inequalities and the Bernoulli measure. Let $\mathrm{E} = \mathbb{R}$. The Gaussian probability measure is

$$d\gamma = (2\pi)^{-1/2} e^{-\|x\|^2/2} dx. \tag{14}$$

Theorem 3 (*PI for the Gaussian on* $\mathbb{R}$) For $f \in \mathbb{H}^1(\mathbb{R}, \gamma)$:

$$\mathrm{Var}_\gamma(f^2) \leq \mathrm{Ener}_\gamma f, \quad \text{i.e. } c_P(\gamma) = 1. \tag{15}$$

Theorem 4 (*LSI for the Gaussian on* $\mathbb{R}$) *For* $f \in \mathbb{H}^1(\mathbb{R}, \gamma)$*:*

$$\mathrm{Ent}_\gamma f^2 \leq 2\, \mathrm{Ener}_\gamma f, \quad \textit{i.e. } c_{LS}(\gamma) = 2. \tag{16}$$

Proof The proof is a step by step transfer of the proof of Theorem 3 using the tensorisation property of entropy. □

Let $\mathbb{E} = \mathbb{R}^p$ and the Gaussian probability measure on $\mathbb{R}^p$,

$$d\gamma^p(x) = (2\pi)^{-p/2} \exp\{-\|x\|^2/2\} dx.$$

The next Theorem 5 gives the best constants for the Poincaré and LSI for the Gaussian measure on $\mathbb{R}^p$, i.e. for the variance of f and the entropy of f^2. Using the following result

$$\begin{aligned} \mathrm{Ener}_{\gamma^p} f = &\ \mathrm{Exp}_{\gamma^p} \|\nabla f\|^2 = \sum_{i=1}^p \mathrm{Exp}_{\gamma^p} \|\partial_i f\|^2 \\ = &\ \sum_{i=1}^p \mathrm{Exp}_{\gamma^p} \left(\mathrm{Exp}_\gamma \|\partial_i f\|^2\right) = \sum_{i=1}^p \mathrm{Exp}_{\gamma^p} \left(\mathrm{Exp}_\gamma (\mathrm{Ener}_\gamma f_i)\right), \end{aligned}$$

the Poincaré and LSI for the Gaussian measure on $\mathbb{R}^p$ can be deduced from Theorem 4. It is interesting to notice the simplicity of the involved constants, with values 1 and 2, for PI and LSI, respectively. Then:

Theorem 5 (*PI and LSI for Gaussian Measure on* $\mathbb{R}^p$) For $f \in \mathbb{H}^1(\mathbb{R}^p, \gamma^p)$ the following are true:

$$\mathrm{Var}_{\gamma^p}(f) \le \mathrm{Ener}_{\gamma^p} f, \quad \text{i.e. } c_P(\gamma^p) = 1, \tag{17}$$

$$\mathrm{Ent}_{\gamma^p} f^2 \le 2\, \mathrm{Ener}_{\gamma^p} f, \quad \text{i.e. } c_{LS}(\gamma^p) = 2. \tag{18}$$

Notice that the values of the constants, as it has already mentioned, are rather nice and easy to be adopted in applications, as the involved constants for the multivariate normal discussed below, see relations (20) and (21). Therefore, there is a simplification in the real life problems.

Consider now the multivariate normal distribution $\mathcal{N}^p(\mu, \Sigma)$ with mean vector $\mu \in \mathbb{R}^p$ and scale matrix $\Sigma \in \mathbb{R}^{p\times p}$, i.e. with p.d.f. of the form

$$f(x) = f(x;\ \mu, \Sigma) = (2\pi)^{-p/2} |\det \Sigma|^{-1/2} \exp\left\{-\tfrac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^{\mathrm{T}}\right\}, \tag{19}$$

with $a^{\mathrm{T}} \in \mathbb{R}^{1\times p}$ being the transpose of the vector $a \in \mathbb{R}^p$. In this general case of the Gaussian measure, the Poincaré and LSI are the following:

$$\mathrm{Var}_{\gamma^p}(f) \le \sigma_\Sigma \mathrm{Exp}_{\gamma^p} \|\nabla f\|^2, \tag{20}$$

$$\mathrm{Ent}_{\gamma^p} f^2 \le 2\sigma_\Sigma \mathrm{Exp}_{\gamma^p} \|\nabla f\|^2, \tag{21}$$

respectively.

Moreover, as far as the entropy of a p-variate random vector X is concerned, say $\mathrm{H}(X)$, considering the following proposition a bound for it is obtained, depending only on the scale matrix.

Proposition 2 *Let the random vector X has zero mean and covariance matrix Σ. Then*

$$\mathrm{H}(X) \le \tfrac{1}{2} \log\left\{(2\pi e)^p |\det \Sigma|\right\},$$

with equality if and only if $X \sim \mathcal{N}(0, \Sigma)$.

This proposition is crucial and clarifies that the entropy for the normal distribution is depending, eventually, only on the variance–covariance matrix, while equality holds when X is following the (multivariate) normal distribution, a result quite often applied in engineering problems, and information systems.

4 The γ-Order Generalized Normal Distribution (γ-GND)

Through the LSI approach, a construction of an exponential power generalization of the usual normal distribution is provided as an extremal of (an Euclidean) LSI. Following [15], the gross logarithm inequality with respect to the Gaussian weight, [14], is of the form

$$\int_{\mathbb{R}^p} \|g\|^2 \log \|g\|^2\, dm \le \tfrac{1}{\pi} \int_{\mathbb{R}^p} \|\nabla g\|^2\, dm, \tag{22}$$

where $\|g\|_2 = 1$, $dm = \exp\{-\pi|x|^2\}dx$ ($\|g\|_2 = \int_{\mathbb{R}^p} \|g(x)\|^2 dx$ is the norm in $\mathcal{L}^2(\mathbb{R}^p, dm)$). Inequality (22) is equivalent to the (Euclidean) LSI,

$$\int\limits_{\mathbb{R}^p} \|u\|^2 \log \|u\|^2 dx \leq \tfrac{p}{2} \log \left\{ \tfrac{2}{\pi p e} \int\limits_{\mathbb{R}^p} \|\nabla u\|^2 dx \right\}, \tag{23}$$

for any function $u \in \mathcal{W}^{1,2}(\mathbb{R}^p)$ with $\|u\|_2 = 1$, see [15] for details. This inequality is is optimal, in the sense that

$$\tfrac{2}{\pi p e} = \inf \left\{ \frac{\int\limits_{\mathbb{R}^p} \|\nabla u\|^2 dx}{\exp\left(\tfrac{2}{n} \int\limits_{\mathbb{R}^p} \|u\|^2 \log \|u\|^2 dx \right)} : \quad u \in \mathcal{W}^{1,2}(\mathbb{R}^n), \quad \|u\|_2 = 1 \right\},$$

see [31]. Extremals for (23) are precisely the Gaussians $u(x) = (\pi\sigma/2)^{-p/4} \exp\{-\sigma^{-1}\|x-\mu\|^2\}$ with $\sigma > 0$ and $\mu \in \mathbb{R}^p$, see [4, 5] for details.

Now, consider the extension of Del Pinto and Dolbeault in [7] for the LSI as in (23). For any $u \in \mathcal{W}^{1,2}(\mathbb{R}^p)$ with $\|u\|_\gamma = 1$, the γ-LSI holds, i.e.

$$\int\limits_{\mathbb{R}^p} \|u\|^\gamma \log \|u\| dx \leq \tfrac{p}{\gamma^2} \log \left\{ K_\gamma \int\limits_{\mathbb{R}^p} \|\nabla u\|^\gamma dx \right\}, \tag{24}$$

with the optimal constant K_γ equals to

$$K_\gamma = \tfrac{\gamma}{p} \left(\tfrac{\gamma-1}{e} \right)^{\gamma-1} \pi^{-\gamma/2} (\xi_\gamma^p)^{\gamma/p}, \tag{25}$$

where

$$\xi_\gamma^p = \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p\frac{\gamma-1}{\gamma}+1)}, \tag{26}$$

and $\Gamma(\,\cdot\,)$ the usual gamma function.

Inequality (24) is optimal and the equality holds when $u(x) = f_{X_\gamma}(x)$ is considered, where X_γ follows the multivariate distribution with p.d.f. f_{X_γ} defined as

$$f_{X_\gamma}(x;\ \mu, \Sigma, \gamma) = C_\gamma^p(\Sigma) \exp\left\{ -\tfrac{\gamma-1}{\gamma} Q_\theta(x)^{\frac{\gamma}{2(\gamma-1)}} \right\}, \quad x \in \mathbb{R}^p, \tag{27}$$

with normalizing factor

$$C_\gamma^p = C_\gamma^p(\Sigma) = \pi^{-p} |\Sigma|^{-1/2} \xi_\gamma^p \left(\tfrac{\gamma-1}{\gamma} \right)^{p\frac{\gamma-1}{\gamma}}, \tag{28}$$

and p-quadratic form $Q_\theta(x) = (x-\mu)\Sigma^{-1}(x-\mu)^{\mathrm{T}}$ where $\theta = (\mu, \Sigma) \in \mathbb{R}^p \times \mathbb{R}^{p \times p}$. The function $\phi(\delta) = f_{X_\delta}(x)^{1/\delta}$ with $\Sigma = (\sigma^2/\delta)^{2(\delta-1)/\delta} \mathbb{I}_p$ corresponds to the

extremal function for the LSI due to [7]. The essential result is that the defined p.d.f f_{X_γ} works as an extremal function to a generalized form of the LSI.

We shall write $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ where $\mathcal{N}_\gamma^p(\mu, \Sigma)$ is an exponential power generalization of the usual normal distribution $\mathcal{N}^p(\mu, \Sigma)$ with mean vector $\mu \in \mathbb{R}^p$, scale matrix $\Sigma \in \mathbb{R}^{p\times p}$ involving a new shape parameter $\gamma \in \mathbb{R} \setminus [0, 1]$. These distributions shall be referred to as the γ-order normal distributions or γ-GND. Notice that for $\gamma = 2$, the second-ordered normal $\mathcal{N}_2^p(\mu, \Sigma)$ is reduced to to the usual multivariate normal $\mathcal{N}^p(\mu, \Sigma)$, i.e. $\mathcal{N}_2^p(\mu, \Sigma) = \mathcal{N}^p(\mu, \Sigma)$. One of the merits of the γ-order normal distribution defined above belongs to the symmetric Kotz type distributions family, [22], as $\mathcal{N}_\gamma^p(\mu, \Sigma) = Kotz_{m,r,s}(\mu, \Sigma)$ with $m = 1$, $r = (\gamma - 1)/\gamma$ and $s = \gamma/(2\gamma - 2)$.

It is commented here that the introduced univariate γ-order normal $\mathcal{N}_\gamma(\mu, \sigma^2) = \mathcal{N}_\gamma^1(\mu, \sigma^2)$ coincides with the existent generalized normal distribution introduced in [25], with density function

$$f(x;\ \mu, \alpha, \beta) = \frac{\beta}{2\alpha\Gamma(1/\beta)} \exp\left\{-\left|\tfrac{x-\mu}{\alpha}\right|^{\beta}\right\},$$

where $\alpha = (\frac{\gamma}{\gamma-1})^{(\gamma-1)/\gamma}\sigma$ and $\beta = \frac{\gamma}{\gamma-1}$, while the multivariate case of the γ-order normal $\mathcal{N}_\gamma^p(\mu, \Sigma)$ coincides with the existent multivariate power exponential distribution $\mathcal{PE}^{\mathcal{P}}(\mu, \Sigma', \beta)$, as introduced in [11], where $\Sigma' = 2^{2(\gamma-1)/\gamma}\Sigma$ and $\beta = \frac{\gamma}{2(\gamma-1)}$. See also [12, 23, 24]. These existent generalizations are technically obtained (involving an extra power parameter β) and not as a theoretical result of a strong mathematical background as the LSI offer.

Recall now the multivariate and elliptically contoured uniform $\mathcal{U}^p(\mu, \Sigma)$ and Laplace $\mathcal{L}^p(\mu, \Sigma)$ distributions, as well as the degenerate Dirac distribution $\mathcal{D}^p(\mu)$ with p.d.f. $f_\mathcal{U}$, $f_\mathcal{L}$, $f_\mathcal{D}$ as follows:

$$f_\mathcal{U}(x) = \frac{\Gamma(\frac{p}{2}+1)}{(\pi^p \det \Sigma)^{1/2}}, \quad x \in \mathbb{R}^p \text{ with } Q_\theta(x) \le 1, \tag{29}$$

$$f_\mathcal{L}(x) = \frac{\Gamma(\frac{p}{2}+1)}{p!(\pi^p \det \Sigma)^{1/2}} \exp\left\{-Q_\theta^{1/2}(x)\right\}, \quad x \in \mathbb{R}^p, \tag{30}$$

$$f_\mathcal{D}(x) = \begin{cases} +\infty, & x = \mu, \\ 0, & x \in \mathbb{R}^p \setminus \mu. \end{cases} \tag{31}$$

The following theorem states that the above distributions, as well as the multivariate normal with p.d.f. $f_\mathcal{N}$ as in (19), are members of the γ-GND family for certain values of the shape parameter γ. Thus, the order γ, eventually, "bridges" distributions with complete different shape as well as "tailing" behaviour.

Theorem 6 *The multivariate γ-GND r.v. X_γ, i.e. $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ with p.d.f. f_{X_γ}, coincides for different values of the shape parameter γ with the uniform, normal,*

Laplace and Dirac distributions, as

$$f_{X_\gamma} = \begin{cases} f_{\mathcal{D}}, & \textit{for } \gamma = 0 \textit{ and } p = 1, 2, \\ 0, & \textit{for } \gamma = 0 \textit{ and } p \geq 3, \\ f_{\mathcal{U}}, & \textit{for } \gamma = 1, \\ f_{\mathcal{N}}, & \textit{for } \gamma = 2, \\ f_{\mathcal{L}}, & \textit{for } \gamma = \pm\infty. \end{cases} \tag{32}$$

Proof From the p.d.f. definition (27) of $\mathcal{N}_\gamma^p(\mu, \Sigma)$, parameter γ is defined over $\mathbb{R} \setminus [0, 1]$, i.e. γ is a real number outside the interval $[0, 1]$. Denote $g = \frac{\gamma-1}{\gamma}$ and E_θ the p–ellipsoid $Q_\theta(x) = 1$, $x \in \mathbb{R}^p$. The following cases are distinguished:

i. The uniform case $\gamma = 1$. From (27) with $x \in \mathbb{R}^p$ inside the p-ellipsoid E_θ, i.e. $Q_\theta(x) \leq 1$, it holds that

$$\begin{aligned} \lim_{\gamma \to 1^+} f_\gamma(x;\ \mu, \Sigma) &= \frac{\Gamma(\frac{p}{2}+1)}{\pi^{p/2}\sqrt{|\det \Sigma|}} \left(\lim_{g \to 0^+} g^g \right) \left(\lim_{g \to 0^+} \exp\left\{ -g Q_\theta(x)^{-1/(2g)} \right\} \right) \\ &= \frac{\Gamma(\frac{p}{2}+1)}{\pi^{p/2}\sqrt{|\det \Sigma|}} \cdot 1 \cdot e^0, \end{aligned}$$

while, for $x \in \mathbb{R}^p$ outside E_θ, i.e. $Q(x) > 1$, it is

$$\begin{aligned} \lim_{\gamma \to 1^+} f_\gamma(x;\ \mu, \Sigma) &= \frac{\Gamma(\frac{p}{2}+1)}{\pi^{p/2}\sqrt{|\det \Sigma|}} \left(\lim_{g \to 0^+} g^g \right) \left(\lim_{g \to 0^+} \exp\left\{ -g Q_\theta(x)^{1/(2g)} \right\} \right) \\ &= \frac{\Gamma(\frac{p}{2}+1)}{\pi^{p/2}\sqrt{|\det \Sigma|}} \cdot 1 \cdot 0, \end{aligned}$$

due to the fact that $g x^{1/g} \to +\infty$ as $g \to 0^+$ for all $x \in \mathbb{R}_+^* = \mathbb{R}_+^* \setminus 0$. Therefore, from (29), the first branch of (32) holds true as $f_{X_1} := \lim_{\gamma \to 1^+} f_{X_\gamma} = f_{\mathcal{U}}$, or $\mathcal{N}_1^p(\mu, \Sigma) := \lim_{\gamma \to 1^+} \mathcal{N}_\gamma^p(\mu, \Sigma) = \mathcal{U}^p(\mu, \Sigma)$. That is, the multivariate first-ordered normal distribution coincides with the elliptically contoured uniform distribution.

ii. The Gaussian case $\gamma = 2$. It is clear that $\mathcal{N}_2^p(\mu, \Sigma) = \mathcal{N}^p(\mu, \Sigma)$, as f_{X_2} coincides with the multivariate (and elliptically contoured) Gaussian density function $f_{\mathcal{N}}$ as in (19). That is, the multivariate second-ordered normal distribution coincides with the usual elliptically contoured normal distribution.

iii. The Laplace case $\gamma = \pm\infty$. For the limiting $g = \frac{\gamma-1}{\gamma} = 1$ (as $\gamma \to \pm\infty$), it holds that that $\mathcal{N}_{\pm\infty}^p(\mu, \Sigma) := \lim_{\gamma \to \pm\infty} \mathcal{N}_\gamma^p(\mu, \Sigma) = \mathcal{L}^p(\mu, \Sigma)$ as $f_{X_{\pm\infty}} := \lim_{\gamma \to \pm\infty} f_{X_\gamma}$ coincides with the multivariate (and elliptically contoured) Laplace density $f_{\mathcal{L}}$ as in (30). That is, the multivariate infinite-ordered normal distribution coincides with the elliptically contoured Laplace distribution.

iv. The degenerate Dirac case $\gamma = 0$. First, it is assumed that $x = \mu$, i.e. $Q_\theta(x) = 0$, and hence, from definition (27),

$$f_{X_\gamma}(\mu) = \pi^{-p/2} |\det \Sigma|^{-1/2} \Gamma\left(\tfrac{p}{2} + 1\right) \frac{g^{pg}}{\Gamma(pg+1)}. \tag{33}$$

From the fact that

$$f_{X_0}(\mu) := \lim_{\gamma \to 0^-} f_{X_\gamma}(\mu) = \lim_{g = \frac{\gamma - 1}{\gamma} \to +\infty} f_{X_\gamma}(\mu) = \lim_{k = [pg] \to \infty} f_{X_\gamma}(\mu),$$

where $[x]$ being the integer value of $x \in \mathbb{R}$, it is

$$f_{X_0}(\mu) = \frac{\Gamma(\frac{p}{2} + 1)}{\pi^{p/2} \sqrt{|\det \Sigma|}} \left(\lim_{k \to \infty} \frac{k^k}{p^k k!} \right). \tag{34}$$

Utilizing now the Stirling's asymptotic formula $k! \approx \sqrt{2\pi k} (\frac{k}{e})^k$ as $k \to \infty$, (34) implies

$$f_{X_0}(\mu) = \frac{\Gamma(\frac{p}{2} + 1)}{\pi^{p/2} \sqrt{|\det \Sigma|}} \left[\lim_{k \to \infty} \frac{1}{\sqrt{2\pi k} (\frac{p}{e})^k} \right], \tag{35}$$

and thus, for $p \geq 3 > e$, (35) implies $f_{X_0}(\mu) = 0$ while, for $p = 1$ or $p = 2$ implies $f_{X_0}(\mu) = +\infty$.
Assuming now $x \neq \mu$ and using (34), it holds that

$$f_{X_0}(x) = \lim_{\gamma \to 0^-} f_{X_\gamma}(\mu) \left[\lim_{g \to +\infty} \exp\left\{ -g Q(x)^{1/(2g)} \right\} \right], \tag{36}$$

hence, for $p \geq 3 > e$, (36) implies $f_{X_0}(x) = 0$ (due to $g x^{1/g} \to 0$ as $g \to +\infty$ for all $x \in \mathbb{R}^*_+$) while, for $p = 1$ or $p = 2$, applying (35) into (36), it is obtained that

$$f_{X_0}(x) = \frac{\Gamma(\frac{p}{2} + 1)}{\pi^{p/2} \sqrt{|\det \Sigma|}} \left[\lim_{k \to \infty} \frac{\exp\left\{ 1 - \frac{1}{p} Q_\theta(x)^{p/(2k)} \right\}}{p^k \sqrt{2\pi k}} \right] = 0.$$

Therefore, for $p = 1, 2$, it is clear that $\mathcal{N}_0^p(\mu, \Sigma) := \lim_{\gamma \to 0^-} \mathcal{N}_\gamma^p(\mu, \Sigma) = \mathcal{D}^p(\mu)$ as f_{X_0} coincides with the multivariate Dirac density function $f_\mathcal{D}$ as in (31), i.e. the univariate and bivariate zero-ordered normals are in fact the (univariate and bivariate) degenerate Dirac distributions, while the p-variate, $p \geq 3$, zero-ordered normals are the degenerate vanishing distributions.

Considering the above cases of (i), (iii) and (iv), the defining values of parameter γ of $\mathcal{N}_\gamma$ distributions can be safely extended to include the limiting values of $\gamma = 0, 1, \pm\infty$, respectively, i.e. γ can now be defined outside the real open interval $(0, 1)$. Eventually, the uniform, normal, Laplace and also the degenerate distributions as the

Dirac or the vanishing ones can be considered as members of the γ-GND family of distributions. $\square$

Notice that $\mathcal{N}_1^1(\mu,\sigma)$ coincides with the known (continuous) uniform distribution $\mathcal{U}(\mu-\sigma,\mu+\sigma)$. Specifically, for every uniform distribution expressed with the usual notation $\mathcal{U}(a,b)$, it holds that $\mathcal{U}(a,b)=\mathcal{N}_1^1(\frac{a+b}{2},\frac{b-a}{2})=\mathcal{U}^1(\mu,\sigma)$. Also $\mathcal{N}_2(\mu,\sigma^2)=\mathcal{N}(\mu,\sigma^2)$, $\mathcal{N}_{\pm\infty}(\mu,\sigma^2)=\mathcal{L}(\mu,\sigma)$ and finally $\mathcal{N}_0(\mu,\sigma)=\mathcal{D}(\mu)$. Therefore, the following holds.

Corollary 1 *The univariate γ-ordered normal distributions $\mathcal{N}_\gamma^1(\mu,\sigma^2)$ for order values $\gamma=0,1,2,\pm\infty$ coincides with the usual (univariate) Dirac $\mathcal{D}(\mu)$, uniform $\mathcal{U}(\mu-\sigma,\mu+\sigma)$, normal $\mathcal{N}(\mu,\sigma^2)$ and Laplace $\mathcal{L}(\mu,\sigma)$ distributions, respectively.*

Notice that for the r.v. X from the p-variate normal and A a given $p\times p$ matrix, it holds

$$X\sim\mathcal{N}^p(\mu,\Sigma)\Rightarrow \mathrm{A}X\sim\mathcal{N}^p(\mathrm{A}\mu,\mathrm{A}\Sigma\mathrm{A}^{\mathrm{T}}). \tag{37}$$

The linear relation described in (37) for the multivariate normal is valid for the γ-GND, in the sense that for given A an appropriate matrix and b an appropriate vector, then

$$X\sim\mathcal{N}_\gamma^p(\mu,\Sigma)\Rightarrow \mathrm{A}X+b\sim\mathcal{N}_\gamma^p(\mathrm{A}\mu+b,\mathrm{A}\Sigma\mathrm{A}^{\mathrm{T}}). \tag{38}$$

Simple calculation also proves that if the matrix A is reduced to an appropriate vector, relation (38) is still valid.

For the multivariate normally distributed $X\sim\mathcal{N}^p(\mu,\Sigma)$, it is clear, from (19), that the maximum density value $\max f_X=f_X(\mu)=(2\pi)^{-p/2}|\det\Sigma|^{-1/2}$ decreases as dimension $p\in\mathbb{N}$ rises, providing “flattened” probability densities. This is also true for the multivariate Laplace distributed $X\sim\mathcal{L}^p(\mu,\Sigma)=\mathcal{N}_{\pm\infty}^p(\mu,\Sigma)$. In fact, from (30), we have that $\max f_X=\pi^{-p/2}\frac{1}{p!}\Gamma(\frac{p}{2}+1)|\det\Sigma|^{-1/2}$ and therefore, the high-dimensional Laplace distribution densities are “flattened”, since the maximum density values decreases as $p\in\mathbb{N}$ increases. This is true because, for dimensions $2p$, with $\max f_X=C_{\pm\infty}^p(\Sigma)=\pi^{-p/2}\frac{1}{(p+1)(p+2)\ldots 2p}|\det\Sigma|^{-1/2}$. Hence, as in the normal distribution case, X provides, in principle, heavy tails as the dimension increases. However, this is not the case for the multivariate (and elliptically contoured) uniform distributed $X\sim\mathcal{U}^p(\mu,\Sigma)=\mathcal{N}_1^p(\mu,\Sigma)$, because the volume of the corresponding p-elliptical-cylinder shape of their density functions, as in (29), must always equal 1, although $\mathcal{U}^p$ have no tails to “absorb” probability mass when dimension increases, as the normal or the Laplace distributions does. Considering the above remark, the following proposition shows that, among all elliptical multivariate uniform distributions $\mathcal{U}^p(\mu,\Sigma)$ with fixed scale matrix Σ, the $\mathcal{U}^5(\mu,\Sigma)$ has the minimum $\max f_X$, see [21].

Theorem 7 *For the elliptically contoured uniformly distributed $X\sim\mathcal{U}^p(\mu,\Sigma)$, we have*

$$\min_{p\in\mathbb{N}}\{\max f_X\}=\tfrac{15}{6\pi^2}|\det\Sigma|^{-1}=\max\mathcal{U}^5(\mu,\Sigma),$$

i.e. the 5-dimensional uniform distribution provides the least of all maximum density values among all $\mathcal{U}^p(\mu,\Sigma)$ with fixed scale matrix Σ.

The γ-GND $\mathcal{N}_\gamma^p(\mu, \Sigma)$ is, in general, an elliptically contoured distribution, and therefore every $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ admits a stochastic representation $X_\gamma = \mu + \sqrt{V}\Sigma^{-1/2}U$ where U is uniformly distributed r.v. on the unit sphere of $\mathbb{R}^p$ and V and U are independent.

Proposition 3 *For the random variable $X_\gamma = \mu + \sqrt{V_\gamma}\Sigma^{-1/2}U \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$, the $2t$–th moments of V_γ are given by*

$$\mathrm{E}(V_\gamma^{2t}) = \frac{\Gamma\Big((p+2t)\frac{\gamma-1}{\gamma}\Big)}{\Gamma\Big(p\frac{\gamma-1}{\gamma}\Big)}\left(\tfrac{\gamma}{\gamma-1}\right)^{2t\frac{\gamma-1}{\gamma}}. \tag{39}$$

Using Theorem 2.8 in [8], the product moments of X are obtained, i.e.

$$\begin{aligned}
\mathrm{E}\left(X_1^{2t_1}\cdots X_\mathrm{p}^{2t_p}\right) &= \frac{\mathrm{E}(V_\gamma^{2t})}{\pi^{p/2}}\frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{p}{2}+t)}\prod_{k=1}^{p}\Gamma\left(\tfrac{1}{2}+t_k\right)\\
&= \pi^{-p/2}\left(\frac{\gamma-1}{\gamma}\right)^{2t\frac{\gamma-1}{\gamma}}\frac{\Gamma\left((p+2t)\frac{\gamma-1}{\gamma}\right)\Gamma(\frac{p}{2})}{\Gamma(p\frac{\gamma-1}{\gamma})\Gamma(\frac{p}{2}+t)}\prod_{k=1}^{p}\Gamma(\tfrac{1}{2}+t_k),
\end{aligned}$$

where $t_i \geq 1$, $i = 1, \ldots, p$ are integers and $t_1 + t_2 + \cdots + t_p = t$.

Consequently, the expected value and the covariance of $X_\gamma = \sqrt{V_\gamma}\Sigma^{-1/2}U$ are respectively $\mathrm{E}(X_\gamma) = \mu$ for every order values $\gamma \in \mathbb{R} \setminus [0, 1]$, and

$$\mathrm{Cov}(X_\gamma) = \frac{\Gamma\left((p+2)\frac{\gamma-1}{\gamma}\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma}\right)}\left(\tfrac{\gamma}{\gamma-1}\right)^{2\frac{\gamma-1}{\gamma}}(\mathrm{rank}\,\Sigma)^{-1}\Sigma. \tag{40}$$

Theorem 8 *An explicit analytic form of the characteristic function φ_{X_γ} of $X_\gamma \sim \mathcal{N}_\gamma^p(0, \mathbb{I}_p)$ is given by*

$$\varphi_{X_\gamma}(t) = e^{-it^{\mathrm{T}}\mu}\frac{\frac{\gamma}{2(\gamma-1)}\Gamma(p/2)}{\Gamma(p\frac{\gamma-1}{\gamma})}\sum_{k=0}^{\infty}(-1)^k\left(\tfrac{\gamma-1}{\gamma}\right)^{k+p\frac{\gamma-1}{\gamma}}q_k\,\|t\|^{\frac{k\gamma}{\gamma-1}-p}, \tag{41}$$

where

$$q_k = \frac{2^{p+k\frac{\gamma}{\gamma-1}}}{\pi k!}\Gamma\left(k\tfrac{\gamma}{2(\gamma-1)}+\frac{p}{2}\right)\Gamma\left(\tfrac{k\gamma}{2(\gamma-1)}+1\right)\sin\left(\pi\left(1+\tfrac{k}{2}\tfrac{\gamma}{\gamma-1}\right)\right).$$

The series in (41) is absolutely convergent for any $t \in \mathbb{R}^p \setminus 0$, see [21] for details. Recall now the cumulative distribution function (c.d.f.) $\Phi_Z(z)$ of the standardized normally distributed $Z \sim \mathcal{N}(0, 1)$, i.e.

$$\Phi_Z(z) = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{erf}\left(\tfrac{z}{2}\right), \quad z \in \mathbb{R}, \tag{42}$$

with erf(·) being the usual error function. For the γ-GND the generalized error function, [13], $\mathrm{Erf}_{\gamma/(\gamma-1)}$ is involved. Indeed, the following holds.

Theorem 9 *Let X be a random variable from the univariate γ-GND, i.e. $X \sim \mathcal{N}_\gamma^p(\mu, \sigma^2)$ with p.d.f. f_γ. If F_X is the c.d.f. of X and Φ_Z the c.d.f. of the standardized $Z = \frac{1}{\sigma}(X-\mu) \sim \mathcal{N}_\gamma(0,1)$, then*

$$F_X(x) = \Phi_Z(\tfrac{x-\mu}{\sigma}) = \tfrac{1}{2} + \frac{\sqrt{\pi}}{2\Gamma(\frac{\gamma-1}{\gamma})\Gamma(\frac{\gamma}{\gamma-1})}\mathrm{Erf}_{\frac{\gamma}{\gamma-1}}\left\{\left(\tfrac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}\tfrac{x-\mu}{\sigma}\right\}, \quad x \in \mathbb{R}. \tag{43}$$

Proof From the definition of the c.d.f. of X it is

$$F_X(x) = \int_0^x f_X(t)dt = C_\gamma^1(\sigma)\int_{-\infty}^x \exp\left\{-\tfrac{\gamma-1}{\gamma}\left|\tfrac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}dt.$$

Applying the linear transformation $w = \frac{t-\mu}{\sigma}$, the above is reduced to

$$F_X(x) = C_\gamma^1(1)\int_{-\infty}^{\frac{x-\mu}{\sigma}} \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw = \Phi_Z(\tfrac{x-\mu}{\sigma}), \tag{44}$$

where Φ_Z is the c.d.f. of the standardized γ-GND with $Z = \frac{1}{\sigma}(X-\mu) \sim \mathcal{N}_\gamma(0,1)$. Moreover, Φ_Z can be expressed in terms of the generalized error function. In particular,

$$\Phi_Z(z) = C_\gamma^1(1)\int_{-\infty}^z \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw = \Phi_Z(0) + C_\gamma^1(1)\int_0^z \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw,$$

and as f_Z is a symmetric density function around zero, we have

$$\Phi_Z(z) = \tfrac{1}{2} + C_\gamma^1(1)\int_0^z \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw = \tfrac{1}{2} + C_\gamma^1(1)\int_0^z \exp\left\{-\left|\left(\tfrac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} w\right|^{\frac{\gamma}{\gamma-1}}\right\}dw,$$

and thus

$$\Phi_Z(z) = \tfrac{1}{2} + C_\gamma^1(1)\left(\tfrac{\gamma}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma}}\int_0^{(\frac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}}z} \exp\left\{-u^{\frac{\gamma}{\gamma-1}}\right\}du. \tag{45}$$

Substituting the normalizing factor, as in (28), it is

$$\Phi_Z(z) = \tfrac{1}{2} + \frac{\sqrt{\pi}}{2\Gamma(\frac{\gamma-1}{\gamma}+1)\Gamma(\frac{2\gamma-1}{\gamma-1})}\mathrm{Erf}_{\frac{\gamma}{\gamma-1}}\left\{\left(\tfrac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} z\right\}, \quad z \in \mathbb{R}, \tag{46}$$

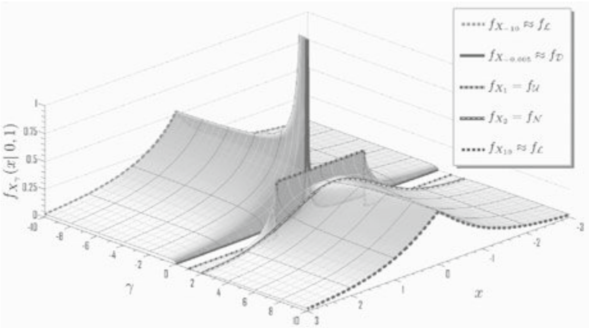

Fig. 1 Graph of all density functions $f_{X_\gamma}(x)$ with $X_\gamma \sim \mathcal{N}_\gamma(0,1)$ along x and γ

through the definition of the generalized error function, i.e. (43) holds. □

Figure 1 illustrates Corollary 1 in a compact form including the density functions $f_{X_\gamma}(x)$ for all $\gamma \in [-10, 0) \cup [1, 10]$, $X_\gamma \sim \mathcal{N}_\gamma(0,1)$ with $x \in [-3, 3]$.

The known densities of uniform ($\gamma = 1$) and normal ($\gamma = 2$) distributions are also depicted. Moreover, the densities of $\mathcal{N}_{\gamma=\pm 10}$ (0, 1) which approximate the density of Laplace distribution $\mathcal{L}(0,1) = \mathcal{N}_{\pm\infty}(0,1)$ as well as the density of $\mathcal{N}_{-0.005}(0,1)$ which approximates the degenerate Dirac distribution $\mathcal{D}(0)$ are clearly presented. Notice also the smooth-bringing between these significant distributions included into the family of the γ-order normals, as shown in Theorem 6.

5 Generalized Entropy Type Fisher's Information Measure

Let X be a multivariate r.v. with parameter vector $\theta = (\theta_1, \theta_2, \ldots, \theta_p) \in \mathbb{R}^p$ and p.d.f. f_θ on $\mathbb{R}^p$. The parametric type Fisher's information matrix $\mathrm{I}_F(X;\theta)$ (also denoted as $\mathrm{I}_\theta(X)$) defined as the covariance of $\nabla_\theta \log f_\theta(X)$ (where ∇_θ is the gradient with respect to the parameters θ_i, $i = 1, 2, \ldots, p$) is a parametric type information measure, expressed also as

$$\begin{aligned}\mathrm{I}_\theta(X) &= \operatorname{Cov}(\nabla_\theta \log f_\theta(X)) = \mathrm{E}_\theta\left(\nabla_\theta \log f_\theta(X) \cdot \nabla_\theta \log f_\theta(X)^{\mathrm{T}}\right) \\ &= \mathrm{E}_\theta\left(\|\nabla_\theta \log f_\theta(X)\|^2\right).\end{aligned}$$

On the other hand the Fisher's entropy type information measure $\mathrm{J}(X)$ of a r.v. X with p.d.f. f on $\mathbb{R}^p$ is defined, as $\mathrm{J}(X) = \mathrm{E}(\|\nabla \log f(X)\|^2)$. Moreover, $\mathrm{J}(X)$ can be written as

$$\begin{aligned}\mathrm{J}(X) &= \int_{\mathbb{R}^p} f(x)\|\nabla \log f(x)\|^2 dx = \int_{\mathbb{R}^p} f(x)^{-1}\|\nabla f(x)\|^2 dx \\ &= \int_{\mathbb{R}^p} \nabla f(x) \cdot \nabla \log f(x) dx = 4\int_{\mathbb{R}^p} \|\nabla \sqrt{f(x)}\|^2 dx.\end{aligned}$$

The generalized Fisher's entropy type information measure, or δ-GFI, is an exponential power generalization of $\mathrm{J}(X)$, defined as

$$\mathrm{J}_\delta(X) = \mathrm{E}\left(\|\nabla \log f(X)\|^\delta\right), \quad \delta \geq 1, \text{ see also [30]} \tag{47}$$

The 2-GFI is reduced to the usual J, i.e. $\mathrm{J}_2(X) = \mathrm{J}(X)$.

From the definition of the δ-GFI above, we can obtain

$$\begin{aligned}\mathrm{J}_\delta(X) &= \int_{\mathbb{R}^p} \|\nabla \log f(x)\|^\delta f(x)dx = \int_{\mathbb{R}^p} \|\nabla f(x)\|^\delta f^{1-\delta}(x)dx \\ &= \delta^\delta \int_{\mathbb{R}^p} \|\nabla f^{1/\delta}(x)\|^\delta dx, \text{see also [16, 17]}\end{aligned} \tag{48}$$

Recall that the Shannon entropy H of a r.v. X is defined as, [6] and [26],

$$\mathrm{H}(X) = \int_{\mathbb{R}^p} f(x) \log f(x)dx, \tag{49}$$

while the entropy power is defined

$$\mathrm{N}(X) = \nu e^{\frac{2}{p}\mathrm{H}(X)}, \tag{50}$$

with $\nu = (2\pi e)^{-1}$. The extension of the entropy power, the generalized entropy power (δ-GEP) is defined for $\delta \in \mathbb{R} \setminus [0, 1]$, as

$$\mathrm{N}_\delta(X) = \nu_\delta e^{\frac{\delta}{p}\mathrm{H}(X)}, \tag{51}$$

where

$$\nu_\delta = \left(\tfrac{\delta-1}{\delta e}\right)^{\delta-1} \pi^{-\delta/2} (\xi_\delta^p)^{\delta/p}, \quad \delta \in \mathbb{R} \setminus [0, 1], \tag{52}$$

with ξ_p^δ as in (26). In technical applications, such as signal I/O systems, the generalized entropy power can still be the power of the white Gaussian noise having the same entropy. Trivially, when $\delta = 2$, (51) is reduced to the existing entropy power $\mathrm{N}(X)$, i.e. $\mathrm{N}_2(X) = \mathrm{N}(X)$ as $\nu_2 = \nu$.

From the δ-GEP, a generalized version of the usual Shannon entropy can be produced, referred as the generalized δ-order Shannon entropy H_δ, i.e. $\mathrm{N}_\delta(X) = \nu \exp\{\frac{2}{p}\mathrm{H}_\delta(X)\}$. Therefore, from (51) a linear relation between the generalized Shannon entropy $\mathrm{H}_\delta(X)$ and the usual Shannon entropy $\mathrm{H}(X)$ is obtained, i.e.

$$\mathrm{H}_\delta(X) = \tfrac{p}{2} \log \tfrac{\nu_\delta}{\nu} + \tfrac{\delta}{2}\mathrm{H}(X). \tag{53}$$

Practically, (53) represents a linear transformation of $\mathrm{H}(X)$ which depends on the parameter δ and dimension the $p \in \mathbb{N}$. It is also clear that the second-ordered Shannon entropy is the usual Shannon entropy, i.e. $\mathrm{H}_2 = \mathrm{H}$.

The following result about the information inequality is essential.

Theorem 10 (*Information Inequality for the δ-GFI*) The information inequality still holds under δ-GFI and δ-GEP, i.e.

$$\mathrm{J}_\delta(X)\mathrm{N}_\delta(X) \geq p. \tag{54}$$

Proof For $u = f^{1/\delta}$, we have $\nabla g = \nabla f^{1/\delta} = \frac{1}{\delta} f^{\frac{1-\delta}{\delta}} \nabla f$ and therefore (24) gives

$$\frac{1}{\delta} \int\limits_{\mathbb{R}^p} f \log f dx \leq \frac{p}{\delta^2} \log \left\{ K_\delta \int\limits_{\mathbb{R}^p} \delta^{-\delta} f^{1-\delta} \|\nabla f\|^\delta dx \right\},$$

while applying (48), we have

$$\int\limits_{\mathbb{R}^p} f \log f dx \leq \log \left\{ K_\delta \delta^{-\delta} \mathrm{J}_\delta(X) \right\}^{p/\delta},$$

or

$$\exp\left\{ \frac{\delta}{p} \int\limits_{\mathbb{R}^p} f \log f dx \right\} \leq K_\delta \delta^{-\delta} \mathrm{J}_\delta(X),$$

while, through ν_δ as in (52),

$$\nu_\delta^{-1} \exp\left\{ \frac{\delta}{p} \int_{\mathbb{R}^p} f \log f dx \right\} \leq \nu_\delta^{-1} K_\delta \delta^{-\delta} \mathrm{J}_\delta(X).$$

Since $\nu_\delta^{-1} K_\delta \delta^{-\delta} = \frac{1}{p}$, (54) is eventually obtained. □

Moreover, the Cramér–Rao inequality can be extended, [15], as

$$\left[\frac{2\pi e}{p} \mathrm{Var}(X)\right]^{1/2} \left[\frac{\nu_\delta}{p} \mathrm{J}_\delta(X)\right]^{1/\delta} \geq 1. \tag{55}$$

Under the normality parameter $\delta = 2$, (55) is reduced to the usual Cramér–Rao inequality form, [6]

$$\mathrm{J}(X)\ \mathrm{Var}(X) \geq p. \tag{56}$$

Furthermore, the classical entropy inequality

$$\mathrm{Var}(X) \geq p\mathrm{N}(X) = \frac{p}{2\pi e} e^{\frac{2}{p}\mathrm{H}(X)} \quad \text{or} \quad \mathrm{H}(X) \leq \frac{p}{2} \log\left\{ \frac{2\pi e}{p} \mathrm{Var}(X) \right\}, \tag{57}$$

can be extended into the form

$$\mathrm{Var}(X) \geq p(2\pi e)^{\frac{\delta-4}{\delta}} \nu_\delta^{2/\delta} \mathrm{N}_\delta^{2/\delta}(X) = p(2\pi e)^{\frac{\delta-2}{\delta}} \nu_\delta^{2/\delta} e^{\frac{4}{p\delta}\mathrm{H}_\delta(X)}, \tag{58}$$

through the generalized Shannon entropy H_δ defined earlier. Under the "normal" parameter value $\delta = 2$, the inequality (58) is reduced to the usual entropy inequality as in (57).

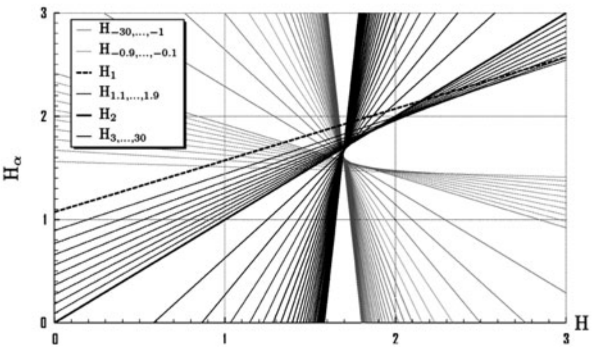

Fig. 2 Graphs of generalized entropy H_α with respect to H for various α parameter values

Figure 2 presents the linear expressions between the generalized H_δ and the usual Shannon entropy H. The area E described by the envelop region of the family of lines $\mathrm{H}_\delta = \mathrm{H}_\delta(\mathrm{H})$ as in (53), indicates no relation between H_δ and H as it lies asymptotically between the lines $\mathrm{H}_0(\mathrm{H})$ and $\mathrm{H}_1(\mathrm{H})$. This was expected in the sense that the parameter $\delta \in \mathbb{R}$ but $\delta \notin [0, 1]$, see (51).

The Blachman–Stam inequality, [3, 4, 29] is generalized through the δ-GFI. Indeed:

Theorem 11 (*Blachman–Stam inequality for the δ-GFI*) For given two p-variate and independent random variables X and Y, it holds

$$\mathrm{J}_\delta\left(\lambda^{1/\delta} X + (1-\lambda)^{1/\delta} Y\right) \leq \lambda \mathrm{J}_\delta(X) + (1-\lambda)\mathrm{J}_\delta(Y), \quad \lambda \in (0,1). \tag{59}$$

The equality holds when X and Y are normally distributed with the same covariance matrix.

Proof Let f_X, f_Y be the density function of X and Y, respectively. Then, if

$$f(x,y) := f_X\left(\lambda^{1/\delta} x - (1-\lambda)^{1/\delta} y\right) f_Y\left((1-\delta)^{1/\delta} y + \lambda^{1/\delta} x\right), \quad \lambda \in (0,1),$$

on $\mathbb{R}^p$, the marginal $\int F(x,y) d^p x$ is the density of $\lambda^{1/\delta} X + (1-\lambda)^{1/\delta} Y$. Also,

$$\int\limits_{\mathbb{R}^p}\int\limits_{\mathbb{R}^p} \left\| \nabla_y f^{1/\delta}(x,y) \right\|^\delta dx dy = \lambda \mathrm{J}_\delta(X) + (1-\lambda)\mathrm{J}_\delta(\mathrm{Y}).$$

Recall now the Minkowski-type inequality of Theorem 2 in [4], i.e.

$$\int\limits_{\mathbb{R}^n} \left\| \nabla_y \left(\int_{\mathbb{R}^m} \| f(x,y) \|^\delta d^m x \right)^{1/\delta} \right\|^\delta d^n y \leq \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^m} \left\| \nabla_y f(x,y) \right\|^\delta d^m x d^n y, \tag{60}$$

for any function f in $\mathcal{L}^p(\mathbb{R}^m \times \mathbb{R}^n, d^m x d^n y) \otimes \mathcal{W}^{1,p}(\mathbb{R}^n)$ where ∇_y denotes the partial distributional gradient for y variables, and whenever there is equality it holds

$\|f(x,y)\| = \|f_1(x)\| \cdot \|f_2(y)\|$. Thus, (60) asserts

$$\int_{\mathbb{R}^p} \left\| \nabla_y \left(\int_{\mathbb{R}^p} \left\| f^{1/\delta}(x,y) \right\|^\delta dx \right)^{1/\delta} \right\|^\delta dy \le \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} \left\| \nabla_y f^{1/\delta}(x,y) \right\|^\delta dxdy,$$

and hence, (59) holds true with equality implying that F is a product of densities in x and y, that occurs only when X and Y are normally distributed with the same covariance (see Theorem 1 in [4]). □

Recall now that corresponding to any orthogonal decomposition $\mathbb{R}^p = \mathbb{R}^r \oplus \mathbb{R}^s$, $p = s + t$, the marginal densities are given by

$$f_1(x) = \int_{\mathbb{R}^s} f(x,y) d^s y, \quad f_2(y) = \int_{\mathbb{R}^t} f(x,y) d^t x. \tag{61}$$

with f being a probability density on $\mathbb{R}^p$. Then, as far as the superadditivity of the δ-GFI is concerned, the following theorem is stated and proved which is a direct analog of the well-known theorem asserting strict subadditivity of the entropy.

Theorem 12 (*Strict Superadditivity for the δ-GFI*) With f, f_1 and f_2 defined and related as above,

$$\mathrm{J}_\delta(f) \ge \mathrm{J}_\delta(f_1) + \mathrm{J}_\delta(f_2), \tag{62}$$

with equality holds when $f(x,y) = f_1(x) f_2(y)$ almost everywhere.

Proof Let $g(x,y) = f^{1/\delta}(x,y)$. Then, from the definition of the δ-GFI (48),

$$\mathrm{J}_\delta(f) = \delta^\delta \int_{\mathbb{R}^p} \|\nabla_x g(x,y)\|^\delta d^s x d^t y + \delta^\delta \int_{\mathbb{R}^p} \left\| \nabla_y g(x,y) \right\|^\delta d^t y d^s x. \tag{63}$$

The inequality (60) gives

$$\int_{\mathbb{R}^p} \|\nabla_x g(x,y)\|^\delta d^s x d^t y \ge \int_{\mathbb{R}^s} \left\| \nabla \left(\int_{\mathbb{R}^t} g^\delta(x,y) d^t y \right)^{1/\delta} \right\|^\delta d^s x,$$

and

$$\int_{\mathbb{R}^p} \|\nabla_y g(x,y)\|^\delta d^s x d^t y \ge \int_{\mathbb{R}^s} \left\| \nabla \left(\int_{\mathbb{R}^t} g^\delta(x,y) d^t y \right)^{1/\delta} \right\|^\delta d^s x,$$

and hence, (63) becomes

$$\mathrm{J}_\delta(f) \ge \delta^\delta \int_{\mathbb{R}^s} \left\| \nabla_x \left(\int_{\mathbb{R}^t} g^\delta(x,y) d^t y \right)^{1/\delta} \right\|^\delta d^s x + \delta^\delta \int_{\mathbb{R}^t} \left\| \nabla_y \left(\int_{\mathbb{R}^s} g^\delta(x,y) d^s x \right)^{1/\delta} \right\|^\delta d^t y$$
$$= \mathrm{J}_\delta(f_1) + \mathrm{J}_\delta(f_2).$$

By the conditions for equality in the previous inequality where (60) were applied, we must have $f(x,y) = f_1(x) f_2(y)$, as f, f_1, f_2 are positive. □

For the "normal" parameter $\delta = 2$, we are reduced to the known supperadditivity of Fisher's information measure, see [4].

6 Entropy and Information Measures for the γ-GND

For the Shannon entropy of an r.v. $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$ it holds,

$$\mathrm{H}(X) = p\tfrac{\gamma-1}{\gamma} - \tfrac{1}{2}\log C_\gamma^p(\Sigma), \tag{64}$$

see [16, 15]) while for the δ-GEP of the γ-GND we have, through (50) and (64), the following.

Theorem 13 *Let X_γ an elliptically contoured γ-GND r.v. $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$. It holds*

$$\mathrm{N}_\delta(X_\gamma) = \left(\tfrac{\delta-1}{e\delta}\right)^{\delta-1}\left(\tfrac{e\gamma}{\gamma-1}\right)^{\delta\frac{\gamma-1}{\gamma}} \xi_{\delta,\gamma}^p |\det\Sigma|^{\frac{\delta}{2p}}, \tag{65}$$

where

$$\xi_{\delta,\gamma}^p = \frac{\xi_\delta^p}{\xi_\gamma^p} = \frac{\Gamma\left(p\frac{\gamma-1}{\gamma}+1\right)}{\Gamma\left(p\frac{\delta-1}{\delta}+1\right)}. \tag{66}$$

Example 3 For the usual entropy power of the γ-GND, i.e. for the second-GEP of the r.v. $X_\gamma \mathcal{N}_\gamma(\mu, \Sigma)$, it holds

$$\mathrm{N}(X_\gamma) = \tfrac{1}{2e}\left(\tfrac{e\gamma}{\gamma-1}\right)^{2\frac{\gamma-1}{\gamma}} (\xi_\gamma^p)^{2/p}|\det\Sigma|^{1/p}. \tag{67}$$

Theorem 14 *The Shannon entropy for the multivariate and elliptically countered uniform, normal and Laplace distributed X (for $\gamma = 1, 2, \pm\infty$, respectively) is given by*

$$\mathrm{H}(X) = \begin{cases} \log\frac{\pi^{p/2}\sqrt{|\det\Sigma|}}{\Gamma(\frac{p}{2}+1)}, & X \sim \mathcal{N}_1^p(\mu,\Sigma) \ = \mathcal{U}^p(\mu,\Sigma), \\ \log\sqrt{(2\pi e)^p|\det\Sigma|}, & X \sim \mathcal{N}_2^p(\mu,\Sigma) \ = \mathcal{N}^p(\mu,\Sigma), \\ \log\frac{p!e\pi^{p/2}\sqrt{|\det\Sigma|}}{\Gamma(\frac{p}{2}+1)}, & X \sim \mathcal{N}_{\pm\infty}^p(\mu,\Sigma) = \mathcal{L}^p(\mu,\Sigma), \end{cases} \tag{68}$$

while $\mathrm{H}(X)$ *is infinite when* $X \sim \mathcal{N}_0^p(\mu, \Sigma)$.

Proof Applying Theorem 6 into (64), we obtain (68). Consider now the limiting case of $\gamma = 0$. We can write (64) in the form

$$\mathrm{H}(X_\gamma) = \log\left\{\frac{\pi^{p/2}\sqrt{|\det\Sigma|}}{\Gamma(\frac{p}{2}+1)}\cdot\frac{\Gamma(pg+1)}{(\frac{g}{e})^{pg}}\right\},$$

where $g = \frac{\gamma-1}{\gamma}$. We then have,

$$\lim_{\gamma\to 0^-}\mathrm{H}(X_\gamma) = \log\left\{\frac{\pi^{p/2}\sqrt{|\det\Sigma|}}{\Gamma(\frac{p}{2}+1)}\lim_{k=p[g]\to\infty}\frac{p^k k!}{(\frac{k}{e})^k}\right\}, \tag{69}$$

and using the Stirling's asymptotic formula $k! \approx \sqrt{2\pi k}(\frac{k}{e})^k$ as $k \to \infty$, (69) finally implies

$$\lim_{\gamma\to 0^-} \mathrm{H}(\mathrm{X}_\gamma) = \log\left\{\sqrt{2\pi|\det\Sigma|}\frac{\pi^{\mathrm{p}/2}}{\Gamma(\frac{\mathrm{p}}{2}+1)}\lim_{\mathrm{k}\to\infty}\mathrm{p}^{\mathrm{k}}\sqrt{\mathrm{k}}\right\} = +\infty,$$

which proves the theorem. □

Example 4 For the univariate case $p = 1$, we are reduced to

$$\mathrm{H}(X) = \begin{cases} \log 2\sigma, & X \sim \mathcal{N}_1(\mu,\sigma) \quad = \mathcal{U}(\mu-\sigma,\mu+\sigma), \\ \log\sqrt{2\pi e}\sigma, & X \sim \mathcal{N}_2(\mu,\sigma^2) \quad = \mathcal{N}(\mu,\sigma^2), \\ 1+\log 2\sigma, & X \sim \mathcal{N}_{\pm\infty}(\mu,\sigma) = \mathcal{L}(\mu,\sigma). \end{cases}$$

Theorem 15 *The generalized Shannon entropy* H_δ *of the multivariate* $X_\gamma \sim \mathcal{N}_\gamma(\mu,\Sigma)$ *is given by*

$$\mathrm{H}_\delta(X_\gamma) = \tfrac{2\gamma-\delta}{2\gamma}p + \tfrac{p}{2}\log\left\{2\pi\left(\tfrac{\delta-1}{\delta}\right)^{\delta-1}\left(\tfrac{\gamma}{\gamma-1}\right)^{\delta\frac{\gamma-1}{\gamma}}\left[\frac{\Gamma(p\frac{\gamma-1}{\gamma}+1)}{\Gamma(p\frac{\delta-1}{\delta}+1)}\right]^{\frac{\delta}{p}}|\det\Sigma|^{\frac{\delta}{2p}}\right\}. \tag{70}$$

For $\delta = \gamma$, *it is* $\mathrm{H}_\gamma(X_\gamma) = \frac{1}{2}\log\{(2\pi e)^p|\det\Sigma|^{\gamma/2}\}$. *Moreover, for a random variable X following the multivariate uniform, normal and Laplace distributions* ($\gamma = 1, 2, \pm\infty$, *respectively), it is*

$$\mathrm{H}_\delta(\mathrm{X}) = \begin{cases} \frac{2-\delta}{2}p + h^p_{\gamma,a}, & X \sim \mathcal{N}^p_1(\mu,\Sigma), \\ p + \frac{\delta}{2}\log\left\{(2/e)^{p/2}\Gamma(\frac{p}{2}+1)\right\} + h^p_{\gamma,\delta}, & X \sim \mathcal{N}^p_2(\mu,\Sigma), \\ p + \frac{p}{2}\log p! + h^p_{\gamma,a}, & X \sim \mathcal{N}^p_{\pm\infty}(\mu,\Sigma), \end{cases} \tag{71}$$

where $h^p_{\gamma,\delta} = \frac{\delta}{2}\log\{(2\pi)^{p/\delta}(\frac{\delta-1}{\delta})^{p(\delta-1)/\delta}[\Gamma(p\frac{\delta-1}{\delta}+1)]^{-1}\sqrt{|\det\Sigma|}\}$. *For the limiting degenerate case of* $\gamma = 0$, *we obtain* $\mathrm{H}_\delta(\mathrm{X}_0) = (\operatorname{sgn}\delta)(+\infty)$, *for* $\delta \neq 0$ *while* $\mathrm{H}_0(X_0) = p\log\sqrt{2\pi}\mathrm{e}$.

Proof Substituting (52) and (64) into (53), we obtain

$$\mathrm{H}_\delta(X_\gamma) = \tfrac{p}{2}\log\left\{2\pi^{\frac{2-\delta}{2}}e^{\frac{2\gamma-\delta}{\gamma}}(\tfrac{\delta-1}{\delta})^{\delta-1}\right\} + \tfrac{\delta}{2}\log\left\{\pi^{p/2}\left(\tfrac{\gamma}{\gamma-1}\right)^{p\frac{\gamma-1}{\gamma}}\frac{\Gamma(p\frac{\gamma-1}{\gamma}+1)}{\Gamma(p\frac{\delta-1}{\delta}+1)}\sqrt{|\det\Sigma|}\right\},$$

and after some algebra we derive (70). In case of $\delta = \gamma$ we have $\mathrm{H}_\gamma(X_\gamma) = \frac{p}{2}\log\{2\pi e|\det\Sigma|^{\gamma/(2p)}\}$.

Recall Theorem 6. For the order values $\gamma = 1$, $\gamma = 2$ and $\gamma = \pm\infty$, the δ-Shannon entropies H_δ of the uniformly, normally and Laplace distributed $X_1 \sim \mathcal{U}^p(\mu,\Sigma)$, $X_2 \sim \mathcal{N}^p(\mu,\Sigma)$ and $X_{\pm\infty} \sim \mathcal{L}^p(\mu,\Sigma)$, respectively, are given by (71).

Consider now the limiting case of $\gamma = 0$. We can write (70) in the form

$$\mathrm{H}_\delta(X_\gamma) = \tfrac{p}{2}(2 - \delta + \gamma\alpha) + \tfrac{p}{2}\log\left\{2\pi(\tfrac{\alpha-1}{\alpha})^{\delta-1}\alpha^{-\alpha\delta}\left[\frac{\Gamma(p\alpha+1)\sqrt{|\det\Sigma|}}{\Gamma(p\frac{\delta-1}{\delta}+1)}\right]^{\frac{\delta}{p}}\right\}$$

$$= \log\left\{(2\pi)^{p/2}\left(\tfrac{\delta-1}{\delta}\right)^{p\frac{\delta-1}{2}}\left[\frac{\Gamma(p\alpha+1)}{(\frac{\alpha}{e})^{p\alpha}\Gamma(p\frac{\delta-1}{\delta}+1)}\right]^{\frac{\delta}{2}}|\det\Sigma|^{\delta}\right\},$$

where $\alpha = \frac{\gamma-1}{\gamma}$. We then have,

$$\lim_{\gamma\to0^-}\mathrm{H}_\delta(X_\gamma) = \log\left\{(2\pi)^{p/2}(\tfrac{\delta-1}{\delta})^{p\frac{\delta-1}{2}}\left[\lim_{k=p[\alpha]\to\infty}\frac{p^k k!}{(\frac{k}{e})^k}\right]^{\frac{\delta}{2}}|\det\Sigma|^{\delta}\right\}. \qquad (72)$$

Using the Stirling's asymptotic formula (similar as in Theorem 14), (72) finally implies

$$\lim_{\gamma\to0^-}\mathrm{H}_\delta(\mathrm{X}_\gamma) = \log\left\{(2\pi)^{\mathrm{p}/2}\left(\tfrac{\delta-1}{\delta}\right)^{\mathrm{p}\frac{\delta-1}{2}}|\det\Sigma|^{\delta}\left(\lim_{\mathrm{k}\to\infty}\mathrm{p}^{\mathrm{k}}\sqrt{\mathrm{k}}\right)^{\frac{\delta}{2}}\right\} = (\mathrm{sgn}\delta)(+\infty),$$

where sgnδ is the sign of parameter δ, which proves the theorem. □

Notice that despite the rather complicated form of the $\mathrm{H}_\delta(X_\gamma)$ with $\delta \neq \gamma$, the generalized Shannon entropy of a δ-order normally distributed X_δ has a very compact expression.

Recall now the known relation of the Shannon entropy of a normally distributed random variable $Z \sim \mathcal{N}(\mu, \Sigma)$, i.e. $\mathrm{H}(Z) = \frac{1}{2}\log\{(2\pi e)^p|\det\Sigma|\}$. Therefore, $\mathrm{H}_\gamma(X_\gamma)$ generalizes $\mathrm{H}(Z) = \mathrm{H}_2(X_2)$ preserving the simple formulation for every γ, as parameter γ affects only the scale matrix Σ (as a power).

Another interesting fact about $\mathrm{H}_\gamma(X_\gamma)$ is that, $\mathrm{H}_0(X_0) = \frac{p}{2}\log\{2\pi e\}$ or $\mathrm{H}_0(X_0) = -\frac{p}{4}\log\nu$. According to Theorem 14 the Shannon entropy diverges to $+\infty$ for the degenerated Dirac distribution $\mathcal{D}(\mu) = \mathcal{N}_0$. However, the 0-Shannon entropy H_0 (in limit) for a Dirac distributed r.v. converges to $\mathrm{H}_0(X_0) = \log\sqrt{2\pi e} = -\frac{1}{2}\log\nu \approx 1.4189$, which is the same value as the Shannon entropy of the standardized normally distributed $Z \sim \mathcal{N}(0, 1)$. Thus, the generalized Shannon entropy can "handle" the Dirac distribution in a more "coherent" way than the usual Shannon entropy (i.e. not diverging to infinity).

We can mention also that (70) expresses the generalized δ-Shannon entropy of the multivariate uniform, normal and Laplace distributions relative to each other. For example, the difference between these entropies of uniform and Laplace is independent of the same scale matrix Σ, i.e. $\mathrm{H}_\delta(X_{\pm\infty}) - \mathrm{H}_\delta(X_1) = p + \frac{p}{p}\log p! + \frac{\delta-2}{\delta}$ while for the usual Shannon entropy, $\mathrm{H}(X_{\pm\infty}) - \mathrm{H}(X_1) = p + \frac{p}{p}\log p!$, i.e. the Shannon entropies differ in a dimension-depending constant.

Theorem 16 *The generalized Fisher's information* J_δ *of an r.v.* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \lambda\Sigma^*)$ *where* $\lambda \in \mathbb{R}_+ \backslash 0$ *and* Σ^* *is a real orthogonal matrix with* $\det \Sigma = 1$*, i.e.* $\Sigma^* \in \mathbb{R}_\perp^{p\times p}$*, is given by*

$$\mathrm{J}_\delta(X_\gamma) = \left(\tfrac{\gamma}{\gamma-1}\right)^{\frac{\delta}{\gamma}} \frac{\Gamma\left(\frac{\delta+p(\gamma-1)}{\gamma}\right)}{\lambda^{\delta/2}\Gamma\left(p\frac{\gamma-1}{\gamma}\right)}. \tag{73}$$

Proof From (48), we have

$$\mathrm{J}_\delta(X_\gamma) = \delta^\delta \int_{\mathbb{R}^p} \left\| \nabla f_{X_\gamma}^{1/\delta}(x) \right\|^\delta dx,$$

while from the definition of the density function f_{X_γ}, in (27), we have

$$\begin{aligned}\mathrm{J}_\delta(X_\gamma) &= \delta^\delta C_\gamma^p \int_{\mathbb{R}^p} \left\| \nabla \exp\left\{-\tfrac{\gamma-1}{\delta\gamma} Q(x)^{\frac{\gamma}{2(\gamma-1)}}\right\}\right\|^\delta dx \\ &= \delta^\delta (\tfrac{\gamma-1}{\delta\gamma})^\delta C_\gamma^p \int_{\mathbb{R}^p} \exp\left\{-\tfrac{\gamma-1}{\gamma} Q^{\frac{\gamma}{2(\gamma-1)}}(x)\right\} \left\| \nabla Q^{\frac{\gamma}{2(\gamma-1)}}(x)\right\|^\delta dx. \end{aligned} \tag{74}$$

For the gradient of the quadratic form $Q(x)$, we have $\nabla Q(x) = \lambda^{-1}\nabla\{(x-\mu)\Sigma^{*-1}(x-\mu)^{\mathrm{T}}\} = 2\lambda^{-1}\Sigma^{*-1}(x-\mu)^{\mathrm{T}}$ while from the fact that Σ^* is an orthogonal matrix, we have $\|\Sigma^{*-1}(x-\mu)^{\mathrm{T}}\| = \|x-\mu\|$. Therefore, (74) can be written as

$$\mathrm{J}_\delta(X_\gamma) = \lambda^{-\delta} C_\gamma^p \int_{\mathbb{R}^p} \exp\left\{-\tfrac{\gamma-1}{\gamma} Q^{\frac{\gamma}{2(\gamma-1)}}(x)\right\} Q^{\frac{\delta\gamma}{2(\gamma-1)}-\delta}(x)\, \|x-\mu\|^\delta\, dx.$$

Applying the linear transformation $z = (x-\mu)(\lambda\Sigma)^{*-1/2}$ in J_δ above, it is $dx = d(x-\mu) = \sqrt{\lambda^p\, |\det \Sigma^*|} dz = \lambda^{p/2} dz$, the quadratic form Q is reduced to

$$Q(x) = (x-\mu)(\lambda\Sigma)^{*-1}(x-\mu)^{\mathrm{T}} = (x-\mu)(\lambda\Sigma^*)^{-1/2}[(x-\mu)(\lambda\Sigma^*)^{-1/2}]^{\mathrm{T}} = \|z\|^2,$$

and thus,

$$\mathrm{J}_\delta(X_\gamma) = \lambda^{(p-\delta)/2} C_\gamma^p \int_{\mathbb{R}^p} \|z\|^{\frac{\delta}{\gamma-1}} \exp\left\{-\tfrac{\gamma-1}{\gamma}\|z\|^{\frac{\gamma}{\gamma-1}}\right\} dz.$$

Switching to hyperspherical coordinates, we get

$$\mathrm{J}_\delta(X_\gamma) = \lambda^{(p-\delta)/2} C_\gamma^p \omega_{p-1} \int_0^{+\infty} \rho^{\frac{\delta}{\gamma-1}} \exp\left\{-\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}\right\} \rho^{p-1} d\rho,$$

where $\omega_{p-1} = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ is the volume of the $(p-1)$-sphere, $\mathbb{S}_{p-1}$, and hence

$$\mathrm{J}_\delta(X_\gamma) = 2\frac{\pi^{p/2}}{\Gamma(\frac{\pi}{2})} \lambda^{(p-\delta)/2} C_\gamma^p \int_0^{+\infty} \rho^{\frac{\delta+(p-1)(\gamma-1)}{\gamma-1}} \exp\left\{-\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}\right\} d\rho.$$

From the fact that $d(\frac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}) = \rho^{\frac{1}{\gamma-1}}d\rho$ and the definition of the gamma function, we obtain successively

$$\begin{aligned}
\mathrm{J}_\delta(X_\gamma) &= 2\frac{\pi^{p/2}}{\Gamma(\frac{\pi}{2})}\lambda^{(p-\delta)/2}C_\gamma^p\int_0^{+\infty}\rho^{\frac{\delta+(p-1)(\gamma-1)}{\gamma-1}-\frac{1}{\gamma-1}}\exp\left\{-\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}\right\}d(\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}})\\
&= 2\frac{\pi^{p/2}}{\Gamma(\frac{\pi}{2})}\lambda^{(p-\delta)/2}C_\gamma^p\int_0^{+\infty}\rho^{\frac{\delta+p\gamma-\gamma-p}{\gamma-1}}\exp\left\{-\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}\right\}d(\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}})\\
&= 2\frac{\pi^{p/2}}{\Gamma(\frac{\pi}{2})}\lambda^{(p-\delta)/2}(\tfrac{\gamma}{\gamma-1})^{\frac{\delta-\gamma+p(\gamma-1)}{\gamma}}C_\gamma^p\times\\
&\int_0^{+\infty}(\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}})^{\frac{\delta-\gamma+p(\gamma-1)}{\gamma}}\exp\left\{-\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}}\right\}d(\tfrac{\gamma-1}{\gamma}\rho^{\frac{\gamma}{\gamma-1}})\\
&= 2\frac{\pi^{p/2}}{\Gamma(\frac{\pi}{2})}\lambda^{(p-\delta)/2}(\tfrac{\gamma}{\gamma-1})^{\frac{\delta-\gamma+p(\gamma-1)}{\gamma}}C_\gamma^p\Gamma(\tfrac{\delta+p(\gamma-1)}{\gamma}),
\end{aligned}$$

and, finally, applying the normalizing factor C_γ^p as in (28), we derive (73) and the theorem has been proved. □

Corollary 2 *The generalized Fisher's information* J_δ *of a spherically contoured r.v.* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \sigma^2\mathbb{I}_p)$ *where* $\sigma \in \mathbb{R}_+ \setminus 0$*, is given by*

$$\mathrm{J}_\delta(X_\gamma) = (\tfrac{\gamma}{\gamma-1})^{\frac{\delta}{\gamma}}\frac{\Gamma\left(\frac{\delta+p(\gamma-1)}{\gamma}\right)}{\sigma^\delta\Gamma\left(p\frac{\gamma-1}{\gamma}\right)}. \tag{75}$$

In the following proposition, we provide some inequalities for the generalized Fisher's entropy type information measure J_δ for the family of the γ-GND distributed r.v. considering parameters $\alpha,\ \gamma > 1$. We denote $\Gamma_{min} \approx 1.4628$ the point of minimum for the positive gamma function, i.e. $\min_{x\in\mathbb{R}_+}\{\Gamma(x)\} = \Gamma(\Gamma_{min})$.

Proposition 4 *The generalized Fisher's information* J_δ *of an r.v.* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \lambda\Sigma^*)$ *where* $\lambda \in \mathbb{R}_+ \setminus 0$ *and* $\Sigma^* \in \mathbb{R}_\perp^{p\times p}$ *with order value* $\gamma > \gamma_p = \frac{p}{p+1-\Gamma_{min}} \approx \frac{2\,p}{2\,p-1}$*, satisfies the inequalities*

$$\mathrm{J}_\delta(X_\gamma)\begin{cases} > p\lambda^{-\delta/2}, & \text{for } \delta > \gamma,\\ = p\lambda^{-\delta/2}, & \text{for } \delta = \gamma,\\ < p\lambda^{-\delta/2}, & \text{for } g_p < \delta < \gamma,\end{cases} \tag{76}$$

where $g_p = \gamma(\Gamma_0 - p) + p \approx \frac{\gamma}{2}(3 - 2\,p) + p$.

Proof For the proof of the first branch of (76), it is assumed that $\delta > \gamma$, i.e. $\frac{\delta}{\gamma} > 1$. Then, it is $\frac{\delta+p(\gamma-1)}{\gamma} > 1 + p\frac{\gamma-1}{\gamma}$. This implies,

$$\Gamma(\tfrac{\delta+p(\gamma-1)}{\gamma}) > \Gamma(1 + p\tfrac{\gamma-1}{\gamma}) = p\tfrac{\gamma-1}{\gamma}\Gamma(p\tfrac{\gamma-1}{\gamma}), \tag{77}$$

if $1 + p\frac{\gamma-1}{\gamma} \geq \Gamma_{min}$. That is, if the inequality $x = 1 + p\frac{\gamma-1}{\gamma} \geq \Gamma_{min}$ holds, then $\Gamma(x) \geq \Gamma(\Gamma_{min})$, as the gamma function is an increasing function for $x \geq \Gamma_0$. Inequality, $1 + p\frac{\gamma-1}{\gamma} \geq \Gamma_{min}$, is equivalent to, $\gamma \geq \frac{p}{p+1-\Gamma_{min}} \approx \frac{p}{p-0.4628} > 1$. As a result, (77) holds indeed, for order values $\gamma \geq \frac{p}{p+1-\Gamma_{min}}$, and so,

$$\frac{\Gamma(\frac{\delta+p(\gamma-1)}{\gamma})}{\Gamma(p\frac{\gamma-1}{\gamma})} > p\frac{\gamma-1}{\gamma}. \tag{78}$$

Our assumption $\frac{\delta}{\gamma} > 1$, together with the fact that $\frac{\gamma}{\gamma-1} > 1$ for all defined order values $\gamma \in \mathbb{R} \setminus [0, 1]$, leads us to $(\frac{\gamma}{\gamma-1})^{\delta/\gamma} > \frac{\gamma}{\gamma-1}$. Then, inequality (78) provides

$$\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\delta}{\gamma}} \frac{\Gamma(\frac{\delta+p(\gamma-1)}{\gamma})}{\Gamma(p\frac{\gamma-1}{\gamma})} > \frac{\gamma}{\gamma-1} p\frac{\gamma-1}{\gamma} = p,$$

and, using (73), it holds that $\mathrm{J}_\delta(X_\gamma) > p\lambda^{-\delta}$ for $\delta > \gamma \geq \gamma_p$, where $\gamma_p = \frac{p}{p+1-\Gamma_{min}}$, i.e. the first branch of (76) holds. The order of inequalities, $\delta > \gamma \geq \gamma_p > 1$, is valid, as $\gamma_p > 1$ is valid. This is true, because $\Gamma_{min} > 1$ implies $p+1-\Gamma_0 < p$, i.e. $\gamma_p = \frac{p}{p+1-\Gamma_{min}} > 1$. The values of γ_p is decreasing and $1 < \gamma_p \leq \gamma_1 \approx 1.8615 < 2$ for all $p \geq 1$. Moreover, $\gamma_p = \frac{p}{p+1-\Gamma_{min}} \approx \frac{p}{p-0.4628} < \frac{p}{p-1/2} = \frac{2\,p}{2\,p-1}$.

For the proof of the third branch of (76), it is assumed now that $\delta < \gamma$, i.e. $\frac{\delta}{\gamma} < 1$, or $\frac{\delta+p(\gamma-1)}{\gamma} < 1 + p\frac{\gamma-1}{\gamma}$. This implies,

$$\Gamma(\tfrac{\delta+p(\gamma-1)}{\gamma}) < \Gamma(1 + p\tfrac{\gamma-1}{\gamma}) = p\tfrac{\gamma-1}{\gamma}\Gamma(p\tfrac{\gamma-1}{\gamma}), \tag{79}$$

if $\Gamma_{min} \leq \frac{\delta}{\gamma} + p\frac{\gamma-1}{\gamma}$. That is, if the inequality $\Gamma_{min} \leq \frac{\delta}{\gamma} + p\frac{\gamma-1}{\gamma} = x$ holds, then $\Gamma(\Gamma_{min}) \leq \Gamma(x)$, as the gamma function is an increasing function for $x \geq \Gamma_{min}$. Inequality, $\Gamma_{min} \leq \frac{\delta}{\gamma} + p\frac{\gamma-1}{\gamma}$, is equivalent to $\delta \geq \gamma(\Gamma_{min} - p) + p$. As a result, (79) holds indeed, for order values γ such that, $\gamma(\Gamma_{min} - p) \leq \delta - p$, and so,

$$\frac{\Gamma(\frac{\delta+p(\gamma-1)}{\gamma})}{\Gamma(p\frac{\gamma-1}{\gamma})} < p\frac{\gamma-1}{\gamma}. \tag{80}$$

From the assumption $\frac{\delta}{\gamma} < 1$, together with the fact that $\frac{\gamma}{\gamma-1} > 1$ for all defined order values γ, leads us to $(\frac{\gamma}{\gamma-1})^{\delta/\gamma} < \frac{\gamma}{\gamma-1}$. Then, inequality (80) provides

$$\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\delta}{\gamma}} \frac{\Gamma(\frac{\delta+p(\gamma-1)}{\gamma})}{\Gamma(p\frac{\gamma-1}{\gamma})} < \frac{\gamma}{\gamma-1} p\frac{\gamma-1}{\gamma} = p,$$

and, using (73), it holds that $\mathrm{J}_\delta(X_\gamma) < p\lambda^{-\delta}$ for $\gamma(\Gamma_{min} - p) + p \leq \delta < \gamma$, i.e. the third branch of (76). These inequalities have a valid order when $\gamma(\Gamma_{min} - p) + p < \gamma$

is valid, i.e. if $\gamma > \gamma_p = \frac{p}{p+1-\Gamma_{min}}$ assumed. Therefore, $g_p \leq \delta < \gamma$, where $g_p = \gamma(\Gamma_{min} - p) + p \approx \frac{\gamma}{2}(3 - 2\,p) + p$ as $\Gamma_{min} \approx 1.4628 \approx 3/2$.

Finally, assuming $\delta = \gamma$, it holds from (73) that $\mathrm{J}_\delta(X_\gamma) = p\lambda^{-\delta}$, i.e. the middle branch of (76) holds true. In this case, the restriction of $\gamma > \gamma_p$ is not needed.

Therefore, Proposition 4 shows that, as the quantity $p\lambda^{-\delta}$ is in fact the known Fisher's information with respect to the multivariate normal distribution, the δ-GFI accepts values greater than $p\lambda^{-\delta}$ when $\delta > \gamma$ and lower than $p\lambda^{-\delta}$ when $g_p < \delta < \gamma$. □

From the above Proposition 4, recall (76). As the number of the involved variables, p, increases then $\gamma_p \to 1$; for example, $\gamma_6 \approx \frac{12}{11} \approx 1.09$. Moreover, $g_p < 1$ as p increases. Therefore, Proposition 4 holds, without practically the restrictions of $\gamma > \gamma_p$ and $g_p < \delta$, for large enough values of dimension p.

Corollary 3 *The generalized entropy type information measure* J_δ *of a random variable* $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \lambda\Sigma^*)$, $\lambda \in \mathbb{R}_+ \setminus 0$, $\Sigma^* \in \mathbb{R}_\perp^{p\times p}$ *and with* $\gamma \geq 2$ *and* $p \geq 2$, *satisfy the inequalities*

$$\mathrm{J}_\delta(X_\gamma)\begin{cases} > p\lambda^{-\delta/2}, for\,\delta > \gamma,\\ = p\lambda^{-\delta/2}, for\,\delta = \gamma,\\ < p\lambda^{-\delta/2}, for\,\delta < \gamma.\end{cases}$$

Proof Applying Proposition 4 for $p \geq 2$, we get $g_p = \gamma(\Gamma_{min} - p) + p < 1$, because, when $\gamma(\Gamma_0 - p) + p > 1$ it holds $\gamma < \frac{p-1}{p-\Gamma_{min}} < \frac{p}{p+1-\Gamma_{min}} = \gamma_p < 2$ (as $1 < \gamma_p < \frac{4}{3}$ holds for $p \geq 2$), which is not valid due to the assumption $\gamma \geq 2$. Moreover, $\gamma \geq 2 > \frac{4}{3} > \gamma_p$, and therefore, from Proposition 4, Corollary 3 indeed holds. □

Due to the classification as in (32) and the above Corollary 3, depicted in Fig. 3, the following result is obtained for the multivariate Laplace distribution, in contrast with the multivariate normal distribution.

Corollary 4 *The generalized Fisher's information measure* J_δ *of an r.v. X following the p-variate,* $p \geq 2$, *Laplace distribution* $\mathcal{L}^p(\mu, \lambda\Sigma^*)$, $\lambda \in \mathbb{R}_+ \setminus 0$ *and* $\Sigma^* \in \mathbb{R}_\perp^{p\times p}$, *is always lower than* $p\lambda^{-\delta}$ *for all the parameter values* δ, *i.e.*

$$\mathrm{J}_\delta(X) < p\lambda^{-\delta/2}, \quad \delta > 1.$$

For the normal case, i.e. for $X \sim \mathcal{N}^p(\mu, \lambda\Sigma)$ *with* $p \geq 2$, *we have*

$$\mathrm{J}_\delta(X)\begin{cases} > p\lambda^{-\delta/2}, for\,\delta > 2,\\ < p\lambda^{-\delta/2}, for\,\delta < 2,\end{cases}$$

while $\mathbf{J}_2(X)$ *is reduced to the known Fisher's information for the multivariate normal, i.e.* $\mathrm{J}_{2(X)} = p\lambda^{-2}$.

Proof The normal case is straightforward from Corollary 3. For the Laplace case, as $\mathcal{N}_{\infty}^{p}(\mu, \lambda\Sigma^*) = \mathcal{L}^p(\mu, \lambda\Sigma^*)$ from 32, it is $\mathrm{J}_\delta(X_\infty) < p\lambda^{-\delta}$ for $\delta < \infty$, i.e. the inequality holds for all the values of δ, and the corollary has been proved. □

Theorem 17 *For the γ-GFI J_δ of a random variable $X_\gamma \sim \mathcal{N}_{\gamma}^{p}(\mu, \lambda\Sigma^*)$, $\lambda \in \mathbb{R}_+\backslash 0$, $\Sigma^* \in \mathbb{R}_{\perp}^{p\times p}$ and with $\gamma \geq 2$ and $p \geq 2$, it holds that*

$$1 < \min_{\delta}\{\mathrm{J}_\delta(X_\gamma)\} \leq \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\sqrt{2/\lambda} < p\sqrt{\lambda}/\lambda. \tag{81}$$

Proof From the proof of the Proposition 4, $\mathrm{J}_\delta(X_\gamma)$ is an increasing function for all $\delta \geq 1$, provided that $\gamma \geq 2$. Thus,

$$\min_{\delta}\{\mathrm{J}_\delta(X_\gamma)\} = \mathrm{J}_1(X_\gamma), \tag{82}$$

and $\mathrm{J}_1(X_\gamma) < \mathrm{J}_\delta(X_\gamma)$ for $\delta \geq 1$, and therefore, together with (76), we get $\min_\delta\{\mathrm{J}_\delta(X_\gamma)\} < p\lambda^{-\delta/2}$.

It is assumed now that, $\gamma \geq p/(p - \Gamma_{min}) \approx 2\,p/(2\,p - 3)$ with $\Gamma_{min}(\approx 3/2)$ being the minimum value point of the positive gamma function. Equivalently, $p\frac{\gamma-1}{\gamma} \geq \Gamma_{min}$. Then, it is $\frac{1}{\gamma} + p\frac{\gamma-1}{\gamma} > p\frac{\gamma-1}{\gamma} \geq \Gamma_{min}$, and thus, $\Gamma(\frac{1}{\gamma} + p\frac{\gamma-1}{\gamma}) > \Gamma(p\frac{\gamma-1}{\gamma})$. As a result, from (73),

$$\mathbf{J}_1(X_\gamma) = (\tfrac{\gamma}{\gamma-1})^{\frac{1}{\gamma}} \frac{\Gamma(\frac{1}{\gamma} + p\frac{\gamma-1}{\gamma})}{\sqrt{\lambda}\Gamma(p\frac{\gamma-1}{\gamma})} > \lambda^{-1/2}(\tfrac{\gamma}{\gamma-1})^{\frac{1}{\gamma}} > \tfrac{\sqrt{\lambda}}{\lambda},$$

and using (82), the left-side inequality of (81) holds for $\gamma \geq \frac{p}{p-\Gamma_{min}} > 4$. Moreover, it is true that

$$\min_{\delta}\{\mathrm{J}_\delta(X_\gamma)\} > \tfrac{\sqrt{\lambda}}{\lambda},$$

not just for $\gamma > 4$ but for $\gamma \geq 2$. We have $\frac{d}{d\gamma}\mathrm{J}_1(X_\gamma) = \gamma^{-2}A_\gamma^p\mathrm{J}_1(X_\gamma)$, where

$$A_\gamma^p = (p-1)\Psi(\tfrac{1}{\gamma} + p\tfrac{\gamma-1}{\gamma}) - p\Psi(p\tfrac{\gamma-1}{\gamma}) - \tfrac{1}{\gamma-1} + \log\tfrac{\gamma-1}{\gamma}, \tag{83}$$

with $\gamma > 1$ and $p \geq 1$. The fact that, $\Psi(x) < \log x$ for every $x > 0$, (83) provides that

$$\begin{aligned} A_\gamma^p &< \log(\tfrac{1}{\gamma} + p\tfrac{\gamma-1}{\gamma})^{p-1} - \log(p\tfrac{\gamma-1}{\gamma})^p - \tfrac{1}{\gamma-1} + \log\tfrac{\gamma-1}{\gamma} \\ &< p\log\left(\tfrac{1}{p(\gamma-1)} + 1\right) - \tfrac{1}{\gamma-1} + \log\tfrac{\gamma-1}{\gamma}, \end{aligned}$$

while using the known logarithmic inequality $\log(x+1) < x$, $x > 0$,

$$A_\gamma^p < p\tfrac{1}{p(\gamma-1)} - \tfrac{1}{\gamma-1} + \log\tfrac{\gamma-1}{\gamma} = \log\tfrac{\gamma-1}{\gamma},$$

as $\frac{\gamma-1}{\gamma} < 1$ for any positive order value $\gamma > 1$. Thus, $A_\gamma^p < 0$, and so $\frac{d}{d\gamma}\mathrm{J}_1(X_\gamma) = A_\gamma^p\mathrm{J}_1(X_\gamma) < 0$, as $\mathrm{J}_1(X_\gamma) > 0$ for every $\gamma > 1$. Therefore, $\mathrm{J}_1(X_\gamma)$ is a decreasing

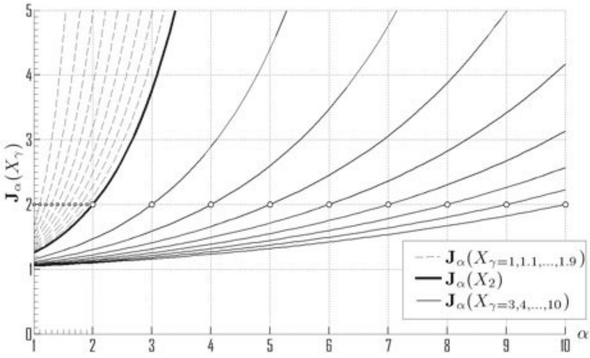

Fig. 3 Graphs of $\mathrm{J}_\alpha(X_\gamma)$ across $\alpha > 1$, for various bivariate γ-ordered normally distributed r.v. $X_\gamma \sim \mathcal{N}_\gamma^2(\mu, \mathbb{I}_2)$

function of $\gamma > 1$. As a result, $\mathrm{J}_1(X_\gamma) > \mathrm{J}_1(X_{+\infty}) = \lim_{\gamma\to+\infty} \mathrm{J}_1(X_\gamma) = 1$ holds for any order value $\gamma > 1$. Therefore, using (82), the left-side inequality of (81) indeed holds for $\gamma \geq 2$.

Finally, due to the fact that $\mathrm{J}_1(X_\gamma)$ is a decreasing function, we have

$$\mathrm{J}_1(X_\gamma) \leq \mathrm{J}_1(X_2) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\sqrt{2/\lambda}, \tag{84}$$

for $\gamma \geq 2$. Using Corollary 3, we get $\mathrm{J}_1(X_2) < p$, and applying (82) to (84), it is concluded that the right-side inequality of (81) indeed holds for $\mathrm{J}_1(X_2) < p$, see Fig. 3. □

From the proof of the above theorem, notice that

$$1 < \mathrm{J}_1(X_\gamma) \leq \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\sqrt{2/\lambda} < p$$

holds for all the positive order values $\gamma > 1$, i.e. without the restriction of $\gamma \geq 2$, as $\mathrm{J}_1(X_\gamma)$ is a decreasing function of any $\gamma > 1$.

Figure 3 depicts the generalized information measure $\mathrm{J}_\delta(X_\gamma)$ with $X_\gamma \sim \mathcal{N}_\gamma^2(\mu, \mathbb{I}_2)$ where $\mathrm{J}_\delta(X_\gamma)$ expressed as a function of the involved parameter $\delta \geq 1$. This figure confirms Theorem 17, for the positive integer order values $\gamma = 2, 3, \ldots, 10$. Moreover, it clearly shows (at least for the bivariate case $p = 2$) that the boundaries as in (81) hold, not only for order values greater than the "normal" order $\gamma = 2$, but for all positive orders $\gamma > 1$.

Corollary 5 *Let $X_\gamma \sim \mathcal{N}_\gamma^p(\mu, \sigma^2\mathbb{I}_p)$ with $p \geq 2$. The lower bound of the generalized entropy type information measure $\mathrm{J}_\delta(X_\gamma)$ with $\delta \geq 2$, is the known Fisher's entropy-type information measure $\mathrm{J}(X_\gamma)$ while it is the upper bound of $\mathrm{J}_\delta(X_\gamma)$ for $1 \leq \delta \leq 2$.*

Proof From the proof of the Proposition 4, $\mathrm{J}_\delta(X_\gamma)$ is an increasing function for all $\delta \geq 1$, provided that $\gamma \geq 2$. Thus, $\mathrm{J}_2(X_\gamma) < \mathrm{J}_\delta(X_\gamma)$ for $\delta \geq 2$, while $\mathrm{J}_2(X_\gamma) > \mathrm{J}_\delta(X_\gamma)$ for $1 \leq \delta \leq 2$. Therefore, corollary has been proved, as J_2 coincides with the usual Fisher's entropy type information measure J. □

7 Discussion

The LSI [28] as well as the PI [2] provide food for thought and a solid mathematical framework for statistics problems, especially when the normal distribution is involved. Briefly speaking the PI is of the form

$$\mathrm{Var}_\mu(f) \le c_p \int |\nabla f|^2 d\mu, \tag{85}$$

for f differentiable function on $\mathbb{R}^p$ with compact support while μ is an appropriate measure, and is related to Fisher's parametric form of information. The constant c_p is known as the Poincaré constant. The Sobolev inequality is of the form

$$\|s\|_q \le c_s \|\nabla f\|_2, \quad q = \tfrac{2p}{p-2}. \tag{86}$$

The constant c_s is known as the Sobolev constant, related to the Fisher's entropy type information. Both PI and LSI are applied to information theory so that to evaluate the appropriate bounds for the variance, entropy, energy, i.e. on statistical measures, see [16, 18].

One of the merits of the family of γ-GND is that includes a number of well-known distributions while the singularity of the Dirac distribution being also one of them. Moreover, the extra parameter γ offers, in principle, different shape approaches and therefore heavy-tailed distributions can easily obtained altering parameter γ which effects kurtosis.

Although a number of papers were presented on the generalized normal, [11, 12, 23], we are still investigating more extensions. We believe we can cover all the possible applications extending the normal distribution case. Recall that there are cases (for example when non-negative time is considered) where a "truncation" of the Normal distribution is needed. Such cases might be possible either for truncation to the right or to the left. We extend this idea to the γ-GND. Let X be a univariate r.v. from $\mathcal{N}_\gamma(\mu, \sigma^2)$ with p.d.f. f_γ as in (27) and c.d.f. F_γ as in (43). We shall say that X follows the γ-GND truncated to the right at $x = \rho$ with p.d.f. $f_{\gamma,\rho}$ when

$$f_{\gamma;\rho}(x) = \begin{cases} 0, & \text{if} \quad x > \rho, \\ \frac{f_\gamma(x)}{F_\gamma(\frac{\rho-\mu}{\sigma})} = \frac{C^1_\gamma(\sigma)}{F_\gamma(\frac{\rho-\mu}{\sigma})} \exp\left\{-\frac{\gamma-1}{\gamma}\left|\frac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}, & \text{if} \quad x \le \rho, \end{cases} \tag{87}$$

Similarly, it would be truncated to the left at $x = \tau$

$$f_{\gamma;\tau}(x) = \begin{cases} 0, & \text{if} \quad x < \tau, \\ \frac{f_\gamma(x)}{1-F_\gamma(\frac{\tau-\mu}{\sigma})} = \frac{C^1_\gamma(\sigma)}{1-F_\gamma(\frac{\tau-\mu}{\sigma})} \exp\left\{-\frac{\gamma-1}{\gamma}\left|\frac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}, & \text{if} \quad x \ge \tau, \end{cases} \tag{88}$$

The lognormal distribution can be also nicely extended to the γ-order lognormal distribution or γ-GLND, in the sense that if $X \sim \mathcal{N}^1_\gamma(\mu, \sigma^2)$ then e^X will follow the γ-GLND, i.e. $e^X \sim \mathcal{LN}_\gamma(\mu, \sigma)$ with p.d.f.

$$g_\gamma(x) = \tfrac{1}{x} f_\gamma(\log x) = C^1_\gamma(\sigma) x^{-1} \exp\left\{-\frac{\gamma-1}{\gamma}\left|\frac{\log x - \mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}, \quad x \in \mathbb{R}^*_+. \tag{89}$$

Moreover, if $X \sim \mathcal{LN}_\gamma(\mu, \sigma)$ then $\log X \sim \mathcal{N}_\gamma^1(\mu, \sigma^2)$. These afore mentioned cases are under investigation for a future work.

In practical problems, such as in Economics where heavy-tailed distributions are needed [10], the γ-GND seems useful. The large positive-ordered GND's provide heavy-tailed distributions as $\mathcal{N}_\gamma^p(\mu, \Sigma)$ approaches the multivariate Laplace distributions, while further heavier-tailed distributions can be extracted through the negative-ordered GND's especially close to zero-ordered GND, i.e. close to the Dirac case. Nevertheless, the higher the dimension gets the heavier the tails become for all multivariate γ-GND's unless γ-GND is considered close to the $\mathcal{N}_1^p(\mu, \Sigma)$, i.e. close to the (elliptically contoured) uniform distribution.

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Applications of Functional Equations to Dirichlet Problem for Doubly Connected Domains

Vladimir Mityushev

Abstract The Dirichlet problem with prescribed vortices for the two-dimensional Laplace equation can be considered as a modification of the classical Dirichlet problem. The modified problem for doubly connected circular domains is reduced to the Riemann–Hilbert boundary value problem and solved by iterative functional equations. The solution of functional equations is derived in terms of the absolutely and uniformly convergent series. The obtained solution can be applied to the minimization of the Ginzburg–Landau functional.

Keywords Multiply connected domain · Riemann–Hilbert boundary value problem · Iterative functional equation · Ginzburg–Landau functional

1 Introduction

The Dirichlet problem for multiply connected circular domains bounded by mutually disjoint circles on the complex plane is one of the fundamental problem of mathematical physics. This problem and the general Riemann–Hilbert boundary value problem were solved in [6, 9]. The crucial point in solution was reduction of the problem to iterative functional equations for analytic functions. The application of successive iterations to the functional equations yields the famous Poincaré θ_2–series associated to the Schottky group [7]. Iterative functional equations in classes of analytic functions were discussed in [4]; see also extended review in the book [9].

In the present paper, we discuss the Dirichlet problem for doubly connected circular domains in a class of functions having prescribed vortices. Let d be a given real constant. Let $\partial/\partial n$ denote the outward normal derivative when $L_k = \{z \in \mathbb{C} : |z - a_k| = r_k\}$ is positively oriented, i.e., leaves the mutually disjoint disk $\mathbb{D}_k = \{z \in \mathbb{C} : |z - a_k| \leq r_k\}$ $(k = 1, 2)$ on the left. Consider the following problem for the function $U(z)$ continuously differentiable in the closure of the doubly

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connected domain $\mathbb{D} = \{z \in \mathbb{C} : \ |z - a_k| > r_k, \ k = 1, 2\}$:

$$\begin{cases} \Delta U = 0, \quad z \in D, \\ U(t) = c_k, \quad t \in L_k \quad (k = 1, 2), \\ U(\infty) = 0, \\ \frac{1}{2\pi} \int_{L_1} \frac{\partial U}{\partial n} ds = -\frac{1}{2\pi} \int_{L_2} \frac{\partial U}{\partial n} ds = d, \end{cases} \tag{1}$$

where Δ stands for the Laplace operator, the constants c_k are undetermined and have to be found during solution to the problem.

This problem (1) generalizes the modified Dirichlet problem [5, 9] and has direct applications to the Ginzburg–Landau functional [1]. Namely, let $H^1(\mathbb{D}; S^1)$ denote the Sobolev space of functions defined in $\mathbb{D}$ and having its values on the unit circle S^1 of the complex plane $\mathbb{C}$. Consider the class of maps

$$V = \left\{ v \in H^1 \left(\mathbb{D}; S^1 \right) : \deg(v, L_k) = d_k \right\}, \tag{2}$$

where $\deg(v, L_k)$ stands for the Brouwer degree, i.e., the winding number of v along the curve L_k. Following [1], we introduce the energy functional

$$E[v] = \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2 dx_1 dx_2, \tag{3}$$

where $z = x_1 + i x_2$ and i denotes the imaginary unit. It is demonstrated in [1] that

$$\inf_{v \in V} E[v] = \frac{1}{2} \int_{\mathbb{D}} |\nabla U|^2 dx_1 dx_2, \tag{4}$$

where U is a solution of the problem (1). It is worth noting that the solution U of the problem (1) is unique up to an arbitrary additive constant and U minimizes the functional

$$F[v] = \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2 dx_1 dx_2 + 2\pi i d \left(v|_{L_1} - v|_{L_2} \right) \tag{5}$$

in the class $\{v \in H^1(\mathbb{D}; \mathbb{R}) : v(t) = constant \ t \in L_k\}$.

In the present paper, we discuss doubly connected domains to show applications of the simple iterative functional equations in classes of analytic functions [4] and to demonstrate in details the method of functional equations. The case of general multiply connected domains will be discussed in a separate paper.

2 $\mathbb{R}$-Linear Problem

The second condition in the problem (1) for doubly connected domains can be presented in the form

$$U(t) = c_1, \quad |t - a_1| = r_1, \tag{6}$$

$$U(t) = c_2, \quad |t - a_2| = r_2. \tag{7}$$

The function $U(z)$ as a function harmonic in the doubly connected domain $\mathbb{D}$ can be presented in the form [9]

$$U(z) = \mathrm{Re}\ \varphi(z) = \mathrm{Re}\ \left(\phi(z) + A \ln \frac{z - a_1}{z - a_2}\right), \tag{8}$$

where A is a real constant, the functions $\varphi(z)$ and $\phi(z)$ are analytic in $\mathbb{D}$ and continuously differentiable in the closure of $\mathbb{D}$, $\phi(z)$ is single-valued. A branch of the logarithm is arbitrary fixed. It does not impact on the result (8) since $U(z)$ depends only on

$$\ln \left| \frac{z - a_1}{z - a_2} \right|.$$

The functions $\varphi(z)$ and $\phi(z)$ vanish at infinity: $\varphi(\infty) = \phi(\infty) = 0$.

We now demonstrate that $A = d$ in (8). Let $\varphi(z) = U(z) + iV(z)$, where $U(z)$ and $V(z)$ stand for the real and imaginary parts of $\varphi(z)$. Let s denote the natural parameter of L_k. It is related with the complex coordinate $t \in L_k$ by formula

$$t = a_k + r_k \exp\left(\frac{is}{r_k}\right). \tag{9}$$

The Cauchy–Riemann equations imply [2] that

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial s}. \tag{10}$$

Calculate the integral

$$\frac{1}{2\pi} \int_{L_1} \frac{\partial U}{\partial n} ds = \frac{1}{2\pi} \int_{L_1} \frac{\partial V}{\partial s} ds = \frac{1}{2\pi} [V]_{L_1}, \tag{11}$$

where $[V]_{L_k}$ denotes the increment of V along L_k. Equation (8) yields

$$\frac{1}{2\pi} [V]|_{L_1} = A, \quad \frac{1}{2\pi} [V]|_{L_2} = -A, \tag{12}$$

since $\phi(z)$ is single-valued and

$$\left[\ln \frac{z - a_1}{z - a_2}\right]_{L_1} = 2\pi i.$$

Equations (10)–(12) and the fourth condition (1) yield $A = d$ in the representation (8).

Using (9) we can calculate the differentials

$$dt = i\ \frac{t - a_k}{r_k} ds, \quad d\overline{t} = -i\ \frac{\overline{t - a_k}}{r_k} ds, \quad t \in L_k, \tag{13}$$

where the bar denotes the complex conjugation. Using (8), we can write the boundary condition (6)–(7) in terms of the analytic function

$$\varphi(t)+\overline{\varphi(t)}=2c_1, \quad |t-a_1|=r_1, \tag{14}$$

$$\varphi(t)+\overline{\varphi(t)}=2c_2, \quad |t-a_2|=r_2. \tag{15}$$

One may differentiate the boundary conditions (14)–(15) on the natural parameter s. Application of (13) yields

$$\frac{t-a_k}{r_k}\psi(t)-\frac{\overline{t-a_k}}{r_k}\overline{\psi(t)}=0, \quad t\in L_k \ (k=1,2), \tag{16}$$

where the function $\psi(z)=\varphi'(z)$ is single-valued in $\mathbb{D}$. Using the relation

$$\overline{t-a_k}=\frac{r_k^2}{t-a_k}, \quad t\in L_k \ (k=1,2) \tag{17}$$

we arrive at the Riemann–Hilbert problem [8]

$$\mathrm{I}m \ (t-a_k)\psi(t)=0, \quad t\in L_k \ (k=1,2) \tag{18}$$

on the function $\psi(z)$ analytic in the domain $\mathbb{D}$ and continuous in its closure. Following [8], one can reduce the Riemann–Hilbert problem to the $\mathbb{R}$–linear problem

$$(t-a_k)\ \psi(t)=(t-a_k)\ \psi_k(t)+\overline{(t-a_k)}\ \overline{\psi_k(t)}+\beta_k, \ |t-a_k|=r_k, \quad k=1,2, \tag{19}$$

where ψ_k are analytic in $|z-a_k|<r_k$ and continuous in $|z-a_k|\le r_k$; β_k are undetermined real constants.

Lemma 1 [8] *The boundary value problems* (18) *and* (19) *are equivalent in the following sense:*

(i) *If $\psi(z)$ and $\psi_k(z)$ are solutions of* (19)*, then $\psi(z)$ satisfies* (18)*.*
(ii) *If $\psi(z)$ is a solution of* (18)*, there exist such functions $\psi_k(z)$ and real constants β_k $(k=1,2)$ that the $\mathbb{R}$–linear condition* (19) *is fulfilled.*

3 Method of Functional Equations

We now proceed to solve the $\mathbb{R}$–linear problem (19) written in the form

$$\psi(t)=\psi_k(t)+\left(\frac{r_k}{t-a_k}\right)^2\overline{\psi_k(t)}+\frac{\beta_k}{t-a_k}, \ |t-a_k|=r_k, \quad k=1,2. \tag{20}$$

The $\mathbb{R}$–linear problem (18) is reduced to functional equations. Consider the inversion with respect to the circle L_k

$$z_{(k)}^{*}:=\frac{r_k^2}{z-a_k}+a_k, \quad (k=1,2). \tag{21}$$

Introduce the function

$$\Phi(z) := \begin{cases} \psi_1(z) - \left(\frac{r_2}{z-a_2}\right)^2 \overline{\psi_2\left(z^*_{(2)}\right)} - \frac{\beta_2}{z-a_2}, \ |z-a_1| \le r_1, \\ \psi_2(z) - \left(\frac{r_1}{z-a_1}\right)^2 \overline{\psi_1\left(z^*_{(1)}\right)} - \frac{\beta_1}{z-a_1}, \ |z-a_2| \le r_2, \\ \psi(z) - \sum_{m=1,2} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} - \sum_{m=1,2} \frac{\beta_m}{z-a_m}, \ z \in \mathbb{D}. \end{cases}$$

The function $\Phi(z)$ is analytic in the disk $|z-a_1| < r_1$ and continuous in $|z-a_1| \le r_1$ since all their components are analytic therein including the function $\overline{\psi_2\left(z^*_{(2)}\right)}$ analytic in $|z-a_2| > r_2$. Along similar lines $\Phi(z)$ is analytic in $|z-a_2| < r_2$, $\mathbb{D}$ and continuous in the closures of the considered domains (in one-side limit topology separately introduced for $|z-a_1| \le r_1$, $|z-a_2| \le r_2$ and $\mathbb{D}$).

Calculate the jump of $\Phi(z)$ across the circle L_k

$$\Delta_k := \Phi^+(t) - \Phi^-(t), \ |t-a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z \to t \ z \in \mathbb{D}} \Phi(z)$, $\Phi^-(t) := \lim_{z \to t \ z \in \mathbb{D}_k} \Phi(z)$. Using (20), we get $\Delta_k = 0$. It follows from the Analytic Continuation Principle that $\Phi(z)$ is analytic in the extended complex plane. Moreover, $\psi(\infty) = 0$ yields $\Phi(\infty) = 0$. Then, Liouville's theorem implies that $\Phi(z) \equiv 0$. The definition of $\Phi(z)$ in $|z-a_k| \le r_k$ yields the following system of functional equations

$$\psi_1(z) = \left(\frac{r_2}{z-a_2}\right)^2 \overline{\psi_2\left(z^*_{(2)}\right)} + \frac{\beta_2}{z-a_2}, \ |z-a_1| \le r_1, \tag{22}$$

$$\psi_2(z) = \left(\frac{r_1}{z-a_1}\right)^2 \overline{\psi_1\left(z^*_{(1)}\right)} + \frac{\beta_1}{z-a_1}, \ |z-a_2| \le r_2. \tag{23}$$

It follows from the definition of $\Phi(z)$ in $\mathbb{D}$ that the general solution of the Riemann–Hilbert problem (18) is constructed via $\psi_k(z)$

$$\psi(z) = \sum_{m=1,2} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + \sum_{m=1,2} \frac{\beta_m}{z-a_m}, \quad z \in \mathbb{D} \cup \partial\mathbb{D}. \tag{24}$$

Introduce the space $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ of functions analytic in the domain $\mathbb{D}_k = \{z \in \mathbb{C} : |z-a_k| < r_k\}$ and continuous in its closure. This is a Banach space endowed with the norm $||f|| = \max_{|t-a_k|=r_k} |f(t)|$. Maximum Principle convergence in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ means uniform convergence in $\mathbb{D}_k$.

Lemma 2 ([9, Lemma 4.8, p. 167]) *The systems* (22) and (23) *have a unique solution in* $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ *(*$k = 1, 2$*). This solution can be found by the method of successive approximations.*

Let $\psi_k(z)$ be a solution to the system of functional Eqs. (22) and (23). Let $w \in \mathbb{D}$ be a fixed point. Introduce the functions

$$\varphi_m(z) = \int_{w^*_{(m)}}^{z} \psi_m(t)dt + \varphi_m(w^*_{(m)}), \quad m = 1, 2 \tag{25}$$

and

$$\omega(z) = -\sum_{m=1,2} \left[\overline{\varphi_m \left(z^*_{(m)}\right)} - \overline{\varphi_m \left(w^*_{(m)}\right)} \right] + \sum_{m=1,2} \beta_m \ln \frac{z - a_m}{w - a_m}. \tag{26}$$

Here, the following relation is used [9]

$$\frac{d}{dz} \left[\overline{\varphi_m \left(z^*_{(m)}\right)} \right] = -\left(\frac{r_k}{z - a_k}\right)^2 \overline{\frac{d\varphi_m}{dz} \left(z^*_{(m)}\right)}, \ |z - a_k| > r_k. \tag{27}$$

The functions $\omega(z)$ and $\varphi_m(z)$ belong to $\mathcal{C}_{\mathcal{A}}(\mathbb{D})$ and to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_m)$, respectively. One can see from (25) that the function $\varphi_m(z)$ is determined by $\psi_m(z)$ up to an additive constant which vanishes in (26). The function $\omega(z)$ vanishes at $z = w$. The function $\omega(z)$ differs from the function $\varphi(z)$ introduced in (8) by an additive constant. One can see that

$$\omega(z) = \varphi(z) - \varphi(w). \tag{28}$$

Therefore, these functions have the same logarithmic coefficients: $\beta_1 = A = d$ and $\beta_2 = -A = -d$. Therefore, the systems of functional Eqs. (22) and (23) become

$$\psi_1(z) = \left(\frac{r_2}{z - a_2}\right)^2 \overline{\psi_2 \left(z^*_{(2)}\right)} - \frac{d}{z - a_2}, \ |z - a_1| \le r_1, \tag{29}$$

$$\psi_2(z) = \left(\frac{r_1}{z - a_1}\right)^2 \overline{\psi_1 \left(z^*_{(1)}\right)} + \frac{d}{z - a_1}, \ |z - a_2| \le r_2. \tag{30}$$

We now demonstrate that the systems (29)–(31) are closely related to the simple iterative functional equation [4]. It follows from (29) that

$$\overline{\psi_2 \left(z^*_{(2)}\right)} = \left(\frac{r_1}{\overline{z^*_{(2)} - a_1}}\right)^2 \psi_1[\alpha(z)] + \frac{d}{\overline{z^*_{(2)} - a_1}}, \ |z - a_2| \ge r_2, \tag{31}$$

where

$$\alpha(z) := \left(z^*_{(2)}\right)^*_{(1)} = \frac{r_1^2 (z - a_2)}{r_2^2 - \overline{(a_1 - a_2)}(z - a_2)} + a_1. \tag{32}$$

The Möbius function $\alpha(z)$ maps the disk $|z - a_1| \le r_1$ into the disk $|z - a_1| < r_1$, since the inversion $z^*_{(2)}$ maps $|z - a_1| \le r_1$ onto a disk lying in $|z - a_2| < r_2$ and the latter disk is mapped by $z^*_{(1)}$ onto a disk in $|z - a_1| < r_1$. The next iterations yield a sequence of shrink disks convergent to a fixed point of $\alpha(z)$. The fixed points z_1 and z_2 of $\alpha(z)$ can be found from the quadratic equation $\alpha(z) = z$ (or equivalently $z^*_{(2)} = z^*_{(1)}$) (Fig. 1):

$$z_1 = \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2} \left[\sqrt{1 + \frac{\left(r_1^2 - r_2^2\right)^2}{4|a_2 - a_1|^4} - 2\frac{r_1^2 + r_2^2}{|a_2 - a_1|^2}} - \frac{r_1^2 - r_2^2}{|a_2 - a_1|^2} \right], \tag{33}$$

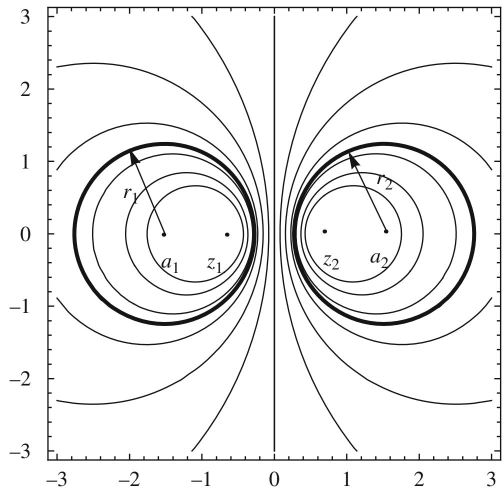

Fig. 1 Two circles (*bold lines*) and level lines of $\alpha(z)$ (*solid lines*) transformed onto concentric circles $|\zeta| = constant$ by (50)

$$z_2 = \frac{a_1 + a_2}{2} - \frac{a_1 - a_2}{2}\left[\sqrt{1 + \frac{\left(r_1^2 - r_2^2\right)^2}{4|a_2 - a_1|^4} - 2\frac{r_1^2 + r_2^2}{|a_2 - a_1|^2}} + \frac{r_1^2 - r_2^2}{|a_2 - a_1|^2}\right].$$

One can see that the fixed point z_1 belongs to $|z - a_1| < r_1$, z_2 belongs to $|z - a_2| < r_2$ and

$$(z_1)^*_{(2)} = z_2, \quad (z_2)^*_{(1)} = z_1. \tag{34}$$

Substituting (31) into (29) yields

$$\psi_1(z) = \alpha'(z)\psi_1[\alpha(z)] + g(z), \quad |z - a_1| \le r_1, \tag{35}$$

where

$$\alpha'(z) = \left(\frac{r_1 r_2}{r_2^2 - \overline{(a_1 - a_2)}(z - a_2)}\right)^2 \tag{36}$$

and

$$g(z) = d\left[\left(\frac{r_2}{z - a_2}\right)^2 \frac{1}{\overline{z^*_{(2)} - a_1}} - \frac{1}{z - a_2}\right] = \frac{d\overline{(a_1 - a_2)}}{r_2^2 - \overline{(a_1 - a_2)}(z - a_2)}. \tag{37}$$

Lemma 3 [4] *The functional equation (35) has a unique solution in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_1)$. This solution can be found by the method of successive approximations uniformly convergent in $|z - a_1| \le r_1$.*

Integrating (35) yields (see (25))

$$\varphi_1(z) = \varphi_1[\alpha(z)] + G(z) - G_0, \quad |z - a_1| \le r_1, \tag{38}$$

where G_0 denotes a constant of integration and

$$G(z) = -d \ln \left[r_2^2 - \overline{(a_1 - a_2)}(z - a_2) \right]. \tag{39}$$

The logarithmic branch can be arbitrarily fixed in $|z - a_1| \le r_1$ since the singular point $z = (a_1)^*_{(2)}$ lies out of the disk $|z - a_1| \le r_1$. Substitution of the fixed point $z = z_1$ into (39) gives formula

$$G_0 = -d \ln \left[r_2^2 - \overline{(a_1 - a_2)}(z_1 - a_2) \right]. \tag{40}$$

The functional equation (38) can be written in the form

$$\varphi_1(z) = \varphi_1[\alpha(z)] + h(z), \quad |z - a_1| \le r_1, \tag{41}$$

where

$$h(z) = -d \ln \frac{r_2^2 - \overline{(a_1 - a_2)}(z - a_2)}{r_2^2 - \overline{(a_1 - a_2)}(z_1 - a_2)}. \tag{42}$$

Lemma 4 ([4, 9]) *Let $\alpha^n(z)$ denotes the nth iteration of $\alpha(z)$. The general solution of the functional equation (41) in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_1)$ is given by formula*

$$\varphi_1(z) = \sum_{k=0}^{\infty} h[\alpha^n(z)] + h_0, \quad |z - a_1| \le r_1, \tag{43}$$

where h_0 is an arbitrary constant. The series (43) converges absolutely and uniformly in the disk $|z - a_1| \le r_1$.

Differentiating (43) terms by terms yields

$$\psi_1(z) = \sum_{k=0}^{\infty} g[\alpha^n(z)][\alpha^n(z)]', \quad |z - a_1| \le r_1, \tag{44}$$

where

$$[\alpha^n(z)]' = \prod_{\ell=1}^{n} \alpha' \left[\alpha^{n-\ell}(z) \right]. \tag{45}$$

Further, the function $\psi_2(z)$ is calculated by (30) and $\psi(z)$ is given by formula (see (24))

$$\psi(z) = \sum_{m=1,2} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m \left(z^*_{(m)} \right)} + \frac{d}{z - a_1} - \frac{d}{z - a_2}, \quad z \in \mathbb{D} \cup \partial\mathbb{D}. \tag{46}$$

The function $\varphi(z)$ is determined by integrating (45) (see formulae (25)–(28)) up to an arbitrary additive constant.

Another method to construct the function $\varphi(z)$ is based on the functional equations obtained by integrating (29) and (30) (see formulae (25) and (27))

$$\varphi_1(z) = -\overline{\varphi_2\left(z^*_{(2)}\right)} - d\ln\frac{z-a_2}{z_1-a_2} + C_1, \; |z-a_1| \le r_1, \tag{47}$$

$$\varphi_2(z) = -\overline{\varphi_1\left(z^*_{(1)}\right)} + d\ln\frac{z-a_1}{z_2-a_1} + C_2, \; |z-a_2| \le r_2, \tag{48}$$

where C_j are undetermined constants. Substituting $z = z_1$ into (47) and $z = z_2$ into (48) implies that $C_2 = \overline{C_1}$. The systems (47) and (48) are reduced to the functional equation (41). Its solution $\varphi_1(z)$ has the form (43). Further, the function $\varphi_2(z)$ is constructed by (48) and $\varphi(z)$ by (27) and (28).

The function $U(z)$ is calculated by (8) and depends on a real arbitrary additive constant.

4 Case of Equal Radii

In the present section, we present a method to find $U(z)$ based on a conformal mapping. For simplicity, we consider the case of the equal radii $r_1 = r_2 = r$ when the centers $a_1 = -a/2$ and $a_2 = a/2$ lie on the real axis ($a > 0$). The latter condition is not an essential restriction on geometry. Then, formula (33) becomes

$$z_1 = -\frac{a}{2}\sqrt{1-4\frac{r^2}{a^2}}, \quad z_2 = \frac{a}{2}\sqrt{1-4\frac{r^2}{a^2}}. \tag{49}$$

The Möbius function

$$\zeta = \frac{z-z_2}{z-z_1} \tag{50}$$

maps the domain $\mathbb{D}$ onto the annulus

$$D = \left\{\zeta \in \mathbb{C} : \frac{1}{R} < |\zeta| < R\right\},$$

where the positive constant R has the form [10]

$$R = \frac{\frac{r}{a} + \frac{1}{2}\left(1+\sqrt{1-\frac{4r^2}{a^2}}\right)}{\frac{r}{a} + \frac{1}{2}\left(1-\sqrt{1-\frac{4r^2}{a^2}}\right)}. \tag{51}$$

The disks $|z-a_1| < r$ and $|z-a_2| < r$ are conformally mapped onto $|\zeta| > R$ and $|\zeta| < 1/R$, respectively, the imaginary axis onto the unit circle $|\zeta| = 1$. The inverse function has the form

$$z = \frac{z_1\zeta - z_2}{\zeta - 1}. \tag{52}$$

Here, we use the property that symmetric points are mapped onto symmetric points by the Möbius transformations. The symmetric points z_1 and z_2 (see (34)) are transformed onto the symmetric points $\zeta = \infty$ and $\zeta = 0$, respectively, which can be symmetric only with respect to the concentric circles. Hence, the circles of symmetry $|z - a_1| = r$ and $|z - a_2| = r$ are transformed onto concentric circles. The radii are found by straight calculations.

Using (52), we introduce the functions $\Phi_1(\zeta) = \varphi_1(z)$, $\Phi(\zeta) = \varphi(z)$ and $\Phi_2(\zeta) = \varphi_2(z)$ analytic in $|\zeta| > R$, $1/R < |\zeta| < R$ and $|\zeta| < 1/R$, respectively. Then, the functional equation (41) becomes

$$\Phi_1(\zeta) = \Phi_1\left(R^4\zeta\right) + H(\zeta), \quad |\zeta| \geq R, \tag{53}$$

where

$$H(\zeta) = -d \ln \frac{r_2^2 - \overline{(a_1 - a_2)}\left(\frac{z_1\zeta - z_2}{\zeta - 1} - a_2\right)}{r_2^2 - \overline{(a_1 - a_2)}(z_1 - a_2)}. \tag{54}$$

The shift $R^4\zeta$ is composed by two inversions $\zeta \to \frac{R^2}{\overline{\zeta}}$ and $\zeta \to \frac{1}{R^2\overline{\zeta}}$. It corresponds to the shift (32) composed by the inversions $z^*_{(2)}$ and $z^*_{(1)}$. One can see that $H(\infty) = 0$, hence, in accordance with Lemma 4, Eq. (53) is solvable and its general solution has the form

$$\Phi_1(\zeta) = \sum_{k=0}^{\infty} H\left[R^{4k}\zeta\right] + H_0, \quad |\zeta| \geq R, \tag{55}$$

where H_0 is an arbitrary constant. One can see that the rate of convergence of the series (55) is equal to R^{-4}.

The functional equations can also be solved with the use of the Taylor expansion near infinity. Let

$$H(\zeta) = \sum_{m=1}^{\infty} H_m \zeta^{-m}. \tag{56}$$

Then

$$\Phi_1(\zeta) = \sum_{m=1}^{\infty} \frac{H_m}{1 - R^{-4\,m}} \zeta^{-m} + H_0. \quad |\zeta| > R. \tag{57}$$

It follows from the Cauchy–Hadamard formula that the series (57) has the same radius of convergence that (56). One can find solution of the functional equation (53) in terms of the elliptic functions in [3].

The function $U(z)$ is constructed by the scheme described at the end of Sect. 3.

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Sign-Changing Solutions for Nonlinear Elliptic Problems Depending on Parameters

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Abstract This chapter is concerned with parametric Dirichlet boundary value problems involving the p-Laplacian operator. Specifically, this chapter gives an account of recent results that establish the existence and multiplicity of solutions according to different types of nonlinearities in the problem. More precisely, we focus on problems exhibiting nonlinearities of concave–convex type and nonlinearities that are asymptotically $(p-1)$-linear. In each situation, we point out significant qualitative properties of the solutions, especially, we establish the existence of sign-changing (that is, nodal) solutions.

Keywords Elliptic equation · Boundary value problem · p-Laplacian · Variational method · Upper and lower solutions · Sign-changing solution

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain, $1 < p < +\infty$ and consider the p-Laplacian operator $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \ (\frac{1}{p} + \frac{1}{p'} = 1)$, which is given by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for all $u \in W_0^{1,p}(\Omega)$. This is expressed as follows

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \text{ for all } u, v \in W_0^{1,p}(\Omega).$$

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In the present chapter, we study the parametric problem

$$\begin{cases} -\Delta_p u = f(x, u(x), \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a parameter belonging to an interval $\Lambda := (0, \overline{\lambda})$ with $\overline{\lambda} > 0$ and the right-hand side of the equation in (1) is described through a function $f : \Omega \times \mathbb{R} \times \Lambda \to \mathbb{R}$.

An important class of problems as in (1) is the one whose right-hand side consists of the so-called concave–convex nonlinearities

$$f(x, s, \lambda) = \lambda |s|^{q-2} s + |s|^{r-2} s \text{ with } 1 < q < p < r < p^*, \tag{2}$$

where p^* denotes the critical exponent of p, i.e., $p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = +\infty$ if $N \leq p$. This class of nonlinearities was first studied by Ambrosetti–Brezis–Cerami [2] in the semilinear case, i.e., for $p = 2$. Their study was then extended to the case of p-Laplacian equations by García Azorero–Manfredi–Peral Alonso [13] (for $1 < p < +\infty$) and by Guo–Zhang [17] (for $p \geq 2$). In these works, the authors establish the existence of two positive solutions and symmetrically two negative solutions of the problem provided the parameter $\lambda > 0$ is sufficiently small.

This chapter is based on the works [19] and [20], which are actually concerned with two generalizations of the nonlinearities in (2). First, our study mainly focuses on the case

$$f(x, s, \lambda) = \lambda h(x, s) + |s|^{r-2} s. \tag{3}$$

Here, h denotes a "concave term" that can be typically of the form $h(x, s) = |s|^{q-2} s$ (see Example 1) but our assumptions also incorporate the case where h is asymptotically $(p-1)$-linear near the origin (see Example 2). Second, we target the situation

$$f(x, s, \lambda) = \lambda |s|^{q-2} s + g(x, s), \tag{4}$$

where g is a Carathéodory function (typically we can take $g(x, s) = |s|^{r-2} s$; see also Example 3). In fact, here, we consider the more general problem

$$\begin{cases} -\Delta_p u = \beta(x) |u(x)|^{q-2} u(x) + g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where the parameter λ is replaced by a nonnegative function $\beta \in L^\infty(\Omega) \setminus \{0\}$ (with sufficiently small L^∞-norm). In both cases, we study the existence of constant sign, extremal constant sign, and sign-changing (that is, nodal) solutions for the corresponding problem (1). In this respect, it is worth mentioning that a fundamental idea to obtain sign-changing solutions is to seek them between extremal opposite constant sign solutions. This approach for p-Laplacian equations originates in Carl-Perera [8] and Carl–Motreanu [6, 7].

Our precise results are formulated in the next section. These results are then proved in Sects. 3 and 4.

2 Statements of Main Results

We first recall basic notation and facts that are used in the statements of our results. Let $\lambda_2 > \lambda_1 > 0$ be the first two eigenvalues of the negative Dirichlet p-Laplacian $-\Delta_p$ on $W_0^{1,p}(\Omega)$. Recall that λ_1 admits the variational characterization

$$\lambda_1 = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \;:\; u \in W_0^{1,p}(\Omega),\; u \neq 0\right\} \tag{6}$$

(where the notation $\|\cdot\|_p$ stands for the norms in both spaces $L^p(\Omega)$ and $L^p(\Omega, \mathbb{R}^N)$), while λ_2 is introduced as

$$\lambda_2 = \inf\{\lambda : \lambda \text{ is an eigenvalue of } -\Delta_p \text{ and } \lambda > \lambda_1\}.$$

By $\hat{u}_1$, we denote the L^p-normalized positive eigenfunction of $-\Delta_p$ corresponding to the first eigenvalue λ_1. Through the strong maximum principle, we know that $\hat{u}_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, where

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \{u \in C_0^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega, \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\}$$

is the interior of the positive cone $C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \Omega\}$ of the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u(x) = 0 \text{ for all } x \in \partial\Omega\}$, where $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$.

In what follows, we state our main results under different sets of hypotheses on the nonlinearity $f(x, s, \lambda)$. We set forth the results into two subsections depending on whether the nonlinearity is of type (3) or (4).

2.1 Results for Nonlinearities of Type (3)

We start with the existence of solutions for problem (1) in the situation where the nonlinearity $f(x, s, \lambda)$ is typically of type (3), in the sense that the considered hypotheses are adequate to the situation of (3).

First we deal with constant sign solutions. Note that, in the following set of hypotheses, we only suppose a polynomial growth condition on the nonlinearity f with arbitrary exponent, not necessarily subcritical (see $\mathrm{H}(f)_1^{\pm}$ (i) below), and nonuniform nonresonance condition at the first eigenvalue λ_1 (see $\mathrm{H}(f)_1^{\pm}$ (ii) below).

$\mathrm{H}(f)_1^{+}$ (i) for every $\lambda \in (0, \bar{\lambda})$, $f(\cdot, \cdot, \lambda)$ is Carathéodory (that is, $f(\cdot, s, \lambda)$ is measurable for all $s \in \mathbb{R}$ and $f(x, \cdot, \lambda)$ is continuous for almost all $x \in \Omega$) with $f(x, 0, \lambda) = 0$ a.e. in Ω, for all $\lambda \in \Lambda$; moreover, there are $a(\lambda) > 0$ with $a(\lambda) \to 0$ as $\lambda \downarrow 0$, and $c > 0$, $r > p$ (both independent of λ) such that

$$|f(x, s, \lambda)| \leq a(\lambda) + c|s|^{r-1} \text{ for a.a. } x \in \Omega, \text{all } s \in \mathbb{R}, \text{all } \lambda \in \Lambda;$$

(ii) for every $\lambda \in \Lambda$, there exists $\eta_\lambda \in L^\infty(\Omega)$ such that $\eta_\lambda \geq \lambda_1$ a.e. in Ω, $\eta_\lambda \neq \lambda_1$ and

$$\eta_\lambda(x) \leq \liminf_{s \downarrow 0} \frac{f(x,s,\lambda)}{s^{p-1}} \text{ uniformly for a.a. } x \in \Omega.$$

Symmetrically, we formulate the following conditions:

$H(f)_1^-$ (i) f satisfies $H(f)_1^+$ (i);
(ii) for every $\lambda \in \Lambda$, there exists $\eta_\lambda \in L^\infty(\Omega)$ such that $\eta_\lambda \geq \lambda_1$ a.e. in Ω, $\eta_\lambda \neq \lambda_1$ and

$$\eta_\lambda(x) \leq \liminf_{s \uparrow 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \text{ uniformly for a.a. } x \in \Omega\,.$$

Proposition 1 (a) *Under* $H(f)_1^+$, *for all* $b > 0$, *there exists* $\lambda^* \in \Lambda$ *such that, for* $\lambda \in (0, \lambda^*)$, *problem* (1) *has a solution* $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *with* $\|u_\lambda\|_\infty < b$. (b) *Under* $H(f)_1^-$, *for all* $b > 0$, *there exists* $\lambda^* \in \Lambda$ *such that, for* $\lambda \in (0, \lambda^*)$, *problem* (1) *has a solution* $v_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *with* $\|v_\lambda\|_\infty < b$.

More insight in the study of existence of constant sign solutions is obtained by producing extremal constant sign solutions for problem (1). To do this, we rely on strengthened versions of hypotheses $H(f)_1^\pm$ which require that the nonlinearity $f(x,\cdot,\lambda)$ is asymptotically $(p-1)$-linear near the origin (see $H(f)_2^\pm$ (ii)).

$H(f)_2^+$ (i) f satisfies $H(f)_1^+$ (i);
(ii) for all $\lambda \in \Lambda$, there exist $\eta_\lambda, \hat{\eta}_\lambda \in L^\infty(\Omega)$ such that $\eta_\lambda(x) \geq \lambda_1$ a.e. in Ω, $\eta_\lambda \neq \lambda_1$ and

$$\eta_\lambda(x) \leq \liminf_{s \downarrow 0} \frac{f(x,s,\lambda)}{s^{p-1}} \leq \limsup_{s \downarrow 0} \frac{f(x,s,\lambda)}{s^{p-1}} \leq \hat{\eta}_\lambda(x)$$

uniformly for a.a. $x \in \Omega$.

Symmetrically, we consider:

$H(f)_2^-$ (i) f satisfies $H(f)_1^+$ (i);
(ii) for all $\lambda \in \Lambda$, there exist $\eta_\lambda, \hat{\eta}_\lambda \in L^\infty(\Omega)$ such that $\eta_\lambda(x) \geq \lambda_1$ a.e. in Ω, $\eta_\lambda \neq \lambda_1$ and

$$\eta_\lambda(x) \leq \liminf_{s \uparrow 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \limsup_{s \uparrow 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \leq \hat{\eta}_\lambda(x)$$

uniformly for a.a. $x \in \Omega$.

Proposition 2 (a) *Under* $H(f)_2^+$, *for all* $b > 0$, *there exists* $\lambda^* \in \Lambda$ *such that for* $\lambda \in (0, \lambda^*)$, *problem* (1) *has a smallest positive solution* $u_{\lambda,+}$ *with* $\|u_{\lambda,+}\|_\infty < b$ *and which in addition satisfies* $u_{\lambda,+} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

(b) *Under* $\mathrm{H}(f)_2^-$*, for all* $b > 0$*, there exists* $\lambda^* \in \Lambda$ *such that for* $\lambda \in (0, \lambda^*)$*, problem* (1) *has a biggest negative solution* $v_{\lambda,-}$ *with* $\|v_{\lambda,-}\|_\infty < b$ *and which in addition satisfies* $v_{\lambda,-} \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$.

Next, we deal with the existence of a nontrivial solution of (1) that is intermediate between the extremal constant sign solutions obtained in Proposition 2. To this end, we need to strengthen conditions $\mathrm{H}(f)_1^\pm$ (ii) to have nonuniform nonresonance from below at the second eigenvalue λ_2 (see $\mathrm{H}(f)_2^\pm$ (ii.a)). In addition, by strengthening conditions $\mathrm{H}(f)_2^\pm$ (ii) (see $\mathrm{H}(f)_2^\pm$ (ii.b) below), the intermediate solution can be chosen to be sign changing. We state:

$\mathrm{H}(f)_3$ (i) f satisfies $\mathrm{H}(f)_1^+$ (i);

(ii) there holds:

(ii.a) for all $\lambda \in \Lambda$, there exists $\theta_\lambda > \lambda_2$ such that

$$\theta_\lambda < \liminf_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \text{ uniformly for a.a. } x \in \Omega;$$

or the stronger condition

(ii.b) for all $\lambda \in \Lambda$, there exist $\theta_\lambda > \lambda_2$ and $\hat{\eta}_\lambda \in L^\infty(\Omega)$ such that

$$\theta_\lambda < \liminf_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \limsup_{s \to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \hat{\eta}_\lambda(x)$$

uniformly for a.a. $x \in \Omega$.

Theorem 1 (a) *Assume that* $\mathrm{H}(f)_3$ *holds. Then, for all* $b > 0$*, there exists* $\lambda^* \in \Lambda$ *such that, for* $\lambda \in (0, \lambda^*)$*, problem* (1) *has at least three distinct, nontrivial solutions:* $u_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$*,* $v_\lambda \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$*, and* $y_\lambda \in C_0^1(\overline{\Omega})$ *with*

$$-b < v_\lambda \le y_\lambda \le u_\lambda < b \text{ in } \overline{\Omega}.$$

(b) *If, in addition,* $\mathrm{H}(f)_3$ (ii.b) *holds, then* y_λ *can be chosen to be sign changing.*

Next, we are concerned with the existence of additional constant sign solutions of (1). This is done in the case where f satisfies a nonuniform version of the so-called Ambrosetti–Rabinowitz condition (see $\mathrm{H}(f)_4^\pm$ (iii) below) and a uniform unilateral sign condition (i.e., in a neighborhood of 0 which is independent of λ, see $\mathrm{H}(f)_4^\pm$ (iv) below). Note that hypothesis $\mathrm{H}(f)_4^\pm$ (iii) below forces the nonlinearity f to be $(p-1)$-superlinear at infinity, but we do not require that $\operatorname{ess\,inf}_{x\in\Omega} \int_0^s f(x,t,\lambda)\,dt > 0$ (contrary to the classical Ambrosetti–Rabinowitz condition). Also, note that a nonuniform sign condition (i.e., satisfied by $f(x,s,\lambda)$ for a fixed λ) is already implied by hypothesis $\mathrm{H}(f)_1^\pm$ (ii).

$\mathrm{H}(f)_4^+$ (i) f satisfies $\mathrm{H}(f)_1^+$ (i) with $r < p^*$;

(ii) f satisfies $\mathrm{H}(f)_1^+$ (ii);

(iii) for every $\lambda \in \Lambda$, there exist $M_\lambda > 0$ and $\mu_\lambda > p$ such that

$$0 < \mu_\lambda F(x,s,\lambda) \le f(x,s,\lambda)s \text{ for a.a. } x \in \Omega, \text{ all } s \ge M_\lambda,$$

where $F(x,s,\lambda) = \int_0^s f(x,t,\lambda)\,dt$;

(iv) there exists $\rho > 0$ such that $f(x, s, \lambda) > 0$ for a.a. $x \in \Omega$, all $s \in (0, \rho)$, all $\lambda \in \Lambda$.

Symmetrically, we state:

$\mathrm{H}(f)_4^-$ (i) f satisfies $\mathrm{H}(f)_4^+$ (i);
(ii) f satisfies $\mathrm{H}(f)_1^-$ (ii);
(iii) for every $\lambda \in \Lambda$, there exist $M_\lambda > 0$ and $\mu_\lambda > p$ such that

$$0 < \mu_\lambda F(x, s, \lambda) \le f(x, s, \lambda)s \text{ for a.a. } x \in \Omega, \text{ all } s \le -M_\lambda ;$$

(iv) there exists $\rho > 0$ such that $f(x, s, \lambda) < 0$ for a.a. $x \in \Omega$, all $s \in (-\rho, 0)$, all $\lambda \in \Lambda$.

Theorem 2 (a) *Under* $\mathrm{H}(f)_4^+$, *for all* $b > 0$, *there exists* $\lambda^* \in \Lambda$ *such that, for* $\lambda \in (0, \lambda^*)$, *problem* (1) *has at least two distinct solutions* $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *with* $u_\lambda \le \hat{u}_\lambda$ *in* Ω *and* $\|u_\lambda\|_\infty < b$. (b) *Under* $\mathrm{H}(f)_4^-$, *for all* $b > 0$, *there exists* $\lambda^* \in \Lambda$ *such that, for* $\lambda \in (0, \lambda^*)$, *problem* (1) *has at least two distinct solutions* $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *with* $\hat{v}_\lambda \le v_\lambda$ *in* Ω *and* $\|v_\lambda\|_\infty < b$.

Combining Theorems 1 and 2, we obtain the existence of five nontrivial solutions. Precisely, we consider the following conditions on $f(x, s, \lambda)$:

$\mathrm{H}(f)_5$ (i) f satisfies $\mathrm{H}(f)_4^+$ (i);
(ii) f satisfies $\mathrm{H}(f)_3$ (ii.a);
(iii) f satisfies $\mathrm{H}(f)_4^+$ (iii) and $\mathrm{H}(f)_4^-$ (iii), that is, for every $\lambda \in \Lambda$, there exist $M_\lambda > 0$ and $\mu_\lambda > p$ such that

$$0 < \mu_\lambda F(x, s, \lambda) \le f(x, s, \lambda)s \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R} \text{ with } |s| \ge M_\lambda ;$$

(iv) f satisfies $\mathrm{H}(f)_4^+$ (iv) and $\mathrm{H}(f)_4^-$ (iv), that is, there exists $\rho > 0$ such that $f(x, s, \lambda)s > 0$ for a.a. $x \in \Omega$, all $s \in [-\rho, \rho]$, all $\lambda \in \Lambda$.

Theorem 3 (a) *Assume that* $\mathrm{H}(f)_5$ *holds. Then, for all* $b > 0$, *there exists* $\lambda^* \in \Lambda$ *such that, for* $\lambda \in (0, \lambda^*)$, *problem* (1) *has at least five distinct, nontrivial solutions:* $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, *and* $y_\lambda \in C_0^1(\overline{\Omega})$ *with*

$$\hat{v}_\lambda \le v_\lambda \le y_\lambda \le u_\lambda \le \hat{u}_\lambda \text{ in } \overline{\Omega}, \quad \|u_\lambda\|_\infty < b, \text{ and } \|v_\lambda\|_\infty < b.$$

(b) *If, in addition,* $\mathrm{H}(f)_3$ (ii.b) *holds, then* y_λ *can be chosen to be sign changing.*

Example 1 As announced in (3), a typical nonlinearity fulfilling $\mathrm{H}(f)_5$ is of the form

$$f(x, s, \lambda) = \lambda h(x, s) + |s|^{r-2}s, \tag{7}$$

where $p < r < p^*$ and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $h(x, 0) = 0$ a.e. in Ω, which satisfies the following conditions

(i) there exist $\hat{c}_0 > 0$ and $1 \le q < p$ such that

$$|h(x,s)| \le \hat{c}_0(1 + |s|^{q-1}) \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}\,;$$

(ii) $\liminf_{s\to 0} \dfrac{h(x,s)}{|s|^{p-2}s} = +\infty$ uniformly for a.a. $x \in \Omega$;
(iii) there exist $M_0 > 0$, $\mu \in (p,r)$, $c_1, c_2 > 0$ and $r_0 \in [0,r)$ such that

$$-c_1|s|^r \le \mu H(x,s) \le h(x,s)s + c_2|s|^{r_0} \text{ for a.a. } x \in \Omega, \text{ all } |s| \ge M_0\,,$$

where $H(x,s) = \int_0^s h(x,t)\,dt$;
(iv) there exists $\rho > 0$ such that $h(x,s)s \ge 0$ for a.a. $x \in \Omega$, all $s \in [-\rho,\rho]$.

Under these conditions, it can be seen that f given in (7) satisfies H$(f)_5$ for $\lambda \in \Lambda := (0, \frac{\mu}{rc_1})$. Thus, Theorem 3 (a) yields five nontrivial solutions for problem (1): two positive, two negative, and an intermediate one. A particular case of h fulfilling (i)–(iv) above is $h(x,s) = |s|^{q-2}s$ with $q \in (1,p)$, so in this case

$$f(x,s,\lambda) = \lambda|s|^{q-2}s + |s|^{r-2}s. \tag{8}$$

Therefore, Theorem 3 (a) extends the corresponding result in García Azorero–Manfredi–Peral Alonso [13] dealing with the case in (8). It also brings new information even in the case of (2) by guaranteeing the existence of five nontrivial solutions for problem (1). In fact, for the particular case of the nonlinearity in (8), more insight will be achieved by Corollary 1 below, showing that actually the intermediate solution can be chosen sign changing.

We can obtain an additional sign-changing solution by strengthening H$(f)_5$:

H$(f)_6$ (i) $f : \overline{\Omega} \times \mathbb{R} \times \Lambda \to \mathbb{R}$ is such that $f(\cdot,\cdot,\lambda)$ is a continuous function, $f(x,0,\lambda) = 0$ for all $x \in \Omega$, all $\lambda \in \Lambda$; moreover, there are $a(\lambda) > 0$ with $a(\lambda) \to 0$ as $\lambda \downarrow 0$, and $c > 0$, $r \in (p,p^*)$ (both independent of λ) such that

$$|f(x,s,\lambda)| \le a(\lambda) + c|s|^{r-1} \text{ for all } x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ all } \lambda \in \Lambda;$$

(ii) for all $\lambda \in \Lambda$, there exist $\theta_\lambda > \lambda_2$ and $\hat{\eta}_\lambda \in L^\infty(\Omega)$ such that

$$\theta_\lambda < \liminf_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \limsup_{s\to 0} \frac{f(x,s,\lambda)}{|s|^{p-2}s} \le \hat{\eta}_\lambda(x)$$

uniformly for all $x \in \Omega$;
(iii) for every $\lambda \in \Lambda$, there exist $M_\lambda > 0$ and $\mu_\lambda > p$ such that

$$0 < \mu_\lambda F(x,s,\lambda) \le f(x,s,\lambda)s \text{ for all } x \in \Omega, \text{ all } s \in \mathbb{R} \text{ with } |s| \ge M_\lambda\,;$$

(iv) there exist $\rho_- < 0 < \rho_+$ such that for all $\lambda \in \Lambda$ we have

$$f(x,\rho_-,\lambda) = 0 = f(x,\rho_+,\lambda) \quad \text{for all } x \in \Omega,$$
$$f(x,s,\lambda)s > 0 \text{ for all } x \in \Omega, \text{all } s \in (\rho_-,\rho_+), s \ne 0.$$

Theorem 4 *Assume that* $\mathrm{H}(f)_6$ *holds. Then, there exists* $\lambda^* \in \Lambda$ *such that, for all* $\lambda \in (0, \lambda^*)$*, problem* (1) *has at least six distinct, nontrivial solutions:* $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$*,* $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$*, and* $y_\lambda, w_\lambda \in C_0^1(\overline{\Omega})$ *both sign-changing.*

Example 2 For $\lambda \in (0, +\infty)$, we consider the following nonlinearity

$$f(x,s,\lambda) = \begin{cases} |s|^{r-2}s + 1 & \text{if } s \le -1, \\ -\theta(x,s)\min\{\lambda, |s|^{p-1}\} & \text{if } -1 < s \le 0, \\ \theta(x,s)\min\{\lambda, s^{p-1}\} & \text{if } 0 < s \le 1, \\ s^{r-1} - 1 & \text{if } s > 1, \end{cases}$$

where $r \in (p, p^*)$ and $\theta : \overline{\Omega} \times [-1,1] \to \mathbb{R}$ is a continuous function satisfying $\theta(x,0) > \lambda_2$, $\theta(x,-1) = \theta(x,1) = 0$ for all $x \in \overline{\Omega}$, and $\theta(x,s) > 0$ for all $x \in \Omega$, all $s \in (-1,1)$. For example, we can take $\theta(x,s) = (e^{|x|} + \lambda_2)(1 - |s|)$. Then, the function $f(x,s,\lambda)$ fulfills $\mathrm{H}(f)_6$ with $\rho_- = -1$, $\rho_+ = 1$. Therefore, Theorem 4 implies that, for the above nonlinearity f and $\lambda > 0$ small, problem (1) admits at least six nontrivial solutions: two positive, two negative, and two sign-changing.

As illustrated by Examples 1 and 2, the sets of hypotheses $\mathrm{H}(f)_1$–$\mathrm{H}(f)_6$ mainly address the case where the nonlinearity $f(x,s,\lambda)$ is of the form (3). In the next subsection, we focus on nonlinearities of type (4).

2.2 *Results for Nonlinearities of Type* (4)

In this subsection, we study problem (5), that is, the considered nonlinearity is of the form

$$\beta(x)|s|^{q-2}s + g(x,s),$$

where $\beta \in L^\infty(\Omega)_+ \setminus \{0\}$ and g is a Carathéodory function. Later in this subsection, we will suppose that $\beta \equiv \lambda$ is constant.

First, we look for constant sign solutions for problem (5). We denote $G(x,s) = \int_0^s g(x,t)\,dt$. We note that we assume that for a.a. $x \in \Omega$, $G(x,\cdot)$ is p-superlinear near $+\infty$ (see $\mathrm{H}(g)_1^+$ (iii.a) below), but we do not require the Ambrosetti–Rabinowitz condition that is common in such cases. In addition, we assume that near zero, $g(x,\cdot)$ satisfies a nonuniform nonresonance condition at the first eigenvalue λ_1 of the negative Dirichlet p-Laplacian (see $\mathrm{H}(g)_1^+$ (ii) below). Precisely, we consider the following hypotheses:

$\mathrm{H}(g)_1^+$ (i) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $g(x,0) = 0$, a.e. in Ω, and there are $c > 0$ and $r \in (p, p^*)$ such that

$$|g(x,s)| \le c(1 + |s|^{r-1}) \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R};$$

(ii) there exist $\vartheta, \hat{\vartheta} \in L^\infty(\Omega)_+$ such that $\vartheta(x) \le \lambda_1$ a.e. in Ω, $\vartheta \ne \lambda_1$, and

$$-\hat{\vartheta}(x) \le \liminf_{s \downarrow 0} \frac{g(x,s)}{s^{p-1}} \le \limsup_{s \downarrow 0} \frac{g(x,s)}{s^{p-1}} \le \vartheta(x)$$

uniformly for a.a. $x \in \Omega$;

(iii) the following asymptotic conditions at $\pm\infty$ are satisfied:

(iii.a) $\lim_{s\to+\infty} \dfrac{G(x,s)}{s^p} = +\infty$ uniformly for a.a. $x \in \Omega$,

(iii.b) there exist $\tau \in ((r-p)\max\{\frac{N}{p}, 1\}, p^*)$, $\tau > q$, and $\gamma_0 > 0$ such that

$$\liminf_{s\to+\infty} \frac{g(x,s)s - pG(x,s)}{s^\tau} \geq \gamma_0 \text{ uniformly for a.a. } x \in \Omega.$$

Theorem 5 *Assume that* $\mathrm{H}(g)_1^+$ *holds. Then, there is* $\lambda^* > 0$ *such that, whenever* $\|\beta\|_\infty < \lambda^*$, *problem* (5) *has two distinct solutions* $u_0, \hat{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Next we are concerned with existence of a smallest positive solution for a restricted version of problem (5) in which $\beta(\cdot) \equiv \lambda$ is constant, namely,

$$\begin{cases} -\Delta_p u = \lambda |u(x)|^{q-2}u(x) + g(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

with $\lambda > 0$ and $q \in (1,p)$. We assume that g satisfies an arbitrary polynomial growth condition and a stronger hypothesis near the origin, in particular, by requiring a local sign condition. Precisely, we consider the following hypotheses:

$\mathrm{H}(g)_2^+$ (i) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $g(x,0) = 0$ a.e. in Ω and there exist $c > 0$ and $r \in [1,+\infty)$ such that

$$|g(x,s)| \leq c(1 + |s|^{r-1}) \text{ for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}\,;$$

(ii) $\lim_{s\downarrow 0} \dfrac{g(x,s)}{s^{p-1}} = 0$ uniformly for a.a. $x \in \Omega$;

(iii) there exists $\delta_0 > 0$ such that $g(x,s) \geq 0$ for a.a. $x \in \Omega$, all $s \in [0,\delta_0]$.

Proposition 3 *Assume that* $\mathrm{H}(g)_2^+$ *holds. Then, there is* $\lambda^* > 0$ *such that, for* $\lambda \in (0,\lambda^*)$, *problem* (9) *has a smallest positive solution* $u_{\lambda,+} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. *Furthermore, it satisfies* $\|u_{\lambda,+}\|_\infty < \delta_0$.

Now, we gather the above hypotheses $\mathrm{H}(g)_1^+$ and $\mathrm{H}(g)_2^+$ together with their counterparts on the negative half-line:

$\mathrm{H}(g)_3$ (i) g satisfies $\mathrm{H}(g)_1^+$ (i);

(ii) $\lim_{s\to 0} \dfrac{g(x,s)}{|s|^{p-1}} = 0$ uniformly for a.a. $x \in \Omega$;

(iii) there exist $\tau \in ((r-p)\max\{\frac{N}{p}, 1\}, p^*)$, $\tau > q$, and $\gamma_0 > 0$ such that

$$\lim_{s\to\pm\infty} \frac{G(x,s)}{|s|^p} = +\infty \text{ and } \liminf_{s\to\pm\infty} \frac{g(x,s)s - pG(x,s)}{|s|^\tau} \geq \gamma_0$$

uniformly for a.a. $x \in \Omega$;

(iv) there exists $\delta_0 > 0$ such that $g(x,s)s \geq 0$ for a.a. $x \in \Omega$, all $s \in [-\delta_0,\delta_0]$.

Theorem 6 *Assume that* $H(g)_3$ *holds. Then, there exists* $\lambda^* > 0$ *such that, for* $\lambda \in (0, \lambda^*)$*, problem* (9) *has at least five distinct, nontrivial solutions:* $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$*,* $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$*, and* $y_\lambda \in C_0^1(\overline{\Omega})$ *sign-changing.*

Remark 1 The nodal solution y_λ in Theorem 6 satisfies the a priori estimate $\|y_\lambda\|_\infty < \delta_0$, with the constant $\delta_0 > 0$ in hypothesis $H(g)_3$ (iv). In Theorem 6, we can choose v_λ to be the biggest negative solution and u_λ the smallest positive solution, and thus we can order the solutions as $\hat{v}_\lambda \le v_\lambda \le y_\lambda \le u_\lambda \le \hat{u}_\lambda$.

Example 3 The functions $g_1(s) = |s|^{r-2}s$ for all $s \in \mathbb{R}$, with $p < r < p^*$, and $g_2(s) = |s|^{p-2}s \ln(1 + |s|^p)$ for all $s \in \mathbb{R}$ satisfy $H(g)_3$. Note that g_1 satisfies the Ambrosetti–Rabinowitz condition, but g_2 does not.

From Theorem 6 and Example 3, we have:

Corollary 1 *Assume that* $f(x, s, \lambda) = \lambda|s|^{q-2}s + |s|^{r-2}s$ *with* $1 < q < p < r < p^*$*. Then, there exists* $\lambda^* > 0$ *such that, for* $\lambda \in (0, \lambda^*)$*, problem* (1) *has at least five distinct, nontrivial solutions:* $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$*,* $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$*, and* $y_\lambda \in C_0^1(\overline{\Omega})$ *sign-changing.*

3 Preliminary Results

3.1 Upper and Lower Solutions Method

This subsection deals with a location result through the upper and lower solutions method for problem (1). The basic definition is the following.

Definition 1 Given $\lambda \in \Lambda$, we say that $u \in W^{1,p}(\Omega)$ is an *upper* (resp. *lower*) *solution* of problem (1) if $u|_{\partial\Omega} \ge 0$ (resp. $u|_{\partial\Omega} \le 0$), $f(\cdot, u(\cdot), \lambda) \in L^{q'}(\Omega)$ ($\frac{1}{q} + \frac{1}{q'} = 1$) for some $q \in (1, p^*)$, and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx - \int_\Omega f(x, u(x), \lambda)v(x)\, dx \text{ is } \ge 0 \text{ (resp. } \le 0)$$

for all $v \in W_0^{1,p}(\Omega)$ with $v \ge 0$ a.e. in Ω.

Let $\lambda \in \Lambda$, and let $\underline{u}_\lambda$ and $\overline{u}_\lambda$ be lower and upper solutions, respectively, such that $\underline{u}_\lambda(x) \le \overline{u}_\lambda(x)$ for a.a. $x \in \Omega$. We define the order interval

$$[\underline{u}_\lambda, \overline{u}_\lambda] := \{u \in W_0^{1,p}(\Omega) : \ \underline{u}_\lambda(x) \le u(x) \le \overline{u}_\lambda(x) \text{ for a.a. } x \in \Omega\}$$

and the Carathéodory function $f_{[\underline{u}_\lambda, \overline{u}_\lambda]} : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$f_{[\underline{u}_\lambda, \overline{u}_\lambda]}(x, s) = \begin{cases} f(x, \underline{u}_\lambda(x), \lambda) & \text{if } s \le \underline{u}_\lambda(x), \\ f(x, s, \lambda) & \text{if } \underline{u}_\lambda(x) < s < \overline{u}_\lambda(x), \\ f(x, \overline{u}_\lambda(x), \lambda) & \text{if } s \ge \overline{u}_\lambda(x) \end{cases} \tag{10}$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$. Setting $F_{[\underline{u}_\lambda,\overline{u}_\lambda]}(x,s) = \int_0^s f_{[\underline{u}_\lambda,\overline{u}_\lambda]}(x,t)\,dt$, we introduce the functional $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]} \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ by

$$\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega F_{[\underline{u}_\lambda,\overline{u}_\lambda]}(x,u(x))\,dx \text{ for all } u \in W_0^{1,p}(\Omega). \tag{11}$$

We start with the following location result.

Proposition 4 *Assume* $\mathrm{H}(f)_1^+$ (i) *(or* $\mathrm{H}(f)_1^-$ (i)*). Given* $\lambda \in \Lambda$*, an upper solution* $\overline{u}_\lambda$ *and a lower solution* $\underline{u}_\lambda$ *of problem* (1) *with* $\underline{u}_\lambda \le \overline{u}_\lambda$ *a.e. in* Ω*, if* $u \in W_0^{1,p}(\Omega)$ *is a critical point of* $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}$*, then* $u \in [\underline{u}_\lambda,\overline{u}_\lambda] \cap C_0^1(\overline{\Omega})$ *and* u *is a solution of* (1)*.*

Proof Let u be a critical point of $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}$, that is u solves the problem

$$\begin{cases} -\Delta_p u = f_{[\underline{u}_\lambda,\overline{u}_\lambda]}(x,u(x)) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Then, the regularity theory (see [18]) implies that $u \in C_0^1(\overline{\Omega})$.

We check that $u \in [\underline{u}_\lambda,\overline{u}_\lambda]$. Since $\underline{u}_\lambda$ is a lower solution of (1), we have in particular that $u - \underline{u}_\lambda \ge 0$ on $\partial\Omega$, hence, $(u-\underline{u}_\lambda)^- \in W_0^{1,p}(\Omega)$ (see, e.g., [9, p. 35]). Acting on (12) with the test function $(u-\underline{u}_\lambda)^-$ and using that $\underline{u}_\lambda$ is a lower solution of (1) yield

$$\begin{aligned}\int_{\{u<\underline{u}_\lambda\}} |\nabla u|^{p-2}\nabla u\cdot\nabla(u-\underline{u}_\lambda)\,dx &= \int_{\{u<\underline{u}_\lambda\}} f(x,\underline{u}_\lambda(x),\lambda)(u(x)-\underline{u}_\lambda(x))\,dx\\ &\le \int_{\{u<\underline{u}_\lambda\}} |\nabla \underline{u}_\lambda|^{p-2}\nabla\underline{u}_\lambda\cdot\nabla(u-\underline{u}_\lambda)\,dx.\end{aligned}$$

Invoking the strict monotonicity of the map $\xi \mapsto |\xi|^{p-2}\xi$ for $\xi \in \mathbb{R}^N$, we obtain that the set $\{x \in \Omega :\ u(x) < \underline{u}_\lambda(x)\}$ has Lebesgue measure zero. Thus, $\underline{u}_\lambda \le u$, a.e. in Ω. Similarly, we can show that $u \le \overline{u}_\lambda$, a.e. in Ω. Thus, $u \in [\underline{u}_\lambda,\overline{u}_\lambda]$.

Finally, we note that $u \in [\underline{u}_\lambda,\overline{u}_\lambda]$ implies that $f_{[\underline{u}_\lambda,\overline{u}_\lambda]}(x,u(x)) = f(x,u(x),\lambda)$ for a.a. $x \in \Omega$. Consequently, from (12), we conclude that u is a solution of (1). □

The next result provides existence of a solution between any lower and upper solutions.

Proposition 5 (a) *Assume* $\mathrm{H}(f)_1^+$*. Given* $\lambda \in \Lambda$*, an upper solution* $\overline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and a lower solution* $\underline{u}_\lambda \in W_0^{1,p}(\Omega)$ *of problem* (1) *with* $\overline{u}_\lambda \ge \underline{u}_\lambda \ge 0$*, a.e. in* Ω*, there exists a solution* $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *of* (1) *satisfying* $u_\lambda \in [\underline{u}_\lambda,\overline{u}_\lambda]$ *and which is a global minimizer of the functional* $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}$*.*
(b) *Assume* $\mathrm{H}(f)_1^-$*. Given* $\lambda \in \Lambda$*, a lower solution* $\underline{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and an upper solution* $\overline{v}_\lambda \in W_0^{1,p}(\Omega)$ *of* (1) *with* $\underline{v}_\lambda \le \overline{v}_\lambda \le 0$*, a.e. in* Ω*, there exists a solution* $v_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *of* (1) *satisfying* $v_\lambda \in [\underline{v}_\lambda,\overline{v}_\lambda]$ *and which is a global minimizer of the functional* $\varphi_{[\underline{v}_\lambda,\overline{v}_\lambda]}$*.*

Proof By $H(f)_1^+$ and using that $\overline{u}_\lambda, \underline{u}_\lambda \in L^\infty(\Omega)$, we have $|f_{[\underline{u}_\lambda,\overline{u}_\lambda]}(x,s)| \leq c_\lambda$ for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, all $\lambda \in \Lambda$, with $c_\lambda > 0$. Using the continuity of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$, we obtain

$$\varphi_{[u_\lambda,\overline{u}_\lambda]}(u) \geq \frac{1}{p}\,\|\nabla u\|_p^p - c_\lambda\|u\|_1 \geq \frac{1}{p}\,\|\nabla u\|_p^p - \tilde{c}_\lambda\|\nabla u\|_p \text{ for all } u\in W_0^{1,p}(\Omega),$$

for some constant $\tilde{c}_\lambda > 0$. Hence, $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}$ is coercive. Since $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}$ is also sequentially weakly lower semicontinuous, it has a global minimizer $u_\lambda \in W_0^{1,p}(\Omega)$. Hence, u_λ is a critical point of $\varphi_{[\underline{u}_\lambda,\overline{u}_\lambda]}$, and so, by Proposition 4, we have $u_\lambda \in [\underline{u}_\lambda, \overline{u}_\lambda] \cap C_0^1(\overline{\Omega})$ and u_λ is a solution of (1).

Let us justify that $u_\lambda \neq 0$. It suffices to check this when $\underline{u}_\lambda = 0$. Letting $\eta_\lambda \in L^\infty(\Omega)_+$ be as in hypothesis $H(f)_1^+$ (ii), we have that

$$\gamma := \lambda_1 - \int_\Omega \eta_\lambda(x)\hat{u}_1(x)^p\,dx = \int_\Omega (\lambda_1 - \eta_\lambda(x))\hat{u}_1(x)^p\,dx < 0.$$

From $H(f)_1^+$ (ii) we know that, for each $\varepsilon \in (0,-\gamma)$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{p}\,(\eta_\lambda(x) - \varepsilon)s^p \leq \int_0^s f(x,t,\lambda)\,dt \text{ for a.a. } x \in \Omega, \text{all s } \in [0,\delta).$$

Since $\overline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, we can find $t \in (0, \frac{\delta}{\|\hat{u}_1\|_\infty})$ such that $0 < t\hat{u}_1(x) \leq \overline{u}_\lambda(x)$ for all $x \in \Omega$. Then, by (11), we have

$$\varphi_{[0,\overline{u}_\lambda]}(t\hat{u}_1) \leq \frac{\lambda_1 t^p}{p} - \frac{t^p}{p}\int_\Omega (\eta_\lambda(x) - \varepsilon)\hat{u}_1(x)^p\,dx \leq \frac{t^p}{p}\,(\gamma + \varepsilon) < 0 = \varphi_{[0,\overline{u}_\lambda]}(0).$$

As u_λ is a global minimizer of $\varphi_{[0,\overline{u}_\lambda]}$, we deduce that $u_\lambda \neq 0$.

Recalling that $u_\lambda \geq 0$, from $H(f)_1^+$, we find a constant $c_0(\lambda) > 0$ such that

$$\Delta_p u_\lambda = -f(\cdot,u_\lambda,\lambda) \leq c_0(\lambda)u_\lambda^{p-1} \text{ in } W^{-1,p'}(\Omega). \tag{13}$$

Then, by the strong maximum principle (see [24]), we conclude that $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. This proves part (a) of Proposition 5(a). Part (b) can be established similarly. □

3.2 Antimaximum Principle

This subsection is devoted to a version of the antimaximum principle for the p-Laplacian operator with weight, which we will need in the proof of Proposition 2. This result is related to the following eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda\xi(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Here, $\xi \in L^\infty(\Omega)_+ \setminus \{0\}$. Let $\hat{\lambda}_1(\xi) > 0$ be the first eigenvalue for problem (14). The next result is due to Motreanu–Motreanu–Papageorgiou [19] and states that the antimaximum principle of Godoy–Gossez–Paczka [14, Theorem 5.1, Remark 5.5] holds L^∞-locally uniformly with respect to the weight.

Theorem 7 *Given $\xi, h \in L^\infty(\Omega)_+ \setminus \{0\}$, there is a number $\delta > 0$ such that, if $\zeta \in L^\infty(\Omega)_+ \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy $\|\zeta - \xi\|_\infty < \delta$ and $\hat{\lambda}_1(\zeta) < \lambda < \hat{\lambda}_1(\zeta) + \delta$, then any weak solution of the problem*

$$\begin{cases} -\Delta_p u = \lambda \zeta(x)|u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $-\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$.

Proof Arguing by contradiction, assume that there exist sequences $\{\zeta_n\}_{n\geq 1} \subset L^\infty(\Omega)_+$ with $\zeta_n \to \xi$ uniformly on Ω, $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}$ with $\hat{\lambda}_1(\zeta_n) < \lambda_n < \hat{\lambda}_1(\zeta_n) + \frac{1}{n}$, and $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta_p u_n = \lambda_n \zeta_n(x)|u_n|^{p-2}u_n + h(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{15}$$

and $u_n \notin -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$. If $\{u_n\}_{n\geq 1}$ were bounded in $L^\infty(\Omega)$ (note that $u_n \in L^\infty(\Omega)$ by the Moser iteration technique), then due to the a priori elliptic estimates (see [18]), $\{u_n\}_{n\geq 1}$ would be bounded in $C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$, so along a subsequence, $u_n \to u$ in $C^1(\overline{\Omega})$, with $u \in C^1(\overline{\Omega})$ solving

$$\begin{cases} -\Delta_p u = \hat{\lambda}_1(\xi)\xi(x)|u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

contradicting [14, Proposition 4.3, Remark 5.5]. Thus, along a relabeled subsequence, we have that $\|u_n\|_\infty \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_\infty}$. By (15), we have that

$$\begin{cases} -\Delta_p v_n = \lambda_n \zeta_n(x)|v_n|^{p-2}v_n + \frac{h(x)}{\|u_n\|_\infty^{p-1}} & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

The sequence $\{v_n\}_{n\geq 1}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ (by [18]), hence, up to considering a subsequence, we have $v_n \to v$ in $C^1(\overline{\Omega})$ as $n \to \infty$, for some $v \in C^1(\overline{\Omega})$. Passing to the limit in (16), we obtain

$$\begin{cases} -\Delta_p v = \hat{\lambda}_1(\xi)\xi(x)|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Moreover, $\|v\|_\infty = \lim_{n\to\infty} \|v_n\|_\infty = 1$, hence, $v \neq 0$. So, v is an eigenfunction corresponding to $\hat{\lambda}_1(\xi)$, and therefore, either $v \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ or $v \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$. The case where $v \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ cannot occur because otherwise we would have $v_n \in C_0^1(\overline{\Omega})_+$ for n large enough, but then (16) contradicts [14, Proposition 4.3, Remark 5.5]. The case $v \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ is also impossible because, as we have $v_n \to v$ in $C^1(\overline{\Omega})$, it implies that $v_n \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ for n large enough, which contradicts the assumption that $u_n \notin -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$. Thus, the proof of the theorem is complete. □

4 Proofs of Main Results

4.1 Proof of Proposition 1

The proof is based on Proposition 5 and the following lemmas.

Lemma 1 *There exists $e \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ such that $-\Delta_p e = 1$ in $W^{-1,p'}(\Omega)$.*

Proof The operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is maximal monotone and coercive, so it is surjective. Hence, there is $e \in W_0^{1,p}(\Omega)$, $e \neq 0$, with $-\Delta_p e = 1$ in $W^{-1,p'}(\Omega)$. It follows that $\|\nabla e^-\|_p^p = \int_\Omega (-e^-)\,dx \leq 0$, thus $e \geq 0$ in Ω. By the regularity theory (see [18]) and the strong maximum principle (see [24]), we infer that $e \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$. □

Lemma 2 *For all $b > 0$, there exists $\lambda^* \in \Lambda$ such that, for all $\lambda \in (0, \lambda^*)$, there is $t_\lambda \in (0, \frac{b}{\|e\|_\infty})$ satisfying*

$$a(\lambda) + c(t_\lambda \|e\|_\infty)^{r-1} < t_\lambda^{p-1},$$

where $a(\lambda), c > 0$ and $r > p$ are as in $\mathrm{H}(f)_1^+$ (i).

Proof Arguing by contradiction, suppose that there exist $b > 0$ and a sequence $\{\lambda_n\}_{n\geq 1} \subset \Lambda$ such that $\lambda_n \to 0$ as $n \to \infty$ and

$$a(\lambda_n) + c(t\|e\|_\infty)^{r-1} \geq t^{p-1} \text{ for all } t \in (0, \frac{b}{\|e\|_\infty}), \text{all } n \geq 1.$$

Because $a(\lambda_n) \to 0$ as $n \to \infty$ (see $\mathrm{H}(f)_1^+$ (i)), letting $n \to \infty$ in the above inequality, we obtain that $c\|e\|_\infty^{r-1} t^{r-p} \geq 1$ for all $t \in (0, \frac{b}{\|e\|_\infty})$. Since $r - p > 0$, we arrive at a contradiction. □

In what follows, we fix $b > 0$.

Lemma 3 *For every $\lambda \in (0, \lambda^*)$ (with λ^* in Lemma 2), problem (1) has an upper solution $\overline{u}_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ with $\|\overline{u}_\lambda\|_\infty < b$.*

Proof Fix $\lambda \in (0, \lambda^*)$ and let $t_\lambda \in (0, \frac{b}{\|e\|_\infty})$ be given by Lemma 2. We set $\overline{u}_\lambda = t_\lambda e$. Then, $\overline{u}_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$, $\|\overline{u}_\lambda\|_\infty < b$, and we have $-\Delta_p \overline{u}_\lambda = t_\lambda^{p-1}$ in $W^{-1,p'}(\Omega)$. By Lemma 2 and hypothesis $\mathrm{H}(f)_1^+$ (i), we see that

$$-\Delta_p \overline{u}_\lambda > a(\lambda) + c\|\overline{u}_\lambda\|_\infty^{r-1} \geq f(x, s, \lambda) \text{ for a.a. } x \in \Omega, \text{ all } s \in [0, \overline{u}_\lambda(x)]. \quad (18)$$

Thus, $\overline{u}_\lambda$ is an upper solution of problem (1). □

Proof of Proposition 1 Part (a) of Proposition 1 follows by applying Proposition 5(a) with the upper solution $\overline{u}_\lambda$ and the lower solution 0. Part (b) of Proposition 1 can be similarly deduced by applying Proposition 5(b) with the upper solution 0. □

4.2 Proof of Proposition 2

We need the following property of lower and upper solutions.

Lemma 4 *Assume* $\mathrm{H}(f)_1^{\pm}$ (i) *and let* $\lambda \in \Lambda$.
(a) *If* $\overline{u}_1, \overline{u}_2 \in L^\infty(\Omega)$ *are upper solutions of problem* (1)*, then,* $\overline{u} := \min\{\overline{u}_1, \overline{u}_2\}$ *is also an upper solution of problem* (1).
(b) *If* $\underline{v}_1, \underline{v}_2 \in L^\infty(\Omega)$ *are lower solutions for problem* (1)*, then,* $\underline{v} := \max\{\underline{v}_1, \underline{v}_2\}$ *is also a lower solution of problem* (1).

Proof We only prove part (a) because part (b) can be similarly established. Given $\varepsilon > 0$, we define $\hat{\tau}_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\tau}_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s \le -\varepsilon \\ s & \text{if } -\varepsilon < s < \varepsilon \\ \varepsilon & \text{if } s \ge \varepsilon\,. \end{cases}$$

Then, for every $u \in W^{1,p}(\Omega)$, we have $\hat{\tau}_\varepsilon(u(\cdot)) \in W^{1,p}(\Omega)$ and

$$\nabla \hat{\tau}_\varepsilon(u) = \begin{cases} 0 & \text{a.e. in } \{x \in \Omega : \ |u(x)| \ge \varepsilon\}, \\ \nabla u & \text{a.e. in } \{x \in \Omega : \ |u(x)| < \varepsilon\}. \end{cases} \tag{19}$$

Let $\psi \in C_c^\infty(\Omega)$ with $\psi \ge 0$ in Ω. Since $\overline{u}_1, \overline{u}_2$ are upper solutions of (1), we have

$$\int_\Omega f(x, \overline{u}_1, \lambda)\hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\psi\, dx \le \langle -\Delta_p \overline{u}_1, \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\psi \rangle, \tag{20}$$

$$\int_\Omega f(x, \overline{u}_2, \lambda)(\varepsilon - \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-))\psi\, dx \le \langle -\Delta_p \overline{u}_2, (\varepsilon - \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-))\psi \rangle. \tag{21}$$

Moreover, in view of (19), we have

$$\begin{aligned} &\langle -\Delta_p \overline{u}_1, \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\psi \rangle + \langle -\Delta_p \overline{u}_2, (\varepsilon - \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-))\psi \rangle \\ &\le \int_\Omega |\nabla \overline{u}_1|^{p-2}(\nabla \overline{u}_1 \cdot \nabla \psi)\, \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\, dx \\ &+ \int_\Omega |\nabla \overline{u}_2|^{p-2}(\nabla \overline{u}_2 \cdot \nabla \psi)\, (\varepsilon - \hat{\tau}_\varepsilon((\overline{u}_1 - \overline{u}_2)^-))\, dx\,. \end{aligned} \tag{22}$$

Adding (20), (21) and using (22), we obtain

$$\int_\Omega f(x,\overline{u}_1,\lambda)\,\frac{1}{\varepsilon}\,\hat{\tau}_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\psi\,dx+\int_\Omega f(x,\overline{u}_2,\lambda)\left(1-\frac{1}{\varepsilon}\,\hat{\tau}_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\right)\psi\,dx$$
$$\leq \int_\Omega |\nabla\overline{u}_1|^{p-2}(\nabla\overline{u}_1\cdot\nabla\psi)\,\frac{1}{\varepsilon}\,\hat{\tau}_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\,dx$$
$$+\int_\Omega |\nabla\overline{u}_2|^{p-2}(\nabla\overline{u}_2\cdot\nabla\psi)\left(1-\frac{1}{\varepsilon}\,\hat{\tau}_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\right)dx. \tag{23}$$

Note that

$$\frac{1}{\varepsilon}\,\hat{\tau}_\varepsilon((\overline{u}_1-\overline{u}_2)^-(x))\to\chi_{\{\overline{u}_1<\overline{u}_2\}}(x) \text{ a.e. in } \Omega \text{ as } \varepsilon\downarrow 0.$$

Hence, passing to the limit as $\varepsilon\downarrow 0$ in (23), we get

$$\int_\Omega f(x,\overline{u},\lambda)\psi\,dx\leq\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\psi\,dx.$$

Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, the above inequality holds for all $\psi\in W_0^{1,p}(\Omega)$ such that $\psi\geq 0$, a.e. in Ω, which completes the proof of the lemma. □

A first step in proving Proposition 2 is to show the existence of extremal constant sign solutions between each lower solution and each upper solution of problem (1).

Proposition 6 (a) *If hypotheses* $\mathrm{H}(f)_1^+$ *hold, then for each* $\lambda\in\Lambda$, *each upper solution* $\overline{u}_\lambda\in\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and each lower solution* $\underline{u}_\lambda\in W_0^{1,p}(\Omega)$ *of* (1) *with* $\overline{u}_\lambda\geq\underline{u}_\lambda\geq 0$, *a.e. in* Ω, $\underline{u}_\lambda\neq 0$, *problem* (1) *admits a smallest solution* u_λ^* *in the ordered interval* $[\underline{u}_\lambda,\overline{u}_\lambda]$. *In addition,* $u_\lambda^*\in\operatorname{int}(C_0^1(\overline{\Omega})_+)$.
(b) *If hypotheses* $\mathrm{H}(f)_1^-$ *hold, then for each* $\lambda\in\Lambda$, *each lower solution* $\underline{v}_\lambda\in-\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *and each upper solution* $\overline{v}_\lambda\in W_0^{1,p}(\Omega)$ *of* (1) *with* $\underline{v}_\lambda\leq\overline{v}_\lambda\leq 0$, *a.e. in* Ω, $\overline{v}_\lambda\neq 0$, *problem* (1) *admits a biggest solution* $v_{\lambda,*}$ *in* $[\underline{v}_\lambda,\overline{v}_\lambda]$. *In addition,* $v_{\lambda,*}\in-\operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Proof We only prove part (a) because part (b) can be obtained similarly. Let $\lambda\in\Lambda$, and let $\overline{u}_\lambda$ and $\underline{u}_\lambda$ be as in the statement. Set

$$\mathcal{S}=\{u\in[u_\lambda,\overline{u}_\lambda]:\ u \text{ is a solution of } (1)\}.$$

By the strong maximum principle of Vázquez [24] (see (13)), since $\underline{u}_\lambda\geq 0$, $\underline{u}_\lambda\neq 0$, we have that $\mathcal{S}\subset\operatorname{int}(C_0^1(\overline{\Omega})_+)$. Moreover, $\mathcal{S}$ is nonempty (by Proposition 5). In order to show the proposition, we need to check that $\mathcal{S}$ has a smallest element. This will be done through the following claims.

Claim 1 For every $u_1,u_2\in\mathcal{S}$, there exists $u\in\mathcal{S}$ such that $u\leq u_1$ and $u\leq u_2$.

Let $u_1,u_2\in\mathcal{S}$. By virtue of Lemma 4 (a), $\hat{u}:=\min\{u_1,u_2\}\in W_0^{1,p}(\Omega)$ is an upper solution of (1). Applying Proposition 5(a) for the pair $\{\underline{u}_\lambda,\hat{u}\}$ of lower and upper solutions, we find a solution u of (1) such that $\underline{u}_\lambda\leq u\leq\hat{u}=\min\{u_1,u_2\}$. This shows Claim 1.

Claim 2 There is $\alpha\in(0,1)$ such that the set $\mathcal{S}$ is a bounded subset of $C^{1,\alpha}(\overline{\Omega})$.

The claim follows from the regularity up to the boundary result of Lieberman [18] because for each $u \in \mathcal{S}$, we have $\|u\|_\infty \le \|\overline{u}_\lambda\|_\infty$.

Let $\{x_k\}_{k\ge1}$ be a dense subset of Ω. For each $k \ge 1$, we let $m_k = \inf_{u\in\mathcal{S}} u(x_k) \ge 0$.

Claim 3 For all $n \ge 1$, there is $u_n \in \mathcal{S}$ such that

$$m_k \le u_n(x_k) \le m_k + \frac{1}{n} \text{ for all } k \in \{1, \dots, n\}.$$

By definition of m_k, we find $u_{n,1}, \dots, u_{n,n} \in \mathcal{S}$ with $u_{n,k}(x_k) \le m_k + \frac{1}{n}$ for all $k \in \{1, \dots, n\}$. By Claim 1, we can find $u_n \in \mathcal{S}$ such that $u_n \le u_{n,k}$ for all $k \in \{1, \dots, n\}$. This function u_n satisfies the Claim 3.

Let $\{u_n\}_{n\ge1} \subset \mathcal{S}$ be the sequence given in Claim 3. By Claim 2, this sequence is bounded in $C^{1,\alpha}(\overline{\Omega})$, so up to considering a subsequence we may assume that $u_n \to u_0$ in $C^1(\overline{\Omega})$ as $n \to \infty$, for some $u_0 \in C^1(\overline{\Omega})$. It is clear that $u_0 \in \mathcal{S}$. Moreover, passing to the limit as $n \to \infty$ in the inequality in Claim 3, we have $u_0(x_k) = m_k$ for all $k \ge 1$. Hence, $u_0(x_k) \le u(x_k)$ for all $k \ge 1$, all $u \in \mathcal{S}$. Since $\{x_k\}_{k\ge1}$ is dense in Ω, we deduce that $u_0 \le u$ for all $u \in \mathcal{S}$. Therefore, u_0 is the smallest element of $\mathcal{S}$. This completes the proof. □

The next step is to produce positive lower solutions and negative upper solutions.

Proposition 7 (a) *Under* $\mathrm{H}(f)_1^+$*, for each* $\lambda \in \Lambda$ *and each* $\overline{u}_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$*, there exists a lower solution* $\underline{u}_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *of* (1) *satisfying* $\overline{u}_\lambda - \underline{u}_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$*. Moreover, for every* $\varepsilon \in (0, 1)$*,* $\varepsilon\underline{u}_\lambda$ *is a lower solution of* (1)*.*
(b) *Under* $\mathrm{H}(f)_1^-$*, for each* $\lambda \in \Lambda$ *and each* $\underline{v}_\lambda \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$*, there exists an upper solution* $\overline{v}_\lambda \in -\mathrm{int}\,(C_0^1(\overline{\Omega})_+)$ *of* (1) *satisfying* $\overline{v}_\lambda - \underline{v}_\lambda \in \mathrm{int}\,(C_0^1(\overline{\Omega})_+)$*. Moreover, for every* $\varepsilon \in (0, 1)$*,* $\varepsilon\overline{v}_\lambda$ *is an upper solution of* (1)*.*

Proof Let $V := \{u \in W_0^{1,p}(\Omega) : \int_\Omega \hat{u}_1^{p-1} u\,dx = 0\}$. We have the direct sum decomposition $W_0^{1,p}(\Omega) = \mathbb{R}\hat{u}_1 \oplus V$. We claim that

$$\lambda_V := \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in V,\ u \ne 0\right\} > \lambda_1. \tag{24}$$

Indeed, arguing by contradiction, assume that there is a sequence $\{u_n\}_{n\ge1} \subset V$ such that $\|u_n\|_p = 1$ and $\|\nabla u_n\|_p^p \to \lambda_1$ as $n \to \infty$ (see (6)). Then, $\{u_n\}_{n\ge1}$ is bounded in $W_0^{1,p}(\Omega)$, so we may assume that $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$, for some $u \in W_0^{1,p}(\Omega)$. Hence, $u \in V$, $\|u\|_p = 1$, and $\|\nabla u\|_p^p \le \lambda_1$. Since the inf in (6) is attained exactly on $\{t\hat{u}_1 : t \in \mathbb{R} \setminus \{0\}\}$, we reach a contradiction with the fact that $u \in V$. This yields (24).

Let $\delta > 0$ be given by Theorem 7 applied for $h := \hat{u}_1^{p-1}$ and $\xi := \lambda_1$. For $\zeta \in L^\infty(\Omega)_+ \setminus \{0\}$, recall that $\hat{\lambda}_1(\zeta) > 0$ denotes the first eigenvalue of $-\Delta_p$ with respect to the weight ζ. Since the map $\zeta \mapsto \hat{\lambda}_1(\zeta)$ is continuous on $L^\infty(\Omega)_+ \setminus \{0\}$, we find $\varepsilon > 0$ such that for all $\zeta \in L^\infty(\Omega)$ with $\|\zeta - \lambda_1\|_\infty \le \varepsilon$ a.e. in Ω, we have $|\hat{\lambda}_1(\zeta) - 1| < \delta$. We may assume that $0 < \varepsilon < \min\{\lambda_V - \lambda_1, \lambda_2 - \lambda_1, \delta\}$. We define

the weight

$$\zeta := \min\{\eta_\lambda, \lambda_1 + \varepsilon\} \in L^\infty(\Omega)_+, \tag{25}$$

with $\eta_\lambda \in L^\infty(\Omega)_+$ as in $\mathrm{H}(f)_1^+$ (ii). Thus, $\lambda_1 \leq \zeta < \lambda_2$, a.e. in Ω, $\zeta \neq \lambda_1$, so we get

$$1 - \delta < \hat{\lambda}_1(\zeta) < \hat{\lambda}_1(\lambda_1) = 1 = \hat{\lambda}_2(\lambda_2) < \hat{\lambda}_2(\zeta), \tag{26}$$

where $\hat{\lambda}_2(\zeta) > 0$ denotes the second eigenvalue of $-\Delta_p$ with respect to ζ. Here, we use the monotonicity properties of $\hat{\lambda}_1(\cdot)$ and $\hat{\lambda}_2(\cdot)$. We consider the auxiliary boundary value problem

$$\begin{cases} -\Delta_p u = \zeta(x)|u|^{p-2}u - \hat{u}_1(x)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

The functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \zeta\,|u|^p\,dx + \int_\Omega \hat{u}_1^{p-1}u\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

is of class C^1 and its critical points are the solutions of (27).

Claim 1 φ_0 satisfies the Palais–Smale condition, that is, every sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that

$$\{\varphi_0(u_n)\}_{n\geq 1} \text{ is bounded and } \varphi_0'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \tag{28}$$

admits a strongly convergent subsequence.

Let $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ be a sequence satisfying (28). First, we show that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. Arguing by contradiction, we assume that along a subsequence $\|\nabla u_n\|_p \to +\infty$ as $n \to \infty$ and set $y_n = \frac{u_n}{\|\nabla u_n\|_p}$ for $n \geq 1$. We may suppose that $y_n \overset{\mathrm{w}}{\to} y$ in $W_0^{1,p}(\Omega)$ and $y_n \to y$ in $L^p(\Omega)$, for some $y \in W_0^{1,p}(\Omega)$. Since $\varphi_0'(u_n) \to 0$, it follows that $\langle -\Delta_p y_n, y_n - y\rangle \to 0$ as $n \to \infty$. Because $-\Delta_p$ is an operator of type $(S)_+$, we deduce that $y_n \to y$ in $W_0^{1,p}(\Omega)$, and so $\|\nabla y\|_p = 1$ and

$$-\Delta_p y = \zeta\,|y|^{p-2}y \text{ in } W^{-1,p'}(\Omega). \tag{29}$$

By (26), we infer that $y = 0$, which is a contradiction. So $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ is bounded, and along a relabeled subsequence, we have $u_n \overset{\mathrm{w}}{\to} u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$, for some $u \in W_0^{1,p}(\Omega)$. As before, we deduce that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Claim 1 is thus proved.

Claim 2 $\varphi_0|_V \geq 0$.

This claim follows from the definition of φ_0 since $\zeta(x) \leq \lambda_1 + \varepsilon < \lambda_V$, a.e. in Ω (see (25)).

Claim 3 For $t > 0$ large, we have $\varphi_0(\pm t\hat{u}_1) < 0$.

Using that $\|\hat{u}_1\|_p = 1$, for $t > 0$, we see that

$$\varphi_0(\pm t\hat{u}_1) = \frac{t^p}{p}\beta \pm t\,, \text{ where } \beta := \int_\Omega (\lambda_1 - \zeta(x))\hat{u}_1(x)^p\,dx.$$

Since $\zeta \geq \lambda_1$ a.e. in Ω, $\zeta \neq \lambda_1$ (see (25)), we have $\beta < 0$. This yields Claim 3.

Claim 4 The auxiliary problem (27) has a solution $\underline{\hat{u}} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Claims 1–3 allow us to apply the saddle point theorem (see [22]), which provides $\underline{\hat{u}} \in W_0^{1,p}(\Omega)$ such that $\varphi_0'(\underline{\hat{u}}) = 0$, thus $\underline{\hat{u}}$ is a solution of problem (27), hence, $\underline{\hat{u}} \neq 0$. Since $\|\zeta - \lambda_1\|_\infty < \delta$ (by (25) and because $0 < \varepsilon < \delta$) and $\hat{\lambda}_1(\zeta) < 1 < \hat{\lambda}_1(\zeta) + \delta$ (see (26)), we can apply Theorem 7 to the function $-\underline{\hat{u}}$, which yields that $\underline{\hat{u}} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. This establishes Claim 4.

Since $\hat{u}_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $\underline{\hat{u}} \in C_0^1(\overline{\Omega})$, we can find $t > 0$ such that

$$\hat{u}_1 - t\underline{\hat{u}} \in \operatorname{int}(C_0^1(\overline{\Omega})_+). \tag{30}$$

By (25) and hypothesis $\mathrm{H}(f)_1^+$ (ii), we can find $\tilde{\delta}_\lambda = \tilde{\delta}_\lambda(t) > 0$ such that

$$(\zeta(x) - t^{p-1})s^{p-1} \leq f(x, s, \lambda) \text{ for a.a. } x \in \Omega, \text{ all } s \in [0, \tilde{\delta}_\lambda]. \tag{31}$$

Finally, since $\overline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $\underline{\hat{u}} \in C_0^1(\overline{\Omega})$, there is $\rho_\lambda > 0$ satisfying

$$\overline{u}_\lambda - \rho_\lambda\underline{\hat{u}} \in \operatorname{int}(C_0^1(\overline{\Omega})_+) \text{ and } 0 \leq \rho_\lambda\underline{\hat{u}}(x) \leq \tilde{\delta}_\lambda \text{ for all } x \in \overline{\Omega}. \tag{32}$$

We set $\underline{u}_\lambda := \rho_\lambda\underline{\hat{u}}$. By Claim 4, we have that $\underline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, whereas (32) yields $\overline{u}_\lambda - \underline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Using (30) and (31), we infer that

$$-\Delta_p\underline{u}_\lambda = \zeta\underline{u}_\lambda^{p-1} - \rho_\lambda^{p-1}\hat{u}_1^{p-1} < (\zeta - t^{p-1})\underline{u}_\lambda^{p-1} \leq f(\cdot, \underline{u}_\lambda(\cdot), \lambda) \text{ a.e. in } \Omega. \tag{33}$$

This implies that $\underline{u}_\lambda$ is a lower solution of problem (1) (see Definition 1). Clearly, $\varepsilon\underline{u}_\lambda$ is also a lower solution of (1) for all $\varepsilon \in (0, 1)$. This proves part (a) of the proposition. The proof of part (b) proceeds in the same way. □

Proof of Proposition 2 We only prove part (a) of Proposition 2, since the proof of part (b) can be performed in a similar way. Let $b > 0$, and $\lambda^* \in \Lambda$ be given by Proposition 1(a), and let $\lambda \in (0, \lambda^*)$. By Proposition 1(a), we know that problem (1) has a solution $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ with $\|u_\lambda\|_\infty < b$. Let $\underline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the lower solution of problem (1) obtained in Proposition 7(a) applied to the upper solution (in fact solution) u_λ. We fix a sequence $\{\varepsilon_n\}_{n\geq 1} \subset (0, 1)$ converging to 0 and, for $n \geq 1$, we set $\underline{u}_{\lambda,n} = \varepsilon_n\underline{u}_\lambda$, which is also a lower solution of (1) by Proposition 7(a). Proposition 6(a) guarantees that problem (1) admits a smallest solution $u_{\lambda,n}^*$ in the ordered interval $[\underline{u}_{\lambda,n}, u_\lambda]$. From the equality $-\Delta_p u_{\lambda,n}^* = f(\cdot, u_{\lambda,n}^*(\cdot), \lambda)$, hypothesis $\mathrm{H}(f)_2^+$ (i), and the fact that $0 < u_{\lambda,n}^* \leq u_\lambda$, we see that the sequence $\{u_{\lambda,n}^*\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$, so

we may assume that $u^*_{\lambda,n} \xrightarrow{w} u_{\lambda,+}$ in $W_0^{1,p}(\Omega)$ and $u^*_{\lambda,n} \to u_{\lambda,+}$ in $L^p(\Omega)$ as $n \to \infty$, for some $u_{\lambda,+} \in W_0^{1,p}(\Omega)$. As in Claim 1 of the proof of Proposition 7, we have

$$u^*_{\lambda,n} \to u_{\lambda,+} \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{34}$$

From (34), it follows that $u_{\lambda,+}$ is a solution of (1). Moreover, up to considering a subsequence, we may assume that we have $u^*_{\lambda,n}(x) \to u_{\lambda,+}(x)$ for a.a. $x \in \Omega$. This implies that $u_{\lambda,+} \in [0, u_\lambda]$ (in particular, $\|u_{\lambda,+}\|_\infty \le \|u_\lambda\|_\infty < b$).

Claim 1 $\quad u_{\lambda,+} \neq 0$.

Arguing by contradiction, suppose that $u_{\lambda,+} = 0$. For $n \ge 1$, we set $y_n = \frac{u^*_{\lambda,n}}{\|\nabla u^*_{\lambda,n}\|_p}$. We may suppose that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(\Omega)$, $y_n \to y$ in $L^p(\Omega)$ as $n \to \infty$, for some $y \in W_0^{1,p}(\Omega)$. Denoting $h_n := \frac{f(\cdot,u^*_{\lambda,n}(\cdot),\lambda)}{\|\nabla u^*_{\lambda,n}\|_p^{p-1}}$, we have

$$-\Delta_p y_n = h_n \text{ in } W^{-1,p'}(\Omega) \text{ for all } n \ge 1. \tag{35}$$

Hypothesis $\mathrm{H}(f)_2^+$ implies that there exists $c_0(\lambda) > 0$ such that

$$|f(x,s,\lambda)| \le c_0(\lambda)s^{p-1} \text{ for a.a. } x \in \Omega, \text{all } s \in [0, \|u_\lambda\|_\infty].$$

Thus, $\{h_n\}_{n\ge1}$ is bounded in $L^{p'}(\Omega)$. Therefore, acting on (35) with the test function $y_n - y \in W_0^{1,p}(\Omega)$, we obtain $\lim_{n\to\infty} \langle -\Delta_p y_n, y_n - y\rangle = 0$, thereby $y_n \to y$ in $W_0^{1,p}(\Omega)$ (because $-\Delta_p$ is an operator of type $(S)_+$) and $\|\nabla y\|_p = 1$. Since $y_n(x) \to y(x)$ for a.a. $x \in \Omega$ (at least along a subsequence), we have $y \ge 0$, a.e. in Ω, $y \neq 0$.

Since $\{h_n\}_{n\ge1}$ is bounded in $L^{p'}(\Omega)$, we may assume that $h_n \xrightarrow{w} h$ in $L^{p'}(\Omega)$, for some $h \in L^{p'}(\Omega)$. For a while, we fix $\varepsilon > 0$. Then, hypothesis $\mathrm{H}(f)_2^+$ (ii) implies that for a.a. $x \in \Omega$,

$$(\eta_\lambda(x) - \varepsilon)y_n(x)^{p-1} \le h_n(x) \le (\hat\eta_\lambda(x) + \varepsilon)y_n(x)^{p-1}$$

for n sufficiently large (recall that $u^*_{\lambda,n}(x) \to 0$ for a.a. $x \in \Omega$). Taking into account that $y_n \to y$ in $W_0^{1,p}(\Omega)$ and $h_n \xrightarrow{w} h$ in $L^{p'}(\Omega)$, invoking Mazur's theorem (see, e.g., [5, p. 61]), we obtain

$$(\eta_\lambda(x) - \varepsilon)y(x)^{p-1} \le h(x) \le (\hat\eta_\lambda(x) + \varepsilon)y(x)^{p-1} \quad \text{for a.a. } x \in \Omega.$$

As $\varepsilon > 0$ is arbitrary, we get

$$\eta_\lambda(x)y(x)^{p-1} \le h(x) \le \hat\eta_\lambda(x)y(x)^{p-1} \quad \text{for a.a. } x \in \Omega.$$

Therefore, $h(x) = \kappa(x)y(x)^{p-1}$ a.e. in Ω with $\kappa \in L^\infty(\Omega)$ such that $\eta_\lambda \le \kappa \le \hat\eta_\lambda$ a.e. in Ω. Passing to the limit as $n \to \infty$ in (35), we obtain that y solves the problem

$$\begin{cases} -\Delta_p y = \kappa y^{p-1} & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $y \neq 0$, we deduce that 1 is an eigenvalue of $-\Delta_p$ with respect to the weight κ and, since y has constant sign, we deduce that $1 = \hat{\lambda}_1(\kappa)$. On the other hand, by $H(f)_2^+$ (ii), we see that $\kappa \geq \lambda_1$ a.e. in Ω with strict inequality on a set of positive measure, hence, by virtue of the monotonicity property of $\hat{\lambda}_1(\cdot)$, we have that $\hat{\lambda}_1(\kappa) < \hat{\lambda}_1(\lambda_1) = 1$, a contradiction. This proves Claim 1.

Claim 2 For every nontrivial solution u of (1) belonging to $[0, u_\lambda]$, we have $u_{\lambda,+} \leq u$ in Ω.

Let u be a nontrivial solution of (1) belonging to $[0, u_\lambda]$. Then, we have $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ (from the regularity theory and strong maximum principle). Using that the sequence $\{\varepsilon_n\}_{n\geq 1}$ converges to 0, for n large enough, we have $\underline{u}_{\lambda,n} = \varepsilon_n \underline{u}_\lambda \leq u \leq u_\lambda$ in Ω. Since $u^*_{\lambda,n}$ is the smallest solution of (1) in $[\underline{u}_{\lambda,n}, u_\lambda]$, we derive that $u^*_{\lambda,n} \leq u$ in Ω. It follows from (34) that $u_{\lambda,+} \leq u$ in Ω, which shows Claim 2.

The proposition is obtained by combining Claims 1 and 2. □

4.3 Proof of Theorem 1

Let $b > 0$ and consider λ^* given by Proposition 1(a). Fix $\lambda \in (0, \lambda^*)$. Then, Proposition 1 shows that problem (1) admits at least two solutions $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $v_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$ such that $\|u_\lambda\|_\infty \leq b$ and $\|v_\lambda\|_\infty \leq b$. Moreover, in the case (b) of Theorem 1 (when $H(f)_3$ (ii.b) is satisfied), u_λ and v_λ can be chosen to be the smallest positive solution and the biggest negative solution of (1), respectively, given by Proposition 2.

We consider the C^1-functionals $\varphi_{[0,u_\lambda]}$, $\varphi_{[v_\lambda,0]}$, and $\varphi_{[v_\lambda,u_\lambda]}$, obtained by truncation with respect to the pairs $\{0, u_\lambda\}$, $\{v_\lambda, 0\}$, and $\{v_\lambda, u_\lambda\}$, respectively (see (11)).

By hypothesis $H(f)_3$ (ii), we find $\mu \in (\lambda_2, \theta_\lambda)$ and $\delta > 0$ such that

$$\frac{f(x,s,\lambda)}{|s|^{p-2}s} > \mu \text{ for a.a. } x \in \Omega, \text{ all } s \in [-\delta, \delta],\ s \neq 0. \tag{36}$$

For $\varepsilon > 0$ with $\varepsilon \hat{u}_1(x) \leq \min\{\delta, u_\lambda(x), -v_\lambda(x)\}$ in Ω, by (36), we see that

$$\max\{\varphi_{[v_\lambda,0]}(-\varepsilon\hat{u}_1),\ \varphi_{[0,u_\lambda]}(\varepsilon\hat{u}_1)\} < \frac{\varepsilon^p}{p}\int_\Omega (\lambda_1 - \mu)\hat{u}_1(x)^p\,dx < 0. \tag{37}$$

Note that, in case (b) of Theorem 1, the minimality of u_λ implies that $0, u_\lambda$ are the only critical points of $\varphi_{[0,u_\lambda]}$ (see Proposition 4) and similarly, $0, v_\lambda$ are the only critical points of $\varphi_{[v_\lambda,0]}$. In case (a) of Theorem 1, we may also suppose that $0, u_\lambda$ are the only critical points of $\varphi_{[0,u_\lambda]}$ and that $0, v_\lambda$ are the only critical points of $\varphi_{[v_\lambda,0]}$ (because otherwise, we deduce that there is a third nontrivial solution of problem (1) belonging either to $[0, u_\lambda]$ or to $[v_\lambda, 0]$, and we are done). From Proposition 5 and (37), we derive that

$$u_\lambda \text{ is the unique global minimizer of } \varphi_{[0,u_\lambda]} \tag{38}$$

and

$$v_\lambda \text{ is the unique global minimizer of } \varphi_{[v_\lambda,0]}. \tag{39}$$

Since the restrictions of the functionals $\varphi_{[0,u_\lambda]}$ and $\varphi_{[v_\lambda,u_\lambda]}$ to $C_0^1(\overline{\Omega})_+$ coincide, from (38), we infer that u_λ is a local minimizer of $\varphi_{[v_\lambda,u_\lambda]}$ with respect to the topology of $C_0^1(\overline{\Omega})$. Then, it turns out that u_λ is a local minimizer of $\varphi_{[v_\lambda,u_\lambda]}$ with respect to the topology of $W_0^{1,p}(\Omega)$ (see [13]). Similarly, we can see that v_λ is a local minimizer of $\varphi_{[v_\lambda,u_\lambda]}$.

Note that we may assume that v_λ, u_λ are isolated critical points of $\varphi_{[v_\lambda,u_\lambda]}$ (because otherwise we find a sequence of distinct solutions of (1) belonging to the order interval $[v_\lambda, u_\lambda]$, so in case (a) of Theorem 1, we infer the existence of a third nontrivial solution $y_\lambda \in [v_\lambda, u_\lambda]$ of (1) whereas in case (b) of Theorem 1 the extremality of v_λ and u_λ implies that y_λ is sign changing).

From Proposition 5, we know that $\varphi_{[v_\lambda,u_\lambda]}$ has a global minimizer $z_\lambda \in [v_\lambda, u_\lambda]$ and we have $\varphi_{[v_\lambda,u_\lambda]}(z_\lambda) < 0$ (see (37)), hence, $z_\lambda \neq 0$. If $z_\lambda \neq u_\lambda$ and $z_\lambda \neq v_\lambda$, then z_λ is the third desired solution of (1) (sign changing in case (b) of Theorem 1).

It remains to study the case where $z_\lambda = u_\lambda$ or $z_\lambda = v_\lambda$. Say $z_\lambda = u_\lambda$ (the other case can be analogously treated). Since v_λ, u_λ are strict local minimizers of $\varphi_{[v_\lambda,u_\lambda]}$ and $\varphi_{[v_\lambda,u_\lambda]}$ satisfies the Palais–Smale condition (because it is coercive and $-\Delta_p$ is an operator of type $(S)_+$), we can apply the mountain pass theorem (see [1]) which yields a critical point $y_\lambda \in W_0^{1,p}(\Omega)$ of $\varphi_{[v_\lambda,u_\lambda]}$ satisfying

$$\varphi_{[v_\lambda,u_\lambda]}(u_\lambda) \leq \varphi_{[v_\lambda,u_\lambda]}(v_\lambda) < \varphi_{[v_\lambda,u_\lambda]}(y_\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} \varphi_{[v_\lambda,u_\lambda]}(\gamma(t))\,, \tag{40}$$

where $\Gamma = \{\gamma \in C([-1,1], W_0^{1,p}(\Omega)) \ : \ \gamma(-1) = v_\lambda,\ \gamma(1) = u_\lambda\}$. Since y_λ is a critical point of $\varphi_{[v_\lambda,u_\lambda]}$, we derive from Proposition 4 that y_λ is a solution of problem (1) belonging to $C_0^1(\overline{\Omega}) \cap [v_\lambda, u_\lambda]$ (see [18]). Clearly, (40) implies that y_λ is distinct of v_λ, u_λ. If we know that $y_\lambda \neq 0$, then y_λ is the desired third nontrivial solution of problem (1) (sign changing in case (b) in view of the extremality of v_λ, u_λ). Hence, to complete the proof of Theorem 1, it remains to check that $y_\lambda \neq 0$. To do this, we show that

$$\varphi_{[v_\lambda,u_\lambda]}(y_\lambda) < 0. \tag{41}$$

Taking (40) into account, to prove (41), it is sufficient to construct a path $\overline{\gamma}_0 \in \Gamma$ such that

$$\varphi_{[v_\lambda,u_\lambda]}(\overline{\gamma}_0(t)) < 0 \text{ for all } t \in [-1,1]. \tag{42}$$

The rest of the proof is devoted to the construction of a path $\overline{\gamma}_0 \in \Gamma$ satisfying (42).

Denote $S = \{u \in W_0^{1,p}(\Omega) : \|u\|_p = 1\}$ endowed with the $W_0^{1,p}(\Omega)$-topology and $S_C = S \cap C_0^1(\overline{\Omega})$ equipped with the $C_0^1(\overline{\Omega})$-topology. Since S_C is dense in S in the $W_0^{1,p}(\Omega)$-topology, setting $\Gamma_0 = \{\gamma \in C([-1,1], S) \ : \ \gamma(-1) = -\hat{u}_1,\ \gamma(1) = \hat{u}_1\}$

and $\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) : \gamma(-1) = -\hat{u}_1,\ \gamma(1) = \hat{u}_1\}$, we have that $\Gamma_{0,C}$ is dense in Γ_0 . Recall from [11], the following variational characterization of λ_2:

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|\nabla u\|_p^p .$$

Since $\mu > \lambda_2$ (see (36)), we can find $\hat{\gamma}_0 \in \Gamma_{0,C}$ such that

$$\max\{\|\nabla u\|_p^p : u \in \hat{\gamma}_0([-1,1])\} < \mu . \tag{43}$$

We see that there exists $\varepsilon > 0$ such that

$$\|\varepsilon u\|_\infty \leq \delta \text{ and } \varepsilon u \in [v_\lambda, u_\lambda] \text{ for all } u \in \hat{\gamma}_0([-1,1]). \tag{44}$$

Indeed, the set $\hat{\gamma}_0([-1,1])$ being compact, it is bounded in $C_0^1(\overline{\Omega})$, and so in $L^\infty(\Omega)$. Thus, we can find $\varepsilon_1 > 0$ satisfying the first inequality in (44). To show the second inequality in (44), note that for each $u \in \hat{\gamma}_0([-1,1])$, we can find a constant $\varepsilon_u > 0$ such that $-v_\lambda - \varepsilon_u u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $u_\lambda - \varepsilon_u u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ (because $-v_\lambda, u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$). Then, there exists a neighborhood $V_u \subset C_0^1(\overline{\Omega})$ such that $-v_\lambda - \varepsilon_u v \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $u_\lambda - \varepsilon_u v \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ for all $v \in V_u$. Since $\hat{\gamma}_0([-1,1])$ is compact, it is covered by a finite number $V_{u_1}, \dots, V_{u_\ell}$ of such neighborhoods. It follows that the number $\varepsilon_2 := \min\{\varepsilon_{u_1}, \dots, \varepsilon_{u_\ell}\}$ satisfies the second inequality in (44). Thus, taking $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we see that (44) holds true.

Fix $\varepsilon > 0$ as in (44). Then, from (36), (43), and since $\hat{\gamma}_0([-1,1]) \subset S$, we obtain

$$\varphi_{[v_\lambda,u_\lambda]}(\varepsilon u) \leq \frac{\varepsilon^p}{p}\|\nabla u\|_p^p - \frac{\varepsilon^p}{p}\mu\|u\|_p^p < 0 \ \text{ for all } u \in \hat{\gamma}_0([-1,1]).$$

So the path $\gamma_0 := \varepsilon\hat{\gamma}_0$ joining $-\varepsilon\hat{u}_1$ and $\varepsilon\hat{u}_1$ verifies

$$\varphi_{[v_\lambda,u_\lambda]}(u) < 0 \text{ for all } u \in \gamma_0([-1,1]). \tag{45}$$

Next we construct a path γ_+ joining $\varepsilon\hat{u}_1$ with u_λ along which $\varphi_{[v_\lambda,u_\lambda]}$ is negative. To do this, we may assume that $u_\lambda \neq \varepsilon\hat{u}_1$ (otherwise the path $\gamma_+ \equiv u_\lambda$ satisfies our requirements). Let $a = \varphi_{[0,u_\lambda]}(u_\lambda)$ and $b = \varphi_{[0,u_\lambda]}(\varepsilon\hat{u}_1)$. Note that $a < b < 0$ (see (37) and (38)). Moreover, u_λ is the only critical point of $\varphi_{[0,u_\lambda]}$ with critical value a (by (37) and (38)) and $(a,b]$ contains no critical value of $\varphi_{[0,u_\lambda]}$ (since $0, u_\lambda$ are the only critical points of $\varphi_{[0,u_\lambda]}$). These properties together with the fact that $\varphi_{[0,u_\lambda]}$ satisfies the Palais–Smale condition (because it is coercive) allow us to apply the second deformation lemma (see [10, p. 23]), which provides a continuous mapping $h : [0,1] \times \varphi^b_{[0,u_\lambda]} \to \varphi^b_{[0,u_\lambda]}$, where $\varphi^b_{[0,u_\lambda]} = \{u \in W_0^{1,p}(\Omega) : \varphi_{[0,u_\lambda]}(u) \leq b\}$, such that for all $u \in \varphi^b_{[0,u_\lambda]}$, we have

$$h(0,u) = u,\ \ h(1,u) = u_\lambda, \text{ and } \varphi_{[0,u_\lambda]}(h(t,u)) \leq \varphi_{[0,u_\lambda]}(u) \ \text{ for all } t \in [0,1]$$

(recall that $\varphi^a_{[0,u_\lambda]} = \{u_\lambda\}$, see (38)). Then, we consider the path $\gamma_+ : [0,1] \to W_0^{1,p}(\Omega)$ defined by

$$\gamma_+(t) = h(t, \varepsilon\hat{u}_1)^+ \ \text{ for all } t \in [0,1].$$

Clearly, γ_+ is continuous and we have $\gamma_+(0) = \varepsilon\hat{u}_1$ and $\gamma_+(1) = u_\lambda$. We see that

$$\varphi_{[v_\lambda,u_\lambda]}(u) < 0 \text{ for all } u \in \gamma_+([0,1]). \tag{46}$$

Indeed, let $u \in \gamma_+([0,1])$, and thus $u = h(t, \varepsilon\hat{u}_1)^+$, for some $t \in [0,1]$. Observing that $F_{[0,u_\lambda]}(-h(t,\varepsilon\hat{u}_1)^-) = 0$, we deduce that $\varphi_{[0,u_\lambda]}(u) \le \varphi_{[0,u_\lambda]}(h(t,\varepsilon\hat{u}_1))$, whence

$$\varphi_{[v_\lambda,u_\lambda]}(u) = \varphi_{[0,u_\lambda]}(u) \le \varphi_{[0,u_\lambda]}(h(t,\varepsilon\hat{u}_1)) \le \varphi_{[0,u_\lambda]}(\varepsilon\hat{u}_1) < 0,$$

where the last inequality follows from (45). Therefore, (46) holds true.

Similarly, applying the second deformation lemma to the functional $\varphi_{[v_\lambda,0]}$, we construct a path $\gamma_- : [0,1] \to W_0^{1,p}(\Omega)$ such that $\gamma_-(0) = -\varepsilon\hat{u}_1$ and $\gamma_-(1) = v_\lambda$, and satisfying

$$\varphi_{[v_\lambda,u_\lambda]}(u) < 0 \text{ for all } u \in \gamma_-([0,1]). \tag{47}$$

Concatenating the paths γ_-, γ_0, γ_+, we obtain a path $\overline{\gamma}_0 \in \Gamma$ which fulfills (42) (see (45)–(47)). This implies (41). The proof of Theorem 1 is complete.

4.4 Proof of Theorem 2

We only prove part (a) of Theorem 2, since the proof of part (b) is similar. Note that, while dealing with $\min\{b,\rho\}$ instead of b, we may assume that $b \le \rho$ where ρ is as in $\mathrm{H}(f)_4^+$ (iv).

Applying Proposition 1(a) to b yields $\lambda^* \in \Lambda$ such that, for every $\lambda \in (0,\lambda^*)$, there exists $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ solution of (1) with $\|u_\lambda\|_\infty < b$ and $u_\lambda \in [0,\overline{u}_\lambda]$, where $\overline{u}_\lambda$ is the upper solution of (1) constructed in the proof of Lemma 3.

Fix $\lambda \in (0,\lambda^*)$. Since $u_\lambda \le \overline{u}_\lambda$ in Ω, we can consider the truncation $f_{[u_\lambda,\overline{u}_\lambda]}$ and the functional $\varphi_{[u_\lambda,\overline{u}_\lambda]}$ (see (10) and (11)). Applying Proposition 5, we find $\tilde{u}_\lambda \in C_0^1(\overline{\Omega}) \cap [u_\lambda,\overline{u}_\lambda]$, global minimizer of $\varphi_{[u_\lambda,\overline{u}_\lambda]}$ and solution of (1).

We may assume that $u_\lambda = \tilde{u}_\lambda$ (otherwise $\tilde{u}_\lambda$ is a second positive solution of (1)), and thus

$$u_\lambda \text{ is a global minimizer of } \varphi_{[u_\lambda,\overline{u}_\lambda]}. \tag{48}$$

Claim 1 $\overline{u}_\lambda - u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Using $\mathrm{H}(f)_4^+$ (iv), the facts that $\|u_\lambda\|_\infty < b \le \rho$ and u_λ is a solution of (1), $\mathrm{H}(f)_4^+$ (i), the fact that $0 \le u_\lambda \le \overline{u}_\lambda$ in Ω, Lemma 2, and the construction of $\overline{u}_\lambda$ in the proof of Lemma 3, we have that

$$0 \le -\Delta_p u_\lambda = f(x,u_\lambda(x),\lambda) < t_\lambda^{p-1} = -\Delta_p \overline{u}_\lambda,$$

for some $t_\lambda \in (0, \frac{b}{\|e\|_\infty})$. Invoking [16, Proposition 2.2], Claim 1 ensues.

Now, we consider the truncation

$$\hat{f}(x,s) = \begin{cases} f(x,u_\lambda(x),\lambda) & \text{if } s \le u_\lambda(x) \\ f(x,s,\lambda) & \text{if } s > u_\lambda(x) \end{cases} \tag{49}$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, the primitive $\hat{F}(x,s) = \int_0^s \hat{f}(x,t)\,dt$, and the corresponding C^1-functional $\hat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\hat{\varphi}(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \hat{F}(x,u(x))\,dx \;\text{ for all } u \in W_0^{1,p}(\Omega),$$

which is well defined due to the growth condition in $\mathrm{H}(f)_4^+$ (i) (where $r \in (p,p^*)$).

Let us show that the functional $\hat{\varphi}$ admits a critical point $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ with $\hat{u}_\lambda \neq u_\lambda$. First, note that the functionals $\hat{\varphi}$ and $\varphi_{[u_\lambda,\overline{u}_\lambda]}$ coincide on the set

$$V := \{u \in C_0^1(\overline{\Omega}) :\; \overline{u}_\lambda - u \in \mathrm{int}\,(C_0^1(\overline{\Omega}))\},$$

which is an open subset of $C_0^1(\overline{\Omega})$. By (48), we have that u_λ is a minimizer of $\varphi_{[u_\lambda,\overline{u}_\lambda]}$ on V. Thus, u_λ is a local minimizer of $\hat{\varphi}$ with respect to the topology of $C_0^1(\overline{\Omega})$. Therefore u_λ is a local minimizer of $\hat{\varphi}$ with respect to the topology of $W_0^{1,p}(\Omega)$ (see [13]). In the case where u_λ is not a strict local minimizer of $\hat{\varphi}$, we deduce the existence of further critical points of $\hat{\varphi}$ and we are done. Hence, we may assume that

$$u_\lambda \text{ is a strict local minimizer of } \hat{\varphi}. \tag{50}$$

Claim 2 The functional $\hat{\varphi}$ satisfies the Palais–Smale condition.

Let $\{w_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ be a sequence such that $\{\hat{\varphi}(w_n)\}_{n\geq 1}$ is bounded and $\hat{\varphi}'(w_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$. Then, we have that

$$\frac{1}{p}\,\|\nabla w_n\|_p^p - \int_\Omega \hat{F}(x,w_n)\,dx \leq M_1 \text{ for all } n \geq 1, \tag{51}$$

for some $M_1 > 0$, and

$$\langle -\Delta_p w_n, v\rangle - \int_\Omega \hat{f}(x,w_n)v\,dx \leq \varepsilon_n \|\nabla v\|_p \;\text{ for all } v \in W_0^{1,p}(\Omega), \text{ all } n \geq 1, \tag{52}$$

with $\varepsilon_n \to 0$ as $n \to \infty$. Acting on (52) with $v = -w_n^- \in W_0^{1,p}(\Omega)$ and using (49), $\mathrm{H}(f)_4^+$ (iv), and the fact that $\|u_\lambda\|_\infty < b \leq \rho$, we see that

$$\|\nabla w_n^-\|_p^p \leq \|\nabla w_n^-\|_p^p + \int_\Omega \hat{f}(x,w_n)w_n^-\,dx \leq \varepsilon_n \|\nabla w_n^-\|_p \;\text{ for all } n \geq 1.$$

Since $p > 1$, it follows that $\{w_n^-\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. Let $\mu_\lambda > p$ and $M_\lambda > 0$ be as in $\mathrm{H}(f)_4^+$ (iii). Taking $v = w_n^+$ in (52), combining with (51), and using (49) and $\mathrm{H}(f)_4^+$ (i), we obtain

$$\begin{aligned}&\left(\frac{\mu_\lambda}{p} - 1\right)\|\nabla w_n^+\|_p^p + \int_{\{w_n \geq M_0\}} (f(x,w_n,\lambda)w_n - \mu_\lambda F(x,w_n,\lambda))\,dx\\ &\leq M_2(1 + \|\nabla w_n^+\|_p) \text{ for all } n \geq 1,\end{aligned}$$

with some $M_2 > 0$, where $M_0 := \max\{M_\lambda, \|u_\lambda\|_\infty\}$. From $\mathrm{H}(f)_4^+$ (iii), we obtain that $\{w_n^+\}_{n\ge1}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, $\{w_n\}_{n\ge1}$ is bounded in $W_0^{1,p}(\Omega)$, so along a relabeled subsequence we have $w_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$, $w_n \to u$ in $L^r(\Omega)$, for some $u \in W_0^{1,p}(\Omega)$. Taking $v = w_n - u$ in (52), it follows that $\langle -\Delta_p w_n, w_n - u\rangle \to 0$ as $n \to \infty$. Then, since $-\Delta_p$ is an operator of type $(S)_+$, we infer that $w_n \to u$ in $W_0^{1,p}(\Omega)$. Therefore, $\hat{\varphi}$ satisfies the Palais–Smale condition, and thus Claim 2 is established.

Claim 3 $\lim\limits_{t\to+\infty} \hat{\varphi}(t\hat{u}_1) = -\infty$.

Note that hypotheses $\mathrm{H}(f)_4^+$ (i), (iii) imply that $F(x,s,\lambda) \ge c_1 s^{\mu_\lambda} - c_2$ for a.a. $x \in \Omega$ and all $s \ge 0$, with $c_1, c_2 > 0$. Whence

$$\hat{F}(x,s) \ge c_1 s^{\mu_\lambda} - \tilde{c}_2 \quad \text{for a.a. } x \in \Omega, \text{all } s \ge 0,$$

for some $\tilde{c}_2 > 0$ (see (49)). We infer that

$$\hat{\varphi}(t\hat{u}_1) \le \frac{t^p}{p}\|\nabla \hat{u}_1\|_p^p - c_1 t^{\mu_\lambda}\|\hat{u}_1\|_{\mu_\lambda}^{\mu_\lambda} + \tilde{c}_2|\Omega|_N \to -\infty \text{ as } t \to +\infty, \tag{53}$$

where $|\Omega|_N$ denotes the Lebesgue measure of Ω. This proves Claim 3.

Combining (50) with Claims 2 and 3, we can apply the mountain pass theorem (see [1]) which yields a critical point $\hat{u}_\lambda \ne u_\lambda$ of the functional $\hat{\varphi}$. As in the proof of Proposition 4, we can show that $\hat{u}_\lambda \ge u_\lambda$, and so $\hat{u}_\lambda$ is a second positive solution of (1). The regularity theory (see [18]) implies that $\hat{u}_\lambda \in C_0^1(\overline{\Omega})$. Since $\hat{u}_\lambda \ge u_\lambda$ and $u_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, we conclude that $\hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

4.5 Proof of Theorem 4

Applying Theorem 1(b) with $b := \min\{\rho_+, |\rho_-|\}$, we find $\lambda^* \in \Lambda$ such that, for $\lambda \in (0, \lambda^*)$, problem (1) admits five solutions $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C_0^1(\overline{\Omega})_+)$, $y_\lambda \in C_0^1(\overline{\Omega})$ sign-changing, and moreover $\|y_\lambda\|_\infty < b$. An additional sign-changing solution $w_\lambda \in C_0^1(\overline{\Omega})$ such that $\|w_\lambda\|_\infty \ge \max\{\rho_+, |\rho_-|\}$ is obtained from Bartsch–Liu–Weth [3, Theorem 1.1] (since hypotheses $\mathrm{H}(f)_6$ are stronger than the ones in [3, Theorem 1.1]). The fact that $\|y_\lambda\|_\infty < b \le \|w_\lambda\|_\infty$ guarantees that $y_\lambda \ne w_\lambda$. The proof of Theorem 4 is complete.

4.6 Proof of Theorem 5

We need the following preliminary result.

Lemma 5 *Let $\zeta \in L^\infty(\Omega)_+$ be such that $\zeta(x) \le \lambda_1$ for a.a. $x \in \Omega$, with strict inequality on a set of positive measure. Then, there exists a constant $c_1 > 0$ such*

that

$$\psi_\zeta(u) := \|\nabla u\|_p^p - \int_\Omega \zeta(x)|u(x)|^p\,dx \geq c_1\|\nabla u\|_p^p \;\; \textit{for all}\; u \in W_0^{1,p}(\Omega).$$

Proof From (6), we have that $\psi_\zeta \geq 0$. Arguing by contradiction, suppose that the lemma is not true. Then, we can find a sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that

$$\|\nabla u_n\|_p = 1 \;\text{ for all } n \geq 1 \;\textit{and}\; \psi_\zeta(u_n) \to 0 \text{ as } n \to \infty.$$

By passing to a relabeled subsequence if necessary, we may assume that

$$u_n \xrightarrow{\text{w}} u \,\text{in}\; W_0^{1,p}(\Omega),\;\; u_n \to u \;\;\text{in}\;\; L^p(\Omega),\;\; u_n(x) \to u(x)\,\text{a.e. in}\;\Omega,$$

and $|u_n(x)| \leq k(x)$ a.e. in Ω, for all $n \geq 1$, with some $k \in L^p(\Omega)_+$. Since

$$\|\nabla u\|_p^p \leq \liminf_{n\to\infty}\|\nabla u_n\|_p^p \;\text{ and }\; \int_\Omega \zeta(x)|u_n(x)|^p\,dx \to \int_\Omega \zeta(x)|u(x)|^p\,dx,$$

from the convergence $\psi_\zeta(u_n) \to 0$, we obtain

$$\|\nabla u\|_p^p \leq \int_\Omega \zeta(x)|u(x)|^p\,dx \leq \lambda_1\|u\|_p^p\,. \tag{54}$$

From (54) and (6), we infer that

$$\|\nabla u\|_p^p = \lambda_1\|u\|_p^p\,,\; \text{and so}\; u = t\hat{u}_1 \;\text{with}\; t \in \mathbb{R}. \tag{55}$$

If $u = 0$, from the fact that $\psi_\zeta(u_n) \to 0$ and since $\int_\Omega \zeta(x)|u_n(x)|^p\,dx \to 0$, it follows that $\|\nabla u_n\|_p \to 0$, which is a contradiction to the fact that $\|\nabla u_n\|_p = 1$ for all $n \geq 1$. Thus, $u = t\hat{u}_1$ with $t \neq 0$. Then, from the first inequality in (54) and since $\zeta < \lambda_1$ on a set of positive measure and $\hat{u}_1(x) > 0$ for all $x \in \Omega$, we deduce $\|\nabla u\|_p^p < \lambda_1\|u\|_p^p$, which contradicts (55). □

Proof of Theorem 5 Let $f(x,s) = \beta(x)|s|^{q-2}s + g(x,s)$ for a.a. $x \in \Omega$, all $s \in \mathbb{R}$. We consider the truncation $\hat{f}_+(x,s) = \beta(x)(s^+)^{q-1} + g(x,s^+)$ and the corresponding functional

$$\hat{\varphi}_+(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega \beta(x)(u^+)^q\,dx - \int_\Omega G(x,u^+)\,dx \;\;\text{for all}\; u \in W_0^{1,p}(\Omega).$$

Step 1 Every nontrivial critical point of $\hat{\varphi}_+$ is a solution of (5) belonging to $\operatorname{int}(C_0^1(\overline{\Omega})_+)$.

As in the proof of Proposition 4, we can see that a critical point $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ of $\hat{\varphi}_+$ is a solution of (5) belonging to $C_0^1(\overline{\Omega})_+$. Moreover, by $\mathrm{H}(g)_1^+$ (i), (ii) and the

boundedness of u, we have that $-\Delta_p u \geq -\tilde{c}u^{p-1}$ in $W^{-1,p'}(\Omega)$, for some $\tilde{c} > 0$. By the strong maximum principle (see [24]), it follows that $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Step 2 $\hat{\varphi}_+$ satisfies the Cerami condition, that is, every sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ satisfying

$$|\hat{\varphi}_+(u_n)| \leq M_1 \quad \text{for all } n \geq 1, \tag{56}$$

with some $M_1 > 0$, and

$$(1 + \|\nabla u_n\|_p)\hat{\varphi}_+'(u_n) \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega) \text{ as } n \to \infty \tag{57}$$

admits a strongly convergent subsequence.

Consider a sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that (56) and (57) hold. From (57), we have

$$\left| \langle -\Delta_p u_n, v\rangle - \int_\Omega \beta(x)(u_n^+)^{q-1} v\, dx - \int_\Omega g(x, u_n^+) v\, dx \right| \leq \frac{\varepsilon_n \|\nabla v\|_p}{1 + \|\nabla u_n\|_p} \tag{58}$$

for all $v \in W_0^{1,p}(\Omega)$, all $n \geq 1$, with $\varepsilon_n \to 0$. Choosing $v = -u_n^- \in W_0^{1,p}(\Omega)$ in (58), we obtain $\|\nabla u_n^-\|_p^p \leq \varepsilon_n$ for all $n \geq 1$, from which we infer that

$$u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{59}$$

Next, we show that

$$\{u_n^+\}_{n\geq 1} \text{ is bounded in } W_0^{1,p}(\Omega). \tag{60}$$

Choosing $v = u_n^+ \in W_0^{1,p}(\Omega)$ in (58), we have

$$-\|\nabla u_n^+\|_p^p + \int_\Omega \beta(x)(u_n^+)^q\, dx + \int_\Omega g(x, u_n^+)u_n^+\, dx \leq \varepsilon_n. \tag{61}$$

On the other hand, from (56), it follows that

$$\|\nabla u_n^+\|_p^p - \frac{p}{q}\int_\Omega \beta(x)(u_n^+)^q\, dx - \int_\Omega pG(x, u_n^+)\, dx \leq pM_1 \text{ for all } n \geq 1. \tag{62}$$

Adding (61) and (62), we obtain

$$\int_\Omega (g(x, u_n^+)u_n^+ - pG(x, u_n^+))\, dx \leq M_2 + \|\beta\|_\infty \left(\frac{p}{q} - 1\right) \|u_n^+\|_q^q \text{ for all } n \geq 1, \tag{63}$$

for some $M_2 > 0$. By means of hypotheses $H(g)_1^+$ (i), (iii.b), we can find constants $\gamma_1 \in (0, \gamma_0)$ and $M_3 > 0$ such that

$$\gamma_1 s^\tau - M_3 \leq g(x, s)s - pG(x, s) \text{ for a.a. } x \in \Omega, \text{ all } s \geq 0. \tag{64}$$

Using (63), (64), and the fact that $\tau > q$, we find $M_4 > 0$ such that

$$\gamma_1 \|u_n^+\|_\tau^\tau \le M_4(1 + \|u_n^+\|_\tau^q) \text{ for all } n \ge 1. \tag{65}$$

From (65) and since $\tau > q$, it follows that

$$\{u_n^+\}_{n\ge 1} \text{ is bounded in } L^\tau(\Omega). \tag{66}$$

Choosing $v = u_n^+ \in W_0^{1,p}(\Omega)$ in (58) and using $\mathrm{H}(g)_1^+$ (i) also show that

$$\|\nabla u_n^+\|_p^p \le \varepsilon_n + M_5(1 + \|u_n^+\|_q^q + \|u_n^+\|_r^r) \text{ for all } n \ge 1, \tag{67}$$

for some $M_5 > 0$. If $\tau \ge r$, then (60) follows from (66), (67), the continuity of the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and the fact that $q < p$. Thus, we may suppose that $\tau < r$. The assumption that $\tau \in ((r-p)\max\{\frac{N}{p}, 1\}, p^*)$ implies that we can always find $\ell \in (r, p^*)$ such that $\ell > \frac{p\tau}{p+\tau-r}$. Since $\tau < r < \ell$, we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{\ell}. \tag{68}$$

By the interpolation inequality (see, e.g., [5, p. 93]), we have $\|u_n^+\|_r \le \|u_n^+\|_\tau^{1-t}\,\|u_n^+\|_\ell^t$ for all $n \ge 1$. Due to (66) and the continuity of the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^\ell(\Omega)$, there is $M_6 > 0$ such that

$$\|u_n^+\|_r^r \le M_6 \|\nabla u_n^+\|_p^{tr} \text{ for all } n \ge 1. \tag{69}$$

The fact that $\ell > \frac{p\tau}{p+\tau-r}$ ensures that the number $t \in (0,1)$ from (68) satisfies $tr < p$. Taking into account (69), the continuity of the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and the fact that $q < p$, we conclude from (67) that (60) holds true.

From (59) and (60), it follows that $\{u_n\}_{n\ge 1}$ is bounded in $W_0^{1,p}(\Omega)$. Then, along a relabeled subsequence, we have

$$u_n \xrightarrow{\mathrm{w}} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{70}$$

Choosing $v = u_n - u$ in (58) and passing to the limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle = 0$. Since $-\Delta_p$ is an operator of type $(S)_+$, we deduce that $u_n \to u$ in $W_0^{1,p}(\Omega)$. This completes Step 2.

Step 3 There exists $\lambda^* > 0$ such that for $\|\beta\|_\infty < \lambda^*$ we find $\rho = \rho(\|\beta\|_\infty) > 0$ with

$$\hat{\eta}_\rho := \inf\{\hat{\varphi}_+(u) : \|\nabla u\|_p = \rho\} > 0.$$

By hypotheses $\mathrm{H}(g)_1^+$ (i), (ii), given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$G(x,s) \le \frac{1}{p}(\vartheta(x) + \varepsilon)s^p + c_\varepsilon s^r \text{ for a.a. } x \in \Omega, \text{ all } s \ge 0. \tag{71}$$

Then, using (71), Lemma 5, and (6), we have

$$\hat{\varphi}_+(u) \geq \frac{1}{p}\left(c_1 - \frac{\varepsilon}{\lambda_1}\right)\|\nabla u\|_p^p - \|\beta\|_\infty c_2 \|\nabla u\|_p^q - c_\varepsilon c_3 \|\nabla u\|_p^r \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

with $c_1, c_2, c_3 > 0$. Choosing $\varepsilon \in (0, c_1\lambda_1)$, we obtain

$$\hat{\varphi}_+(u) \geq \left(c_4 - \|\beta\|_\infty c_2 \|\nabla u\|_p^{q-p} - c_5 \|\nabla u\|_p^{r-p}\right)\|\nabla u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega), \quad (72)$$

with constants $c_4, c_5 > 0$ (depending on the choice of ε). Consider the function $\sigma : (0, +\infty) \to \mathbb{R}$ defined by

$$\sigma(t) = \|\beta\|_\infty c_2 t^{q-p} + c_5 t^{r-p} \text{ for all } t > 0. \quad (73)$$

There is a unique $t_0 > 0$ such that $\sigma(t_0) = \inf\limits_{(0,+\infty)} \sigma$, namely

$$t_0 = \left(\frac{\|\beta\|_\infty c_2(p-q)}{c_5(r-p)}\right)^{\frac{1}{r-q}}.$$

Then, estimating $\sigma(t_0)$ (from (73)), we can find $\lambda^* > 0$ such that $\sigma(t_0) < c_4$ whenever $\|\beta\|_\infty < \lambda^*$. From (72), it follows that $\inf\{\hat{\varphi}_+(u) : \|\nabla u\|_p = \rho\} > 0$ for $\rho = \rho(\|\beta\|_\infty) := t_0$. This completes the proof of Step 3.

Step 4 For every $u \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$, we have $\hat{\varphi}_+(tu) \to -\infty$ as $t \to +\infty$.

By hypotheses $\mathrm{H}(g)_1^+$ (i), (iii.a), given $M > 0$, we find $M_7 = M_7(M) > 0$ such that

$$G(x, s) \geq Ms^p - M_7 \quad \text{for a.a. } x \in \Omega, \text{all } s \geq 0.$$

Thus

$$\hat{\varphi}_+(tu) \leq \frac{t^p}{p}\|\nabla u\|_p^p - Mt^p\|u\|_p^p + M_7|\Omega|_N \text{ for all } t \geq 0,$$

where $|\Omega|_N$ denotes the Lebesgue measure of Ω. Since $M > 0$ is arbitrary, we can choose it such that $M\|u\|_p^p > \frac{1}{p}\|\nabla u\|_p^p$. The conclusion of Step 4 follows.

Step 5 $\hat{\varphi}_+$ admits a critical point $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $\hat{\varphi}_+(u_0) > 0$.

Steps 2–4 permit the application of the mountain pass theorem (see [1]), which yields $u_0 \in W_0^{1,p}(\Omega)$ critical point of $\hat{\varphi}_+$ such that

$$\hat{\varphi}_+(u_0) \geq \hat{\eta}_\rho > 0 = \hat{\varphi}_+(0).$$

This completes Step 5.

Step 6 $\hat{\varphi}_+$ admits a local minimizer $\hat{u} \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $\hat{\varphi}_+(\hat{u}) < 0$.

Let $\rho, \hat{\eta}_\rho > 0$ be as in Step 3. We consider the ball $B_\rho(0) = \{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p < \rho\}$. In view of $\mathrm{H}(g)_1^+$ (i), we know that $\inf\limits_{\overline{B_\rho(0)}} \hat{\varphi}_+ \in (-\infty, 0]$. Thus, we have $\eta_0 := \hat{\eta}_\rho - \inf\limits_{\overline{B_\rho(0)}} \hat{\varphi}_+ > 0$. Let $\varepsilon \in (0, \eta_0)$. By the Ekeland variational principle (see [12]), there exists $v_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\hat{\varphi}_+(v_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} \hat{\varphi}_+ + \varepsilon \quad (74)$$

and

$$\hat{\varphi}_+(v_\varepsilon) \le \hat{\varphi}_+(y) + \varepsilon \|\nabla(y - v_\varepsilon)\|_p \text{ for all } y \in \overline{B_\rho(0)}. \tag{75}$$

Since $\varepsilon < \eta_0$, from (74), we have $\hat{\varphi}_+(v_\varepsilon) < \hat{\eta}_\rho$, hence, $v_\varepsilon \in B_\rho(0)$. So, for any $h \in W_0^{1,p}(\Omega)$, we have that $v_\varepsilon + th \in B_\rho(0)$ whenever $t > 0$ is sufficiently small. Taking $y = v_\varepsilon + th$ in (75), dividing by t, and then letting $t \to 0$, we obtain $-\varepsilon\|\nabla h\|_p \le \langle \hat{\varphi}'_+(v_\varepsilon), h\rangle$. This establishes that

$$\|\hat{\varphi}'_+(v_\varepsilon)\| \le \varepsilon. \tag{76}$$

Consider a sequence $\varepsilon_n \downarrow 0$ and denote $u_n = v_{\varepsilon_n}$. Then, from (76), we have $\hat{\varphi}'_+(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ and also $(1 + \|\nabla u_n\|_p)\hat{\varphi}'_+(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$ (recall that $u_n \in B_\rho(0)$ for all $n \ge 1$). Step 2 implies that we may assume that $u_n \to \hat{u}$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$, for some $\hat{u} \in \overline{B_\rho(0)}$. From (74), we have

$$\hat{\varphi}_+(\hat{u}) = \inf_{\overline{B_\rho(0)}} \hat{\varphi}_+ \le 0. \tag{77}$$

Since $\inf\limits_{\partial B_\rho(0)} \hat{\varphi}_+ = \hat{\eta}_\rho > 0$, we have $\hat{u} \in B_\rho(0)$, thus $\hat{u}$ is a local minimizer of $\hat{\varphi}_+$.

We claim that

$$\inf_{\overline{B_\rho(0)}} \hat{\varphi}_+ < 0. \tag{78}$$

By virtue of hypothesis $\mathrm{H}(g)_1^+$ (ii), we can find $c_6 > 0$ and $\hat{\delta} > 0$ such that

$$G(x,s) \ge -c_6 s^p \text{ for a.a. } x \in \Omega, \text{ all } s \in [0, \hat{\delta}]. \tag{79}$$

Let $v \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ with $\|v\|_\infty \le \hat{\delta}$. Due to (79), for $t \in (0,1)$, we have

$$\hat{\varphi}_+(tv) \le \frac{t^p}{p}\|\nabla v\|_p^p - \frac{t^q}{q}\int_\Omega \beta(x) v^q\,dx + t^p c_6 \|v\|_p^p.$$

Since $q < p$, choosing $t \in (0,1)$ small, we have $\hat{\varphi}_+(tv) < 0$ and $tv \in B_\rho(0)$. This yields (78). Finally, comparing (77) and (78), we obtain that $\hat{u}$ fulfills the requirements of Step 6.

Theorem 5 follows by combining Steps 1, 5, and 6. □

4.7 *Proof of Proposition 3*

Let $e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ be the unique solution of the equation $-\Delta_p e = 1$ in $W^{-1,p'}(\Omega)$ (see Lemma 1). We fix $\varepsilon \in (0, \frac{1}{\|e\|_\infty^{p-1}})$. By $\mathrm{H}(g)_2^+$ (ii), we can find $\delta_\varepsilon \in (0, \delta_0)$ (see $\mathrm{H}(g)_2^+$ (iii)) such that

$$0 \le g(x,s) \le \varepsilon s^{p-1} \text{ for a.a. } x \in \Omega, \text{ all } s \in [0, \delta_\varepsilon]. \tag{80}$$

Set $\lambda^* = \delta_\varepsilon^{p-q}(\|e\|_\infty^{1-p} - \varepsilon) > 0$ and fix $\lambda \in (0, \lambda^*)$. It is straightforward to check that the number $\eta_\lambda := (\lambda\|e\|_\infty^{q-1}(1 - \varepsilon\|e\|_\infty^{p-1})^{-1})^{\frac{1}{p-q}}$ satisfies

$$0 < \eta_\lambda\|e\|_\infty < \delta_\varepsilon \text{ and } \lambda(\eta_\lambda\|e\|_\infty)^{q-1} + \varepsilon(\eta_\lambda\|e\|_\infty)^{p-1} = \eta_\lambda^{p-1}. \tag{81}$$

Let $\overline{u}_\lambda = \eta_\lambda e \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Then, by (80) and (81), we see that

$$-\Delta_p\overline{u}_\lambda = \eta_\lambda^{p-1} = \lambda(\eta_\lambda\|e\|_\infty)^{q-1} + \varepsilon(\eta_\lambda\|e\|_\infty)^{p-1} \geq \lambda\overline{u}_\lambda^{q-1} + g(x, \overline{u}_\lambda)$$

in $W^{-1,p'}(\Omega)$, hence, $\overline{u}_\lambda$ is an upper solution of problem (9). Moreover, we have $\|\overline{u}_\lambda\|_\infty < \delta_\varepsilon < \delta_0$.

Note that the function $f(x, s, \lambda) = \lambda|s|^{q-2}s + g(x, s)$ fulfills hypothesis $\mathrm{H}(f)_1^+$. Thus, we can apply Proposition 7(a) which yields $\underline{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ satisfying $\underline{u}_\lambda \leq \overline{u}_\lambda$ in Ω and such that $\tilde{\varepsilon}\underline{u}_\lambda$ is a lower solution of problem (9) whenever $\tilde{\varepsilon} \in (0, 1]$. Then, we fix a sequence $\{\tilde{\varepsilon}_n\}_{n\geq 1} \subset (0, 1]$ with $\tilde{\varepsilon}_n \to 0$ as $n \to \infty$ and we let $\underline{u}_{\lambda,n} = \tilde{\varepsilon}_n\underline{u}_\lambda$. From Proposition 6(a), we know that problem (9) has a smallest solution $u_{\lambda,n}^*$ in the order interval $[\underline{u}_{\lambda,n}, \overline{u}_\lambda]$ and in addition $u_{\lambda,n}^* \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Thus

$$-\Delta_p u_{\lambda,n}^* = \lambda(u_{\lambda,n}^*)^{q-1} + g(x, u_{\lambda,n}^*) \text{ in } W^{-1,p'}(\Omega), \quad \text{for all } n \geq 1. \tag{82}$$

From (82), the fact that $0 \leq u_{\lambda,n}^* \leq \overline{u} < \delta_\varepsilon$ in Ω, and (80), we see that $\{u_{\lambda,n}^*\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$, thus there is $u_{\lambda,+} \in W_0^{1,p}(\Omega)$ such that

$$u_{\lambda,n}^* \overset{\mathrm{w}}{\to} u_{\lambda,+} \text{ in } W_0^{1,p}(\Omega) \text{ and } u_{\lambda,n}^* \to u_{\lambda,+} \text{ in } L^p(\Omega) \text{ as } n \to \infty \tag{83}$$

along a relabeled subsequence. Acting on (82) with $u_{\lambda,n}^* - u_{\lambda,+} \in W_0^{1,p}(\Omega)$, then letting $n \to \infty$ and using (80) and (83), we obtain $\lim\limits_{n\to\infty}\langle -\Delta_p u_{\lambda,n}^*, u_{\lambda,n}^* - u_{\lambda,+}\rangle = 0$. Since $-\Delta_p$ is an operator of type $(S)_+$, it follows that

$$u_{\lambda,n}^* \to u_{\lambda,+} \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{84}$$

Passing to the limit in (82) and using (84), we obtain that $u_{\lambda,+}$ is a solution of (9).

We show that $u_{\lambda,+} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. To this end, note that there is $\tilde{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ such that

$$-\Delta_p\tilde{u}(x) = \lambda\tilde{u}(x)^{q-1} \text{ in } W^{-1,p'}(\Omega)$$

(see [21]). Since $u_{\lambda,n}^* \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, we know that there exists $t > 0$ such that $t\tilde{u} \leq u_{\lambda,n}^*$ in Ω. Let $t_n = \max\{t > 0 : t\tilde{u} \leq u_{\lambda,n}^* \text{ in } \Omega\}$ for all $n \geq 1$. We claim that $t_n \geq 1$ for all $n \geq 1$. Suppose that there is $n \geq 1$ with $t_n < 1$. Using $\mathrm{H}(g)_2^+$ (iii) and the fact that $0 \leq u_{\lambda,n}^* \leq \overline{u}_\lambda < \delta_0$ in Ω, we have that

$$-\Delta_p u_{\lambda,n}^* = \lambda u_{\lambda,n}^*(x)^{q-1} + g(x, u_{\lambda,n}^*(x)) \geq \lambda(t_n\tilde{u}(x))^{q-1} > \lambda t_n^{p-1}\tilde{u}(x)^{q-1} = -\Delta_p(t_n\tilde{u})$$

a.e. in Ω. Invoking [16, Proposition 2.2], we infer that $u_{\lambda,n}^* - t_n\tilde{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, which contradicts the maximality of t_n. Therefore, we obtain that $t_n \geq 1$ for all

$n \geq 1$. Hence, we have $u^*_{\lambda,n} \geq \tilde{u}$ in Ω for all $n \geq 1$. Letting $n \to \infty$, we derive that $u_{\lambda,+} \geq \tilde{u}$ in Ω. Since $\tilde{u} \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$, we deduce that $u_{\lambda,+} \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$.

Finally, we claim that $u_{\lambda,+}$ is the smallest positive solution of (9). To justify this, let $u \in W^{1,p}_0(\Omega)$ be a nontrivial solution of (9) such that $u \geq 0$ a.e. in Ω. As in Step 1 of the proof of Theorem 5, we have that $u \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$. In view of Lemma 4, we note that $\overline{u}_0 := \min\{u, \overline{u}_\lambda\}$ is an upper solution of (9). Using that $u, \overline{u}_\lambda \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$, for $n \geq 1$, large we have $\underline{u}_{\lambda,n} = \tilde{\varepsilon}_n \underline{u}_\lambda \leq \overline{u}_0$ in Ω. By Proposition 5(a), there exists a solution $\tilde{u}_n$ of (9) in the ordered interval $[\underline{u}_{\lambda,n}, \overline{u}_0]$. Since $u^*_{\lambda,n}$ is the smallest solution of (9) in $[\underline{u}_{\lambda,n}, \overline{u}_\lambda]$, it follows that $u^*_{\lambda,n} \leq \tilde{u}_n \leq \overline{u}_0 \leq u$ in Ω, which yields $u_{\lambda,+} \leq u$ in Ω. This proves the minimality of $u_{\lambda,+}$.

4.8 Proof of Theorem 6

From Theorem 5 and Proposition 3, we know that there exists $\lambda^* > 0$ such that, given $\lambda \in (0, \lambda^*)$, problem (9) admits two distinct positive solutions $u_\lambda, \hat{u}_\lambda \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$ as well as a smallest positive solution $u_{\lambda,+} \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$ with $\|u_{\lambda,+}\|_\infty < \delta_0$ (possibly equal to u_λ or $\hat{u}_\lambda$). Since the hypotheses are symmetric with respect to the origin, the same reasoning as in Theorem 5 and Proposition 3 shows that, up to choosing $\lambda^* > 0$ smaller, there exist $v_\lambda, \hat{v}_\lambda \in -\operatorname{int}(C^1_0(\overline{\Omega})_+)$ distinct solutions of (9) as well as a biggest negative solution $v_{\lambda,-} \in -\operatorname{int}(C^1_0(\overline{\Omega})_+)$ with $\|v_{\lambda,-}\|_\infty < \delta_0$. It remains to show that we can find a solution $y_\lambda \in C^1_0(\overline{\Omega})$ of (9) in the ordered interval $[v_{\lambda,-}, u_{\lambda,+}]$ distinct from $0, v_{\lambda,-}, u_{\lambda,+}$, because then the extremality property of $v_{\lambda,-}, u_{\lambda,+}$ will ensure that y_λ must be sign changing.

Recall that we denote $f(x,s,\lambda) = \lambda|s|^{q-2}s + g(x,s)$. We consider the Carathéodory function $f_{[v_{\lambda,-},u_{\lambda,+}]}$ obtained by truncation:

$$f_{[v_{\lambda,-},u_{\lambda,+}]}(x,s) = \begin{cases} \lambda|v_{\lambda,-}(x)|^{q-2}v_{\lambda,-}(x) + g(x, v_{\lambda,-}(x)) & \text{if } s < v_{\lambda,-}(x) \\ \lambda|s|^{q-2}s + g(x,s) & \text{if } v_{\lambda,-}(x) \leq s \leq u_{\lambda,+}(x) \\ \lambda u_{\lambda,+}(x)^{q-1} + g(x, u_{\lambda,+}(x)) & \text{if } s > u_{\lambda,+}(x) \end{cases}$$

and the corresponding C^1-functional $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ defined as in (11). According to Proposition 4, it suffices to show that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ admits a critical point distinct from 0, $v_{\lambda,-}$, $u_{\lambda,+}$. We may assume that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ has only a finite number of critical points (otherwise we are done).

Claim 1 $v_{\lambda,-}$ and $u_{\lambda,+}$ are strict local minimizers of $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$.

We only argue for $u_{\lambda,+}$ (the proof in the case of $v_{\lambda,-}$ is similar). Consider the truncation $\varphi_{[0,u_{\lambda,+}]}$ (see (11)). From Proposition 5(a), we know that $\varphi_{[0,u_{\lambda,+}]}$ admits a global minimizer $v \in C^1_0(\overline{\Omega}) \cap [0, u_{\lambda,+}]$. Arguing as at the end of Step 6 in the proof of Theorem 5, we can see that $\varphi_{[0,u_{\lambda,+}]}(tu_{\lambda,+}) < 0$ for $t \in (0,1)$ small, which guarantees that $v \neq 0$. By the minimality of $u_{\lambda,+}$ and Proposition 4, we get that $u_{\lambda,+} = v$ is the unique global minimizer of $\varphi_{[0,u_{\lambda,+}]}$. Since the functionals $\varphi_{[0,u_{\lambda,+}]}$ and $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$

coincide on $C_0^1(\overline{\Omega})_+$, we have that $u_{\lambda,+}$ is a local minimizer of $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ with respect to the topology of $C_0^1(\overline{\Omega})$ and so $u_{\lambda,+}$ is a local minimizer of $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ with respect to the topology of $W_0^{1,p}(\Omega)$ (see [13]). In fact, $u_{\lambda,+}$ is a strict local minimizer because $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ is assumed to have only a finite number of critical points. This proves Claim 1.

The rest of the proof relies on techniques of Morse theory based on the notion of critical groups that we recall first. Given two topological spaces $A \subset Y$ and an integer $k \geq 0$, we denote by $H_k(Y, A)$ the kth singular homology group with integer coefficients (see, e.g., [23] for the definition and the properties of the singular homology). Given a Banach space X, a functional $\varphi \in C^1(X, \mathbb{R})$, and an isolated critical point $x \in X$ of φ with $\varphi(x) = c$, the kth critical group of φ at x is defined as

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x\}),$$

where $\varphi^c = \{y \in X : \varphi(y) \leq c\}$, and $U \subset X$ is any neighborhood of x which does not contain other critical points of φ (the excision property of singular homology guarantees that the definition is independent of the choice of U).

Claim 2 There is $y_\lambda \in W_0^{1,p}(\Omega)$ critical point of $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ which is distinct from $v_{\lambda,-}$ and $u_{\lambda,+}$ such that $C_1(\varphi_{[v_{\lambda,-},u_{\lambda,+}]}, y_\lambda) \neq 0$.

Say that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(v_{\lambda,-}) \leq \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+})$ (the analysis is similar in the other situation). Since $u_{\lambda,+}$ is a strict local minimizer of $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ (see Claim 1), we can find $\rho_0 > 0$ such that

$$\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) > \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+}) \text{ for all } u \in B_{\rho_0}(u_{\lambda,+}) \setminus \{u_{\lambda,+}\}. \tag{85}$$

Then, there exists $\rho > 0$ such that for all $\rho \in (0, \rho_0)$ we have

$$\eta_\rho := \inf\{\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) : \|\nabla(u - u_{\lambda,+})\|_p = \rho\} > \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+}). \tag{86}$$

To see this, we argue by contradiction. Assume that $\eta_\rho = \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+})$ for some $\rho \in (0, \rho_0)$. It follows that we can find a sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that $\|\nabla(u_n - u_{\lambda,+})\|_p = \rho$ and $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_n) \leq \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+}) + \frac{1}{n^2}$ for all $n \geq 1$. By the Ekeland variational principle (see [12]), there is a sequence $\{v_n\}_{n\geq 1}$ such that

$$\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(v_n) \leq \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_n), \quad \|\nabla(v_n - u_n)\|_p \leq \frac{1}{n}, \text{ and } \|\varphi'_{[v_{\lambda,-},u_{\lambda,+}]}(v_n)\| \leq \frac{1}{n} \tag{87}$$

for all $n \geq 1$. For $n > \frac{1}{\rho_0 - \rho}$, we have $\|\nabla(v_n - u_{\lambda,+})\|_p \leq \|\nabla(u_n - u_{\lambda,+})\|_p + \frac{1}{n} < \rho_0$, and so $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+}) \leq \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(v_n) \leq \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_n) \leq \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+}) + \frac{1}{n^2}$. It follows that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(v_n) \to \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+})$ as $n \to \infty$. From this and the third relation in (87), since $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ satisfies the Palais–Smale condition (because it is coercive), we obtain that the sequence $\{v_n\}_{n\geq 1}$ admits a strongly convergent subsequence $\{v_{n_k}\}_{k\geq 1}$ whose limit, denoted by v_0, satisfies $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(v_0) = \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_{\lambda,+})$. Moreover, by the second relation in (87), $u_{n_k} \to v_0$ as $k \to \infty$,

hence, $\|\nabla(v_0 - u_{\lambda,+})\|_p = \rho$, which contradicts (85). This establishes (86). Now, Claim 2 follows in view of (86) (see [10, p. 90]).

Claim 3 $C_k(\varphi_{[v_{\lambda,-},u_{\lambda,+}]}, 0) = 0$ for all $k \geq 0$.

By H$(g)_3$ (i), (ii) and the boundedness of $v_{\lambda,-}$ and $u_{\lambda,+}$, we have

$$F_{[v_{\lambda,-},u_{\lambda,+}]}(x,s) := \int_0^s f_{[v_{\lambda,-},u_{\lambda,+}]}(x,t)\,dt = \frac{\lambda}{q}\,|s|^q + G(x,s) \geq \frac{\lambda}{q}\,|s|^q - c_1|s|^p$$

for a.a. $x \in \Omega$ and all $s \in [v_{\lambda,-}(x), u_{\lambda,+}(x)]$, with a constant $c_1 > 0$. Recalling that $u_{\lambda,+}, -v_{\lambda,-} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, for each $u \in W_0^{1,p}(\Omega)$, we can find $t^* = t^*(u) > 0$ such that $v_{\lambda,-}(x) \leq tu(x) \leq u_{\lambda,+}(x)$ for a.a. $x \in \Omega$ and all $t \in (0, t^*)$. Since $q < p$, corresponding to each $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, we choose $t^* = t^*(u) > 0$ smaller if necessary such that

$$\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) \leq \frac{1}{p}\,t^p\|\nabla u\|_p^p - \frac{\lambda}{q}\,t^q\|u\|_q^q + c_1 t^p\|u\|_p^p < 0 \text{ for all } t \in (0,t^*). \quad (88)$$

Setting

$$T_x(s) = \begin{cases} v_{\lambda,-}(x) & \text{if } s < v_{\lambda,-}(x) \\ s & \text{if } v_{\lambda,-}(x) \leq s \leq u_{\lambda,+}(x) \\ u_{\lambda,+}(x) & \text{if } u_{\lambda,+}(x) < s, \end{cases}$$

we note that $|T_x(s)| \leq |s|$ and

$$f_{[v_{\lambda,-},u_{\lambda,+}]}(x,s) = g(x, T_x(s)) + \lambda|T_x(s)|^{q-2}T_x(s).$$

Fix $\mu \in (q,p)$. Using hypotheses H$(g)_3$ (ii), (iv), there exist constants $c_2, c_3 > 0$ and $\delta \in (0, \delta_0)$ such that

$$\begin{aligned} \mu F_{[v_{\lambda,-},u_{\lambda,+}]}(x,s) - f_{[v_{\lambda,-},u_{\lambda,+}]}(x,s)s &\geq \lambda\left(\frac{\mu}{q} - 1\right)|T_x(s)|^q - g(x,T_x(s))T_x(s) \\ &\geq c_2|T_x(s)|^q - c_3|T_x(s)|^p \geq 0 \end{aligned}$$

for a.a. $x \in \Omega$ and all $|s| < \delta$. Then, taking into account H$(g)_3$ (i), as well as the boundedness of $v_{\lambda,-}$ and $u_{\lambda,+}$, we obtain

$$\mu F_{[v_{\lambda,-},u_{\lambda,+}]}(x,s) - f_{[v_{\lambda,-},u_{\lambda,+}]}(x,s)s \geq -c_4|s|^r \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

for a constant $c_4 > 0$. Then, for $u \in W_0^{1,p}(\Omega)$ with $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) = 0$, we have

$$\begin{aligned} \frac{d}{dt}\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu)\Big|_{t=1} &= \langle\varphi'_{[v_{\lambda,-},u_{\lambda,+}]}(u), u\rangle - \mu\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) \\ &= \left(1 - \frac{\mu}{p}\right)\|\nabla u\|_p^p + \int_\Omega (\mu F_{[v_{\lambda,-},u_{\lambda,+}]}(x,u(x)) - f_{[v_{\lambda,-},u_{\lambda,+}]}(x,u(x))u(x))\,dx \\ &\geq \left(1 - \frac{\mu}{p}\right)\|\nabla u\|_p^p - c_5\|\nabla u\|_p^r, \end{aligned}$$

with $c_5 > 0$. Since $r \in (p, p^*)$, we can find $\rho > 0$ such that

$$\frac{d}{dt}\,\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu)\Big|_{t=1} > 0 \text{ for all } u \text{ with } 0 < \|\nabla u\|_p < \rho \text{ and } \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) = 0. \tag{89}$$

We claim that

$$B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0 \text{ is contractible in itself,} \tag{90}$$

where $B_\rho(0) = \{w \in W_0^{1,p}(\Omega) : \|\nabla w\|_p < \rho\}$ and $(\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0 = \{w \in W_0^{1,p}(\Omega) : \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(w) \le 0\}$. Let $u \in W_0^{1,p}(\Omega)$ with $0 < \|\nabla u\|_p < \rho$ and $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) \le 0$. We show that

$$\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) \le 0 \text{ for all } t \in [0,1]. \tag{91}$$

Arguing indirectly, assume that there exists $t_0 \in (0,1)$ such that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(t_0 u) > 0$. Since $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) \le 0$ and $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ is continuous, we can define

$$t_1 = \min\{t \in (t_0, 1] : \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) = 0\} > t_0 > 0,$$

which results in

$$\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) > 0 \text{ for all } t \in [t_0, t_1). \tag{92}$$

Let $v = t_1 u$. We have $0 < \|\nabla v\|_p \le \|\nabla u\|_p < \rho$ and $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(v) = 0$. Therefore, by virtue of (89), we have

$$\frac{d}{dt}\,\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tv)\Big|_{t=1} > 0. \tag{93}$$

On the other hand, from (92), we have $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(t_1 u) = 0 < \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu)$ for all $t \in [t_0, t_1)$, and thus

$$\frac{d}{dt}\,\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tv)\Big|_{t=1} = t_1 \frac{d}{dt}\,\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu)\Big|_{t=t_1} = t_1 \lim_{t \uparrow t_1} \frac{\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu)}{t - t_1} \le 0. \tag{94}$$

Comparing (93) and (94), we reach a contradiction. This proves (91).

Let $h : [0,1] \times (B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0) \to B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0$ be defined by $h(t,u) = (1-t)u$. By (91), we see that h is well defined and continuous, so h is a homotopy between $h_0(0,\cdot) = \mathrm{id}_{B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0}$ and $h_0(1,\cdot) = 0$. This establishes (90).

Given $u \in B_\rho(0) \setminus \{0\}$ such that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) > 0$, we claim that there exists $t(u) \in (0,1]$ (necessarily unique) such that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(t(u)u) = 0$ and

$$\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) < 0 \text{ if } t \in (0, t(u)) \text{ and } \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) > 0 \text{ if } t \in (t(u), 1]. \tag{95}$$

Indeed, set $t(u) = \sup\{t \in (0,1] : \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) \le 0\}$. By (88), we have that $t(u) \in (0,1]$. By construction we have $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(t(u)u) = 0$ and $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) > 0$

for $t \in (t(u), 1]$, whereas (91) implies that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(tu) \leq 0$ for $t \in (0, t(u))$. If there is $\hat{t} \in (0, t(u))$ such that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(\hat{t}u) = 0$, then, using (91), we see that

$$\frac{d}{dt}\,\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(t\hat{t}u)\Big|_{t=1} = \lim_{t\downarrow 1} \frac{\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(t\hat{t}u) - \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(\hat{t}u)}{t-1} \leq 0,$$

which contradicts (89). We have shown (95).

We further set $t(u) = 1$ if $u \in B_\rho(0) \setminus \{0\}$ is such that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) \leq 0$. The so-obtained map $t : B_\rho(0) \setminus \{0\} \to (0, 1]$ is well defined.

We claim that the map $u \mapsto t(u)$ is continuous on $B_\rho(0) \setminus \{0\}$. It is sufficient to check the continuity of t on the closed subsets $\{u \in B_\rho(0) \setminus \{0\} : \ \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) \leq 0\}$ and $\{u \in B_\rho(0) \setminus \{0\} : \ \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u) \geq 0\}$ of $B_\rho(0) \setminus \{0\}$. The continuity on the first subset is immediate, so it remains to check the continuity on the second subset. Let $\{u_n\}_{n\geq 1} \subset B_\rho(0) \setminus \{0\}$ be such that $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(u_n) \geq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} u_n = u \in B_\rho(0) \setminus \{0\}$. Up to taking a subsequence, we may assume that $t(u_n) \to \bar{t} \in [0, 1]$. Assume by contradiction that $\bar{t} < t(u)$, hence fixing $\hat{t} \in (\bar{t}, t(u))$, for every $n \geq 1$ large enough, we have $t(u_n) < \hat{t}$, and so (95) implies $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(\hat{t}u_n) > 0$. Thereby, $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}(\hat{t}u) = \lim_{n\to\infty} \varphi_{[v_{\lambda,-},u_{\lambda,+}]}(\hat{t}u_n) \geq 0$, which contradicts (95). This yields $\bar{t} \geq t(u)$, and similarly we can prove that $\bar{t} \leq t(u)$, so $\bar{t} = t(u)$. This proves the continuity of $u \mapsto t(u)$ on $B_\rho(0) \setminus \{0\}$.

By the continuity of $u \mapsto t(u)$, the map $\zeta : B_\rho(0) \setminus \{0\} \to B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0 \setminus \{0\}$ defined by $\zeta(u) = t(u)u$ is a well-defined retraction. Since $W_0^{1,p}(\Omega)$ is infinite dimensional, $B_\rho(0) \setminus \{0\}$ is contractible (see [4]). From this and (90), for $\rho > 0$ small enough, we derive that

$$C_k(\varphi_{[v_{\lambda,-},u_{\lambda,+}]}, 0) = H_k(B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0, B_\rho(0) \cap (\varphi_{[v_{\lambda,-},u_{\lambda,+}]})^0 \setminus \{0\}) = 0 \quad (96)$$

for all $k \geq 1$ (see e.g., [15, p. 389]). Claim 3 ensues.

Comparing Claims 2 and 3, we obtain that y_λ is a critical point of $\varphi_{[v_{\lambda,-},u_{\lambda,+}]}$ distinct from $v_{\lambda,-}, u_{\lambda,+}, 0$. The proof of Theorem 6 is complete.

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On Strongly Convex Functions and Related Classes of Functions

Kazimierz Nikodem

Abstract Many results on strongly convex functions and related classes of functions obtained in the last few years are collected in the paper. In particular, Jensen, Hermite–Hadamard- and Fejér-type inequalities for strongly convex functions are presented. Counterparts of the classical Bernstain–Doetsch and Sierpiński theorems for strongly midconvex functions are given. New characterizations of inner product spaces involving strong convexity are obtained. A representation of strongly Wright-convex functions and a characterization of functions generating strongly Schur-convex sums are presented. Strongly n-convex and Jensen n-convex functions are investigated. Finally, a relationship between strong convexity and generalized convexity in the sense of Beckenbach is established.

Keywords Strongly convex (midconvex, Wright-convex, Schur-convex, h-convex, n-convex) function · Jensen (Hermite–Hadamard, Fejér) inequality · Inner product space · Generalized convex function

1 Introduction

Convexity is one of the most natural, fundamental, and important notions in mathematics. Convex functions were introduced by J. L. W. V. Jensen over 100 years ago and since then they were a subject of intensive investigations. There are many papers, books, and monographs devoted to the theory and various applications of convex functions (cf. e.g., [19, 20, 27, 35, 48] and the references therein).

In this paper we investigate strongly convex functions, that is functions satisfying the following condition stronger than the usual convexity.

Let $(X, \|\cdot\|)$ be a normed space, D be a convex subset of X, and c be a positive constant. A function $f : D \to \mathbb{R}$ is called:

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- *Strongly convex with modulus c* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{1}$$

  for all $x, y \in D$ and $t \in [0,1]$;
- *Strongly midconvex* (or *strongly Jensen convex) with modulus c* if (1) is assumed only for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \quad x, y \in D. \tag{2}$$

We say that f is *strongly convex* or *strongly midconvex* if it satisfies the condition (1) or (2), respectively, with some $c > 0$. The usual notions of convex and midconvex functions correspond to relations (1) and (2) with $c = 0$, respectively.

Strongly convex functions have been introduced by Polyak [44] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [22, 32, 43, 44, 48, 53, 55]).

The aim of this chapter is to collect and bring together many results on strongly convex functions and other related classes of functions obtained by the author with coauthors in the last few years in the papers [4–6, 18, 30, 31, 40, 41]. In Sect. 2 we present a support theorem and counterparts of the discrete and integral Jensen inequalities for strongly convex functions. We give also conditions under which two functions can be separated by a strongly convex function and, as a consequence, obtain a Hyers–Ulam-type stability result for strongly convex functions. In Sect. 3 we discuss properties of strongly midconvex functions. We present, in particular, some versions of the classical theorems of Bernstein–Doetsch, Ostrowski, and Sierpiński. We give also a counterpart of the theorem of Kuhn, stating that strongly t-convex functions are strongly midconvex. Section 4 contains new characterizations of inner product spaces among normed spaces involving the notion of strong convexity. In particular, it is shown that a normed space $(X, \|\cdot\|)$ is an inner product space if and only if every function $f : X \to \mathbb{R}$ strongly convex with modulus $c > 0$ is of the form $f = g + c\|\cdot\|^2$ with a convex function g. Section 5 is devoted to the Hermite–Hadamard and Fejér inequalities for strongly convex functions. In Sect. 6 we introduce, motivated by recent results of S. Varošanec, the notion of strongly h-convex functions and present a Hermite–Hadamard-type inequality for them. Section 7 is devoted to strongly Wright-convex functions. We present there an Ng-type representation theorem for such functions. In Sect. 8 we establish a relationship between strongly Wright-convex functions and the strong Schur-convexity. Referring to the classical result of Hardy, Littlewood, and Pólya, we show that strongly convex functions generate strongly Schur-convex sums and prove a counterpart of the Ng theorem on functions generating strongly Schur-convex sums. In Sect. 9 the notion of strongly n-convex functions is investigated. Relationships between such functions and n-convex functions in the sense of Popoviciu and characterizations via derivatives are presented. Some results on strongly Jensen n-convex functions are also given. Finally, in Sect. 10, a relationship between strong convexity and generalized convexity in the sense of Beckenbach is shown.

2 Strongly Convex Functions

Strongly convex functions have properties useful in optimization, mathematical economics and other branches of pure and applied mathematics. For instance, if $f : I \to \mathbb{R}$ is strongly convex, then it is bounded from below, its level sets $\{x \in I : f(x) \leq \lambda\}$ are bounded for each λ and f has a unique minimum on every closed subinterval of I (cf. [48, p. 268]). Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just "stronger versions" of known properties of convex functions. For instance, a function $f : I \to \mathbb{R}$ is strongly convex with modulus c if and only if for every $x_0 \in intI$ there exists a number $l \in \mathbb{R}$ such that

$$f(x) \geq c(x - x_0)^2 + l(x - x_0) + f(x_0), \quad x \in I, \tag{3}$$

i.e., f has a quadratic support at x_0. For differentiable f, f is strongly convex with modulus c if and only if f' is strongly increasing, i.e., $(f'(x) - f'(y))(x - y) \geq 2c(x - y)^2, \;\; x, y \in I$. For twice differentiable f, f is strongly convex with modulus c if and only if $f'' \geq 2c$ (cf. [48, p. 268]; see also [20] for counterparts of these properties in $\mathbb{R}^n$). In this section we present further properties of strongly convex functions.

We start with a useful characterization of strongly convex functions defined on a convex set $D \subset X$ in the case where X is a real inner product space (that is, the norm $\|\cdot\|$ in X is induced by an inner product: $\|x\|^2 = \langle x, x\rangle$). In the case $X = \mathbb{R}^n$ this result can be found in [20, Proposition 1.1.2].

Lemma 1 [40] *Let $(X, \|\cdot\|)$ be a real inner product space, D be a convex subset of X, and c be a positive constant. A function $f : D \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is convex.*

Proof Assume that f is strongly convex with modulus c. Using elementary properties of the inner product we get

$$\begin{aligned} g(tx + (1-t)y) &= f(tx + (1-t)y) - c\|tx + (1-t)y\|^2 \\ &\leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2 - c\|tx + (1-t)y\|^2 \\ &\leq tf(x) + (1-t)f(y) - c\big(t(1-t)(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2) \\ &+ t^2\|x\|^2 + 2t(1-t)\langle x|y\rangle + (1-t)^2\|y\|^2\big) \\ &= tf(x) + (1-t)f(y) - ct\|x\|^2 - c(1-t)\|y\|^2 \\ &= tg(x) + (1-t)g(y), \end{aligned}$$

which proves that g is convex. Conversely, if g is convex, then

$$\begin{aligned} f(tx + (1-t)y) &= g(tx + (1-t)y) + c\|tx + (1-t)y\|^2 \\ &\leq tg(x) + (1-t)g(y) + c\left(t^2\|x\|^2 + 2t(1-t)\langle x|y\rangle + (1-t)^2\|y\|^2\right) \end{aligned}$$

$$= t\left(g(x) + c\|x\|^2\right) + (1-t)\left(g(y) + c\|y\|^2\right)$$
$$- ct(1-t)\left(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2\right)$$
$$= f(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2,$$

which shows that f is strongly convex with modulus c.

Remark 1 It is shown in Sect. 4 that the assumption that $(X, \|\cdot\|)$ is an inner product space is not redundant in the above lemma. Moreover, the condition that for every $f : D \to \mathbb{R}$, f is strongly convex if and only if $f - \|\cdot\|^2$ is convex, characterizes inner product spaces among all normed spaces.

Now, recall that a function $h : D \to \mathbb{R}$ is said to be a support for the function $f : D \to \mathbb{R}$ at a point $x_0 \in D$, if $h(x_0) = f(x_0)$ and $h(x) \le f(x)$ for all $x \in D$.

As a consequence of Lemma 1 we get the following support theorem. In the case where $X = \mathbb{R}$ this result reduces to (3) and can be found in [48, p. 268].

Theorem 1 *Let $(X, \|\cdot\|)$ be a real inner product space, let D be an open convex subset of X, and let $c > 0$. A function $f : D \to \mathbb{R}$ is strongly convex with modulus c if and only if, at every point $x_0 \in D$, f has support of the form*

$$h(x) = c\|x - x_0\|^2 + L(x - x_0) + f(x_0),$$

where $L : X \to \mathbb{R}$ is a linear function (depending on x_0).

Proof Suppose that $f : D \to \mathbb{R}$ is strongly convex with modulus c and fix $x_0 \in D$. Then, by Lemma 1, there exists a convex function $g : D \to \mathbb{R}$ such that

$$f(x) = g(x) + c\|x\|^2$$

for all $x \in D$. Being convex g has support at x_0 of the form

$$h_1(x) = L_1(x - x_0) + g(x_0), \quad x \in D,$$

where $L_1 : X \to \mathbb{R}$ is a linear function. Hence, the function $h : D \to \mathbb{R}$ defined by

$$h(x) := c\|x\|^2 + L_1(x - x_0) + g(x_0)$$

supports f at x_0. Since $g(x_0) = f(x_0) - c\|x_0\|^2$, we can express h in the form

$$h(x) = c\left(\|x\|^2 - \|x_0\|^2\right) + L_1(x - x_0) + f(x_0)$$
$$= c\|x - x_0\|^2 + 2\,c\,\langle x_0, x - x_0\rangle + L_1(x - x_0) + f(x_0)$$
$$= c\|x - x_0\|^2 + L(x - x_0) + f(x_0),$$

where $L := L_1 + 2\,c\,\langle x_0, \cdot\rangle$ is also a linear function.

To prove the converse, fix arbitrary $x, y \in D$ and $t \in (0, 1)$. Put $z_0 := tx + (1-t)y$ and take a support of f at z_0 of the form

$$h(z) = c\|z - z_0\|^2 + L(z - z_0) + f(z_0), \quad z \in D.$$

Then

$$f(x) \geq c\left(\|x - z_0\|^2\right) + L(x - z_0) + f(z_0)$$

and

$$f(y) \geq c\left(\|y - z_0\|^2\right) + L(y - z_0) + f(z_0).$$

Hence

$$\begin{aligned} tf(x) + (1-t)f(y) \;\geq\; & c(t\|x - z_0\|^2 + (1-t)\|y - z_0\|^2) \\ & + t(L(x - z_0) + (1-t)L(y - z_0)) + f(z_0). \end{aligned}$$

Since

$$t\|x - z_0\|^2 + (1-t)\|y - z_0\|^2 \;=\; t(1-t)\|x - y\|^2,$$

and the linearity of L implies that

$$tL(x - z_0) + (1-t)L(y - z_0) \;=\; 0,$$

we conclude that

$$f(tx + (1-t)y) = f(z_0) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2,$$

which proves that f is strongly convex with modulus c.

Now we will present Jensen-type inequalities for strongly convex functions. Let $x_1, x_2 \in I$, $t \in [0,1]$ and $\bar{x} = tx_1 + (1-t)x_2$. Since

$$t(1-t)\|x_1 - x_2\|^2 = t\|x_1 - \bar{x}\|^2 + (1-t)\|x_2 - \bar{x}\|^2$$

we can rewrite condition (1) in the definition of strongly convex functions in the form

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - c\left(t\|x_1 - \bar{x}\|^2 + (1-t)\|x_2 - \bar{x}\|^2\right).$$

Extending this relation to convex combinations of n points we obtain the following version of the classical discrete Jensen inequality (for $X = \mathbb{R}$ see [30]).

Theorem 2 *Let $(X, \|\cdot\|)$ be a real inner product space, let D be an open convex subset of X, and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly convex with modulus c, then*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i) - c\sum_{i=1}^{n} t_i \left(x_i - \bar{x}\right)^2,$$

for all $x_1, \ldots, x_n \in D$, $t_1, \ldots, t_n > 0$ with $t_1 + \cdots + t_n = 1$ and $\bar{x} = t_1x_1 + \cdots + t_nx_n$.

Proof Fix $x_1, \ldots, x_n \in D$ and $t_1, \ldots, t_n > 0$ such that $t_1 + \cdots + t_n = 1$. Put $\bar{x} = t_1x_1 + \cdots + t_nx_n$ and take a function $g : D \to \mathbb{R}$ of the form $g(x) = c\|x - \bar{x}\|^2 + L(x - \bar{x}) + f(\bar{x})$ supporting f at $\bar{x}$. Then, for every $i = 1, \ldots, n$, we have

$$f(x_i) \geq g(x_i) = c\|x_i - \bar{x}\|^2 + a(x_i - \bar{x}) + f(\bar{x}).$$

Multiplying both sides by t_i and summing up we get

$$\sum_{i=1}^{n} t_i f(x_i) \geq c \sum_{i=1}^{n} t_i \|x_i - \bar{x}\|^2 + a \sum_{i=1}^{n} t_i (x_i - \bar{x}) + f(\bar{x}).$$

Since $\sum_{i=1}^{n} t_i (x_i - \bar{x}) = 0$, we obtain

$$f(\bar{x}) \leq \sum_{i=1}^{n} t_i f(x_i) - c \sum_{i=1}^{n} t_i \|x_i - \bar{x}\|^2,$$

which was to be proved.

In a similar way we can prove a counterpart of the integral Jensen inequality for strongly convex functions defined on $I \subset \mathbb{R}$.

Theorem 3 [30] *Let (X, Σ, μ) be a probability measure space, I be an open interval and $\varphi : X \to I$ be a Lebesgue square-integrable function. If $f : I \to \mathbb{R}$ is strongly convex with modulus c, then*

$$f\left(\int_X \varphi(x) d\mu\right) \leq \int_X f(\varphi(x)) d\mu - c \int_X (\varphi(x) - m)^2 d\mu,$$

where $m = \int_X \varphi(x) d\mu$.

Proof Put $m = \int_X \varphi(x) d\mu$ and take a function $g : I \to \mathbb{R}$ of the form $g(x) = c(x - m)^2 + l(x - m) + f(m)$ supporting f at m. Then $f(\varphi(x)) \geq g(\varphi(x))$, for all $x \in X$. Integrating both sides over X, we obtain

$$\int_X f(\varphi(x)) d\mu \geq c \int_X (\varphi(x) - m)^2 d\mu + l \int_X (\varphi(x) - m) d\mu + \int_X f(m) d\mu.$$

Hence, using the fact that

$$\int_X (\varphi(x) - m) d\mu = 0 \text{ and } \int_X f(m) d\mu = f(m),$$

we obtain

$$f(m) \leq \int_X f(\varphi(x)) d\mu - c \int_X (\varphi(x) - m)^2 d\mu,$$

which finishes the proof.

We will present also a probabilistic characterization of strong convexity obtained recently by Rajba and Wąsowicz [46]. Given a random variable X we denote by $E[X]$ and $D^2[X]$ the expected value and the variance of X, respectively (in what follows we assume that $E[X]$ and $D^2[X]$ do exist). It is known that if a function $f : I \to \mathbb{R}$ is convex then for every random variable X taking values in I

$$f(E[X]) \leq E[f(X)]. \tag{4}$$

Conversely, if (4) holds for every X, then f is convex. For strongly convex functions we have the following counterpart of this result.

Theorem 4 [46] *A function $f : I \to \mathbb{R}$ is strongly convex with modulus c if and only if*

$$f(E[X]) \leq E[f(X)] - cD^2[X] \tag{5}$$

for any random variable X taking values in I.

Proof By Lemma 1 f is strongly convex with modulus c if and only if $g(x) = f(x) - cx^2$ is convex. By (4) this is equivalent to

$$f(E[X]) - c\,(E[X])^2 \leq E[f(X)] - cE[X^2].$$

Because $E[X^2] - (E[X])^2 = D^2[X]$, the proof is finished.

Now we will present a sandwich theorem and a Hyers–Ulam stability theorem for strongly convex functions. It is proved in [7] that two functions $f, g : I \to \mathbb{R}$ can be separated by a convex function if and only if

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y), \; x, y \in I, \; t \in [0, 1].$$

The following theorem is a counterpart of that result for strongly convex functions.

Theorem 5 [30] *Let $f, g : I \to \mathbb{R}$ and $c > 0$. There exists a strongly convex function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$ on I if and only if*

$$\begin{aligned} &f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) - ct(1-t)(x-y)^2, \\ &x, y \in I, \; t \in [0, 1]. \end{aligned} \tag{6}$$

Proof The "only if" part is obvious. To prove the "if" part assume that f, g satisfy (6) and consider the functions $f_1, g_1 : I \to \mathbb{R}$ defined by

$$f_1(x) = f(x) - cx^2, \;\; g_1(x) = g(x) - cx^2, \;\; x \in I.$$

Using (6) we get

$$\begin{aligned} f_1(tx + (1-t)y) &= f(tx + (1-t)y) - c\,(tx + (1-t)y))^2 \\ &\leq tg(x) + (1-t)g(y) - ct(1-t)(x-y)^2 - c(tx + (1-t)y))^2 \\ &= tg(x) + (1-t)g(y) - ctx^2 - c(1-t)y^2 = tg_1(x) + (1-t)g_1(y), \end{aligned}$$

for all $x, y \in I, \; t \in [0, 1]$. Hence, by the Baron–Matkowski–Nikodem theorem [7], there exists a convex function $h_1 : I \to \mathbb{R}$ such that $f_1 \leq h_1 \leq g_1$ on I. Define $h(x) = h_1(x) + cx^2, \; x \in I$. Then, by Lemma 1, h is strongly convex with modulus c and $f \leq h \leq g$ on I.

As a consequence of the above sandwich theorem we obtain the following Hyers–Ulam-type stability result for strongly convex functions (see [21] for the classical

Hyers–Ulam theorem). Let $\varepsilon > 0$. We say that a function $f : I \to \mathbb{R}$ is *ε-strongly convex with modulus c* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 + \varepsilon,$$

for all $x, y \in I$, $t \in [0,1]$.

Corollary 1 [30] *If $f : I \to \mathbb{R}$ is ε-strongly convex with modulus c, then there exists a function $h : I \to \mathbb{R}$ strongly convex with modulus c such that*

$$|f(x) - h(x)| \le \frac{\varepsilon}{2},\ x \in I.$$

Proof Put $g = f + \varepsilon$. By the ε-strong convexity of f it follows that f and g satisfy (6). Hence, according to Theorem (5), there exists a function $h_1 : I \to \mathbb{R}$ strongly convex with modulus c and such that $f \le h_1 \le g = f + \varepsilon$ on I. Putting $h = h_1 - \frac{\varepsilon}{2}$, we get

$$|f(x) - h(x)| \le \frac{\varepsilon}{2},\ \ x \in I,$$

and, clearly, h is also strongly convex with modulus c.

3 Strongly Midconvex and t-Convex Functions

In this section we present some results on strongly midconvex functions. Condition (2) defining such functions appears in [48] and [54], but no properties are stated. Obviously, every strongly convex function is strongly midconvex, but not conversely. For instance, if $a : \mathbb{R} \to \mathbb{R}$ is an additive discontinuous function and $f : \mathbb{R} \to \mathbb{R}$ is given as $f(x) := a(x) + x^2$, then f is strongly midconvex with modulus 1, but it is not strongly convex (with any modulus) because it is not continuous. In the class of continuous functions, strong midconvexity is equivalent to strong convexity because of the following lemma.

Lemma 2 [6] *Let D be a convex subset of a normed space $(X, \|\cdot\|)$ and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c then*

$$f\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y\right) \le \frac{k}{2^n}f(x) + \left(1 - \frac{k}{2^n}\right)f(y) - c\frac{k}{2^n}\left(1 - \frac{k}{2^n}\right)\|x - y\|^2, \tag{7}$$

for all $x, y \in D$ and all $k, n \in \mathbb{N}$ such that $k < 2^n$.

Proof The proof is by induction on n. For $n = 1$ (7) reduces to (2). Assuming (7) to hold for some $n \in \mathbb{N}$ and all $k < 2^n$, we will prove it for $n + 1$. Fix $x, y \in D$ and take $k < 2^{n+1}$. Without loss of generality we may assume that $k < 2^n$. Then, by (2) and the induction assumption, we get

$$
\begin{aligned}
f\left(\frac{k}{2^{n+1}}x+\left(1-\frac{k}{2^{n+1}}\right)y\right) &= f\left(\frac{1}{2}\left(\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y\right)+\frac{1}{2}y\right)\\
&\le \frac{1}{2}f\left(\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y\right)+\frac{1}{2}f(y)-\frac{c}{4}\|\frac{k}{2^n}x+\left(1-\frac{k}{2^n}\right)y-y\|^2\\
&\le \frac{1}{2}\left(\frac{k}{2^n}f(x)+\left(1-\frac{k}{2^n}\right)f(y)-c\frac{k}{2^n}\left(1-\frac{k}{2^n}\right)\|x-y\|^2\right)\\
&+\frac{1}{2}f(y)-\frac{c}{4}\frac{k^2}{2^{2n}}\|x-y\|^2\\
&\le \frac{k}{2^{n+1}}f(x)+\left(1-\frac{k}{2^{n+1}}\right)f(y)-c\frac{k}{2^{n+1}}\left(1-\frac{k}{2^{n+1}}\right)\|x-y\|^2,
\end{aligned}
$$

which finishes the proof.

Since the set of dyadic numbers from $[0, 1]$ is dense in $[0, 1]$, we get the following result as an immediate consequence of Lemma 2 .

Corollary 2 [6] *Let D be a convex subset of a normed space and $c > 0$. Assume that $f : D \to \mathbb{R}$ is continuous. Then f is strongly convex with modulus c if and only if it is strongly midconvex with modulus c.*

In fact, strong convexity can be deduced from strong midconvexity under conditions formally much weaker than continuity. We present a few results of such type. They are versions of the classical theorems of Bernstein–Doetsch, Ostrowski, and Sierpiński (see [27, 48]).

Theorem 6 [6] *Let D be an open convex subset of a normed space and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c and bounded from above on a set with nonempty interior, then it is continuous and strongly convex with modulus c.*

Proof Being strongly midconvex, f is also midconvex. Since f is bounded from above on a set with nonempty interior, it is continuous in view of the Bernstein–Doetsch theorem. Consequently, by Corollary 2, it is strongly convex with modulus c.

Theorem 7 [6] *Let D be an open convex subset of $\mathbb{R}^n$ and let $c > 0$. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c and bounded from above on a set $A \subset D$ with positive Lebesgue measure, then it is continuous and strongly convex with modulus c.*

Proof Suppose that $f \le M$ on A. Since f is strongly midconvex

$$
f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2 \le M
$$

for all $x, y \in A$. This means that f is bounded from above on the set $\frac{A+A}{2}$. Since $\lambda(A) > 0$, it follows, by the classical theorem of Steinhaus (cf. [27]), that $int(\frac{A+A}{2}) \neq \emptyset$. This proves the theorem in view of Theorem 6.

Theorem 8 [6] *Let D be an open convex subset of $\mathbb{R}^n$ and let $c > 0$. If $f : D \to \mathbb{R}$ is Lebesgue measurable and strongly midconvex with modulus c, then it is continuous and strongly convex with modulus c.*

Proof For each $m \in \mathbb{N}$, define the set $A_m := \{x \in D : f(x) \leq m\}$. Since $D = \bigcup A_m$, there exists $m_0 \in \mathbb{N}$ such that $\lambda(A_{m_0}) > 0$. Hence, f is bounded from above on a set of positive Lebesgue measure, which in view of Theorem 7 completes the proof.

Let t be a fixed number in $(0, 1)$ and let $c > 0$. We say that a function $f : D \to \mathbb{R}$ is *strongly t-convex with modulus c* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2 \tag{8}$$

for all $x, y \in D$. It is known by Kuhn's Theorem [28] that t-convex functions (i.e., those that satisfy (8) with $c = 0$) are midconvex. The following result is a counterpart of that theorem for strongly t-convex functions. In the proof we apply the idea used in [12].

Theorem 9 [6] *Let D be a convex subset of a normed space X, and let $t \in (0, 1)$ be a fixed number. If $f : D \to \mathbb{R}$ is strongly t-co*e.g.*nvex with modulus c, then it is strongly midconvex with modulus c.*

Proof Fix $x, y \in D$ and put $z := \frac{x+y}{2}$.
Consider the points $u := t\,x + (1-t)\,z$ and $v := t\,z + (1-t)\,y$. Then, one can easily check that

$$z = (1-t)\,u + t\,v.$$

Applying condition (8) three times in the definition of strong t-convexity, we obtain

$$\begin{aligned}
f(z) &= (1-t)\,f(u) + t\,f(v) - c\,t\,(1-t)\,\|u - v\|^2 \\
&\leq (1-t)\,[t\,f(x) + (1-t)\,f(z) - c\,t\,(1-t)\,\|x - z\|^2] \\
&+ t[t\,f(z) + (1-t)\,f(y) - c\,t\,(1-t)\,\|z - y\|^2] \\
&- t(1-t)\,\|u - v\|^2 \\
&= t(1-t)\,[f(x) + f(y)] + \left[(1-t)^2 + t^2\right] f(z) \\
&- c\,t\,(1-t)\,[(1-t)\,\|x - z\|^2 + t\|z - y\|^2 + \|u - v\|^2],
\end{aligned}$$

and from this last inequality, after regrouping and simplifying, we get

$$2\,f(z) \leq f(x) + f(y) - c\,[(1-t)\,\|x - z\|^2 + t\|z - y\|^2 + \|u - v\|^2]. \tag{9}$$

Now, since $\|x - z\| = \|z - y\| = \|u - v\| = \frac{\|x-y\|}{2}$, we have

$$(1-t)\,\|x - z\|^2 + t\|z - y\|^2 + \|u - v\|^2 = \frac{\|x - y\|^2}{2}.$$

Consequently, inequality (9) can be written as

$$f\left(\frac{x+y}{2}\right) = f(z) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2,$$

which shows that f is strongly midconvex with modulus c. This finishes the proof.

It is well known that convex functions are characterized by having affine support at every point of their domains (see e.g., [48]). An analogous result for midconvex functions, stating that they have Jensen support (that is, an additive function plus a constant), is due to Rodé [49] (cf. also [26, 38] for simpler proofs). We present a counterpart of that result for strongly midconvex functions. In the proof we will use the following characterization of strongly midconvex functions in inner product spaces.

Lemma 3 [39] *Let X be an inner product space, let D be a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is midconvex.*

Proof Assume first that $f : D \to \mathbb{R}$ is strongly midconvex with modulus c. Define

$$g(x) := f(x) - c\|x\|^2.$$

Then, applying the Jordan–von Neumann parallelogram law, we obtain

$$\begin{aligned} g\left(\frac{x+y}{2}\right) &= f\left(\frac{x+y}{2}\right) - c\|\frac{x+y}{2}\|^2 \\ &\le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2 - \frac{c}{4}\|x+y\|^2 \\ &= \frac{f(x)+f(y)}{2} - \frac{c}{4}\left(2\|x\|^2 + 2\|y\|^2\right) \\ &= \frac{g(x)+g(y)}{2} \end{aligned}$$

which proves that g is midconvex.
The converse implication follows analogously.

Remark 2 It is shown in the next section that the assumption that $(X, \|\cdot\|)$ is an inner product space is essential in Lemma 3. Moreover, the condition that for every $f : D \to \mathbb{R}$, f is strongly midconvex if and only if $f - \|\cdot\|^2$ is midconvex, characterizes inner product spaces among all normed spaces.

Using the above lemma we obtain the following support theorem.

Theorem 10 [6] *Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space, let D be an open convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is strongly midconvex with modulus c if and only if, at every point $x_0 \in D$, f has support of the form*

$$h(x) = c\|x - x_0\|^2 + a(x - x_0) + f(x_0),$$

where $a : X \to \mathbb{R}$ is an additive function (depending on x_0).

Proof Suppose that $f : D \to \mathbb{R}$ is strongly midconvex with modulus c and fix $x_0 \in D$. Then, by Lemma 3, there exists a midconvex function $g : D \to \mathbb{R}$ such that

$$f(x) = g(x) + c\|x\|^2$$

for all $x \in D$. By Rodé's Theorem, the function g has support at x_0 of the form

$$h_1(x) = a_1(x - x_0) + g(x_0), \quad x \in D,$$

where $a_1 : X \to \mathbb{R}$ is an additive function. Hence, the function $h : D \to \mathbb{R}$ defined by

$$h(x) := c\|x\|^2 + a_1(x - x_0) + g(x_0)$$

supports f at x_0. Now, since $g(x_0) = f(x_0) - c\|x_0\|^2$, we can express h as

$$\begin{aligned} h(x) &= c\left(\|x\|^2 - \|x_0\|^2\right) + a_1(x - x_0) + f(x_0) \\ &= c\|x - x_0\|^2 + 2\,c\,\langle x_0, x - x_0\rangle + a_1(x - x_0) + f(x_0) \\ &= c\|x - x_0\|^2 + a(x - x_0) + f(x_0), \end{aligned}$$

where $a := a_1 + 2\,c\,\langle x_0, \cdot\rangle$ is also an additive function.

To prove the converse, fix arbitrary $x, y \in D$, put $z_0 := \frac{x+y}{2}$ and take a support of f at z_0 of the form

$$h(z) = c\|z - z_0\|^2 + a(z - z_0) + f(z_0), \quad z \in D.$$

Then

$$f(x) \geq c(\|x - z_0\|^2) + a(x - z_0) + f(z_0)$$

and

$$f(y) \geq c(\|y - z_0\|^2) + a(y - z_0) + f(z_0).$$

Hence

$$\frac{f(x) + f(y)}{2} \;\geq\; \frac{c}{2}\left(\|x - z_0\|^2 + \|y - z_0\|^2\right) + \frac{1}{2}(a(x - z_0) + a(y - z_0)) + f(z_0).$$

Finally, since

$$\frac{c}{2}\left(\|x - z_0\|^2 + \|y - z_0\|^2\right) \;=\; \frac{c}{4}\|x - y\|^2,$$

and the additivity of a implies that

$$a(x - z_0) + a(y - z_0) \;=\; 0,$$

we conclude that

$$f\left(\frac{x + y}{2}\right) = f(z_0) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2,$$

which proves that f is strongly midconvex with modulus c.

As an application of the above support theorem we get the following version of the Jensen inequality for strongly midconvex functions.

Theorem 11 [6] *Let D be an open and convex subset of an inner product space X. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c, then for all $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in D$:*

$$f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{c}{n}\sum_{i=1}^{n} \|x_i - s\|^2,$$

where $s = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Proof Fix $x_1, x_2, \ldots, x_n \in D$ and put $s := \frac{1}{n}\sum_{i=1}^{n} x_i$. By Theorem 3 there exists an additive function a such that f has at s support of the form

$$h(x) = c\|x - s\|^2 + a(x - s) + f(s).$$

Thus, for each $i = 1, 2, \ldots, n$,

$$f(x_i) \ge h(x_i) = c\|x_i - s\|^2 + a(x_i - s) + f(s).$$

Summing up these n inequalities, and using the fact that

$$\sum_{i=1}^{n} a(x_i - s) = a\left(\sum_{i=1}^{n} x_i - ns\right) = 0,$$

we have

$$\sum_{i=1}^{n} f(x_i) \ge c\sum_{i=1}^{n} \|x_i - s\|^2 + nf(s),$$

or

$$f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) = f(s) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{c}{n}\sum_{i=1}^{n} \|x_i - s\|^2,$$

which was to be proved.

Now we extend the above result to convex combinations with arbitrary rational coefficients.

Theorem 12 [6] *Let D be an open and convex subset of an inner product space X. If $f : D \to \mathbb{R}$ is strongly midconvex with modulus c, then*

$$f\left(\sum_{i=1}^{n} q_i x_i\right) \le \sum_{i=1}^{n} q_i f(x_i) - c\sum_{i=1}^{n} q_i\|x_i - s\|^2,$$

for all $x_1, \ldots, x_n \in D$, $q_1, \ldots, q_n \in \mathbb{Q} \cap (0,1)$ *with* $q_1 + \cdots + q_n = 1$ *and* $s = \sum_{i=1}^{n} q_i x_i$.

Proof Fix $x_1, \ldots, x_n \in D$ and $q_1 = k_1/l_1, \ldots, q_n = k_n/l_n \in \mathbb{Q} \cap (0,1)$ with $q_1 + \cdots + q_n = 1$. Without loss of generality we may assume that $l_1 = \cdots = l_n =: l$. Then $k_1 + \cdots + k_n = l$. Put $y_{11} = \cdots = y_{1k_1} =: x_1$, $y_{21} = \cdots = y_{2k_2} =: x_2, \ldots$, $y_{n1} = \cdots = y_{nk_n} =: x_n$. Then

$$s = \sum_{i=1}^{n} q_i x_i = \frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} y_{ij}.$$

Hence, using Theorem 11, we obtain

$$\begin{aligned} f\left(\sum_{i=1}^{n} q_i x_i\right) &= f\left(\frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} y_{ij}\right) \le \frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} f\left(y_{ij}\right) - \frac{c}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} \|y_{ij} - s\|^2 \\ &= \sum_{i=1}^{n} q_i f(x_i) - c \sum_{i=1}^{n} q_i \|x_i - s\|^2, \end{aligned}$$

which finishes the proof.

4 Characterizations of Inner Product Spaces Involving Strong Convexity

It is well known that in a normed space $(X, \|\cdot\|)$ the following Jordan–von Neumann parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X,$$

holds if and only if the norm $\|\cdot\|$ is derivable from an inner product. In the literature one can find many other conditions characterizing inner product spaces among normed spaces. A rich collection of such characterizations is contained in the celebrated book of D. Amir [3] (cf. also [1, Chap. 11], [2, 47]). In this section we present a new result of this type involving strongly convex and strongly midconvex functions.

We already know (see Lemmas 1 and 3) that for functions defined on a convex subset D of a real inner product space $(X, \|\cdot\|)$ the following characterization holds: A function $f : D \to \mathbb{R}$ is strongly convex (strongly midconvex) with modulus c if and only if the function $g = f - c\|\cdot\|^2$ is convex (midconvex).

The following example shows that the assumption that X is an inner product space is essential in that result.

Example 1 Let $X = \mathbb{R}^2$ and $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2)$. Take $f = \|\cdot\|^2$. Then $g = f - \|\cdot\|^2$ is convex being the zero function. However, f is neither strongly

convex nor strongly midconvex with modulus 1. Indeed, for $x = (1, 0)$ and $y = (0, 1)$ we have

$$f\left(\frac{x+y}{2}\right) = 1 > 0 = \frac{f(x) + f(y)}{2} - \frac{1}{4}\|x - y\|^2,$$

which contradicts (2).

It appears that something stronger can be proved: the assumption that X is an inner product space is necessary in Lemmas 1 and 3. Namely, the following characterizations of inner product spaces hold.

Theorem 13 [40] *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:*

1. *For all $c > 0$ and for all functions $f : D \to \mathbb{R}$, f is strongly convex with modulus c if and only if $g = f - c\|\cdot\|^2$ is convex;*
2. *For all $c > 0$ and for all functions $f : D \to \mathbb{R}$, f is strongly midconvex with modulus c if and only if $g = f - c\|\cdot\|^2$ is midconvex;*
3. *There exists $c > 0$ such that, for all functions $g : D \to \mathbb{R}$, g is convex if and only if $f = g + c\|\cdot\|^2$ is strongly convex with modulus c;*
4. *There exists $c > 0$ such that, for all functions $g : D \to \mathbb{R}$, g is midconvex if and only if $f = g + c\|\cdot\|^2$ is strongly midconvex with modulus c;*
5. *$\|\cdot\|^2 : X \to \mathbb{R}$ is strongly convex with modulus 1;*
6. *$\|\cdot\|^2 : X \to \mathbb{R}$ is strongly midconvex with modulus 1;*
7. *$(X, \|\cdot\|)$ is an inner product space.*

Proof We will show the following chains of implications: *1* $\Rightarrow$ *3* $\Rightarrow$ *5* $\Rightarrow$ *7* $\Rightarrow$ *1* and *2* $\Rightarrow$ *4* $\Rightarrow$ *6* $\Rightarrow$ *7* $\Rightarrow$ *2*.

Implications *1* $\Rightarrow$ *3* and *2* $\Rightarrow$ *4* are obvious. To show *3* $\Rightarrow$ *5* and *4* $\Rightarrow$ *6* take $g = 0$. Then $f = c\|\cdot\|^2$ is strongly convex (resp. strongly midconvex) with modulus c. Consequently, $\frac{1}{c} f = \|\cdot\|^2$ is strongly convex (resp. strongly midconvex) with Modulus 1.

To see that *5* $\Rightarrow$ *7* and *6* $\Rightarrow$ *7* also hold, observe that, by the strong convexity or strong midconvexity with modulus 1 of $\|\cdot\|^2$ we have

$$\left\|\frac{x+y}{2}\right\|^2 \le \frac{\|x\|^2 + \|y\|^2}{2} - \frac{1}{4}\|x - y\|^2$$

and hence

$$\|x + y\|^2 + \|x - y\|^2 \le 2\|x\|^2 + 2\|y\|^2 \tag{10}$$

for all $x, y \in X$. Now, putting $u = x + y$ and $v = x - y$ in (10), we get

$$2\|u\|^2 + 2\|v\|^2 \le \|u + v\|^2 + \|u - v\|^2, \qquad u, v \in X. \tag{11}$$

Conditions (10) and (11) mean that the norm $\|\cdot\|$ satisfies the parallelogram law, which implies that $(X, \|\cdot\|)$ is an inner product space.

Implications *7* $\Rightarrow$ *1* and *7* $\Rightarrow$ *2* follow by Lemmas 1 and 3.

5 Hermite–Hadamard and Fejér Inequalities

In this section we present counterparts of the classical Hermite–Hadamard and Féjer inequalities for strongly convex functions. If a function $f : I \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{12}$$

for all $a, b \in I,\ a < b$. This classical Hermite–Hadamard inequality plays an important role in convex analysis and in the theory of inequalities, and it has a huge literature dealing with its applications, various generalizations, and refinements (see for instance [9, 14, 35], and the references therein). It is also known that if f is continuous, then each of the two sides of (12) characterizes the convexity of f (cf. [10, 35]). In this section we present a counterpart of the Hermite–Hadamard inequality for strongly convex functions.

Theorem 14 [30] *If a function $f : I \to \mathbb{R}$ is strongly convex with modulus c then*

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2, \tag{13}$$

for all $a, b \in I,\ a < b$.

Proof The right-hand side of (13) (denoted by (R)) follows by integrating the inequality (1) over the interval $[0, 1]$.

To prove the left-hand side of (13) (denoted by (L)), fix $a, b \in I,\ a < b,$ and put $s = \frac{a+b}{2}$. Take a function $g : I \to \mathbb{R}$ of the form $g(x) = c(x-s)^2 + m(x-s) + f(s)$ supporting f at s and integrate both sides of the inequality $g(x) \le f(x)$ over $[a, b]$.

Remark 3 Similarly as in the case of the classical Hermite–Hadamard inequality, each of the two sides of (13) characterizes strongly convex functions under the continuity assumption. Indeed, if f is continuous and satisfies (L) or (R), then $g : I \to \mathbb{R}$ given by $g(x) = f(x) - cx^2,\ x \in I,$ is also continuous and satisfies the left- or the right-hand side of the Hermite–Hadamard inequality, respectively. In both cases this implies that g is convex. Consequently, by Lemma 1, f is strongly convex with modulus c.

Now we present a refinement of the above Hermite–Hadamard-type inequalities (13) for strongly convex functions. A similar result for convex functions can be found in [35, Remark 1.9.3].

Theorem 15 [5] *If a function $f : [a, b] \to \mathbb{R}$ is strongly convex function with modulus c, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{c}{48}(b-a)^2 \\ &\le \frac{1}{b-a}\int_a^b f(x)dx \end{aligned} \tag{14}$$

$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{c}{24}(b-a)^2 \leq \frac{f(a)+f(b)}{2}-\frac{c}{6}(b-a)^2.$$

Proof Applying the Hermite–Hadamard-type inequalities (13) on each of the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ we obtain

$$f\left(\frac{3a+b}{4}\right)+\frac{c}{48}(b-a)^2 \leq \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx \leq \frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}-\frac{c}{24}(b-a)^2$$

and

$$f\left(\frac{a+3b}{4}\right)+\frac{c}{48}(b-a)^2 \leq \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx \leq \frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}-\frac{c}{24}(b-a)^2.$$

Summing up these inequalities we get

$$\begin{aligned} f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+\frac{2c}{48}(b-a)^2 &\leq \frac{2}{b-a}\int_a^b f(x)\,dx \\ &\leq \frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2c}{24}(b-a)^2. \qquad (15) \end{aligned}$$

Now, using the strong convexity of f and (15), we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2 &= f\left(\frac{\frac{3a+b}{4}+\frac{a+3b}{4}}{2}\right)+\frac{c}{12}(b-a)^2 \\ &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{c}{4}\left(\frac{b-a}{2}\right)^2+\frac{c}{12}(b-a)^2 \\ &= \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{c}{48}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

Similarly, using once more (15) and the strong convexity of f, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{c}{24}(b-a)^2 \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+\frac{f(a)+f(b)}{2}-\frac{c}{4}(b-a)^2\right]-\frac{c}{24}(b-a)^2 \\ &= \frac{f(a)+f(b)}{2}-\frac{c}{6}(b-a)^2, \end{aligned}$$

which finishes the proof.

Remark 4 As a consequence of the above theorem we obtain that in the Hermite–Hadamard-type inequalities (13) the left-hand side inequality is stronger than the right-hand one, that is

$$\frac{1}{b-a}\int_a^b f(x)\,dx - \left[f\Big(\frac{a+b}{2}\Big) + \frac{c}{12}(b-a)^2\right] \le \left[\frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2\right] - \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It follows immediately from the third inequality in (14). For the classical Hermite–Hadamard inequalities an analogous observation is given in [42, p. 140].

It is known (see [45]; cf. also [42, p. 145]) that if a function $f : I \to \mathbb{R}$ is convex and $x_1 < x_2 < \ldots < x_n$ are equidistant points in I then the following discrete analogues of the Hermite–Hadamard inequalities are valid:

$$f\left(\frac{x_1+x_n}{2}\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) \le \frac{f(x_1)+f(x_n)}{2}.$$

The following theorem is a counterpart of that result for strongly convex functions.

Theorem 16 [5] *Let $f : [a,b] \to \mathbb{R}$ be a strongly convex function with modulus c and $a = x_1 < x_2 < \ldots < x_n = b$ be equidistant points. Then*

$$f\left(\frac{a+b}{2}\right) + \frac{c(n+1)}{12(n-1)}(b-a)^2 \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) \le \frac{f(a)+f(b)}{2} - \frac{c(n-2)}{6(n-1)}(b-a)^2. \qquad (16)$$

Proof Since the points $x_1, \ldots, x_n$ are equidistant, we have $\frac{1}{n}\sum\limits_{i=1}^{n} x_i = \frac{x_1+x_n}{2}$. Hence, by Theorem 11, we get

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{c}{n}\sum_{i=1}^{n}(x_i-s)^2, \qquad (17)$$

where $s = \frac{1}{n}\sum\limits_{i=1}^{n} x_i = \frac{a+b}{2}$. To finish the left-hand side inequality in (16) we will show that

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-s)^2 = \frac{n+1}{12(n-1)}(b-a)^2.$$

Putting $h = \frac{b-a}{n-1}$, we have $x_i = a + (i-1)h,\ i = 1, \ldots, n$. From here

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-s)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i)^2 - s^2 = \frac{1}{n}\sum_{i=1}^{n}\left(a^2 + 2ah(i-1) + (i-1)^2\,h^2\right) - s^2$$

$$= a^2 + \frac{2ah}{n} \sum_{i=1}^{n} (i-1) + \frac{h^2}{n} \sum_{i=1}^{n} (i-1)^2 - s^2.$$

Consequently, using the formulas

$$\sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} (i-1)^2 = \frac{(n-1)n(2n-1)}{6},$$

we obtain

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - s)^2 = a^2 + a(b-a) + \frac{2n-1}{6(n-1)}(b-a)^2 - \left(\frac{a+b}{2}\right)^2$$
$$= \frac{n+1}{12(n-1)}(b-a)^2,$$

which was to be proved.

To show the right-hand inequality in (16) note that

$$x_i = (1-q_i)a + q_i b, \quad \text{where} \quad q_i = \frac{i-1}{n-1}, \; i = 1, \ldots, n.$$

Hence, by the strong convexity of f,

$$f(x_i) = f((1-q_i)a + q_i b) \le (1-q_i) f(a) + q_i f(b) - c q_i (1-q_i)(b-a)^2.$$

Summing up the above inequalities and using the fact that the numbers $(1 - q_i) f(a) + q_i f(b)$ are terms of an arithmetic sequence, we get

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \le \frac{f(a) + f(b)}{2} - \frac{c}{n(n-1)^2} \sum_{i=1}^{n} (i-1)(n-i)(b-a)^2.$$

Now, applying the formula

$$\sum_{i=1}^{n} (i-1)(n-i) = \frac{(n-2)(n-1)n}{6},$$

we obtain

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \le \frac{f(a) + f(b)}{2} - \frac{c(n-2)}{6(n-1)}(b-a)^2,$$

which finishes the proof.

Remark 5 Note that the sums $\frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$ are the Riemann approximate sums of the integral $\int_a^b f(x)\,dx$. Therefore, letting $n \to \infty$ in (16), we get the Hermite–Hadamard-type inequalities (13).

The Hermite–Hadamard double inequality (14) was generalized by Fejér [16] by proving that if $g : [a,b] \to [0,\infty)$ is a *symmetric density function on* $[a,b]$ (that is, $g(a+b-x) = g(x)$ for all $x \in [a,b]$, and $\int_a^b g(x)\,dx = 1$), and a function $f : [a,b] \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}. \tag{18}$$

Of course, if $g(x) = \frac{1}{b-a}$, then (18) coincides with (14).

However, the example below shows that the Fejér-type generalization of (13) of the form

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2, \tag{19}$$

does not hold, in general, for any symmetric density function $g : [a,b] \to [0,\infty)$ and a strongly convex function $f : I \to \mathbb{R}$.

Example 2 Let $f(x) = x^2$ and $[a,b] = [-1,1]$. Clearly, f is strongly convex with modulus $c = 1$. Take the density function g on $[-1,1]$ given by

$$g(x) = \begin{cases} 1, & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{if } x \in \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then

$$\int_{-1}^{1} x^2\, g(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx = \frac{1}{12} < \frac{1}{3} = f\left(\frac{-1+1}{2}\right) + \frac{1}{12}(1+1)^2,$$

which shows that the left-hand side inequality in (19) does not hold.

Now, take the density function g on $[-1,1]$ defined by

$$g(x) = \begin{cases} 1, & \text{if} x \in \left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right] \\ 0, & \text{if} x \in \left(-\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

Then

$$\int_{-1}^{1} x^2\, g(x)\,dx = 2\int_{\frac{1}{2}}^{1} x^2\,dx = \frac{7}{12} > \frac{1}{3} = \frac{f(-1)+f(1)}{2} - \frac{1}{6}(1+1)^2,$$

which shows that the right-hand side inequality in (19) does not hold.

The following theorem is a counterpart of the Fejér inequalities for strongly convex functions.

Theorem 17 [5] *Let* $g : [a,b] \to [0,\infty)$ *be a symmetric density function on* $[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *be a strongly convex function with modulus* $c > 0$. *Then*

$$f\left(\frac{a+b}{2}\right) + c\left[\int_a^b x^2\, g(x)\,dx - \left(\frac{a+b}{2}\right)^2\right] \le \int_a^b f(x)g(x)\,dx$$

$$\leq \frac{f(a)+f(b)}{2} - c\left[\frac{a^2+b^2}{2} - \int_a^b x^2\, g(x)\,dx\right]. \tag{20}$$

Remark 6 Using the Fejér inequalities (18) for the function $f(x)=x^2$, we get

$$\left(\frac{a+b}{2}\right)^2 \leq \int_a^b x^2\, g(x)\,dx \leq \frac{a^2+b^2}{2}$$

for every symmetric density function g on $[a,b]$. Therefore the terms

$$\int_a^b x^2\, g(x)\,dx - \left(\frac{a+b}{2}\right)^2 \quad \text{and} \quad \frac{a^2+b^2}{2} - \int_a^b x^2\, g(x)\,dx$$

on the left- and the right-hand side of (20) are nonnegative. Consequently, inequalities (20) are a strengthening of the Fejér inequalities (18). Note also that inequalities (20) generalize the Hermite–Hadamard-type inequalities (13). Indeed, for $g(x)=\frac{1}{b-a}$ we have

$$\int_a^b x^2\, g(x)\,dx - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12} \quad \text{and} \quad \frac{a^2+b^2}{2} - \int_a^b x^2\, g(x)\,dx = \frac{(b-a)^2}{6}$$

and then (20) reduces to (13).

Remark 7 If g is any symmetric density function on $[a,b]$, then

$$\int_a^b xg(x)\,dx = \frac{a+b}{2}.$$

Indeed, putting $s=\frac{a+b}{2}$ and using the fact that $g(2s-x)=g(x)$, we obtain

$$\begin{aligned}\int_a^b xg(x)\,dx &= \int_a^s xg(x)\,dx + \int_s^b yg(y)\,dy\\ &= \int_a^s xg(x)\,dx + \int_a^s (2\,s-x)g(x)\,dx = 2\,s\int_a^s g(x) = s = \frac{a+b}{2}.\end{aligned}$$

Proof of Theorem 17 To prove the left-hand side of (20) put $s=\frac{a+b}{2}$, and take a function $h:[a,b]\to\mathbb{R}$ of the form $h(x)=c(x-s)^2+m(x-s)+f(s)$ supporting f at s. Then

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\geq \int_a^b h(x)g(x)\,dx\\ &= c\int_a^b x^2\, g(x)\,dx + (-2cs+m)\int_a^b xg(x)\,dx\\ &\quad + (cs^2-ms+f(s))\int_a^b g(x)\,dx.\end{aligned}$$

Hence, using the integrals

$$\int_a^b g(x)\,dx = 1 \quad\text{and}\quad \int_a^b xg(x)\,dx = \frac{a+b}{2} = s, \tag{21}$$

we obtain

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\geq c\int_a^b x^2\,g(x)\,dx - cs^2 + f(s)\\ &= f\left(\frac{a+b}{2}\right) + c\left[\int_a^b x^2\,g(x)\,dx - \left(\frac{a+b}{2}\right)^2\right].\end{aligned}$$

In the proof of the right-hand side of (20) we use inequality (1).

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &= \int_a^b f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right)g(x)\,dx\\ &\leq \int_a^b \left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) - c\frac{(b-x)(x-a)}{(b-a)^2}(b-a)^2\right)g(x)\,dx\\ &= \int_a^b \left(\frac{bf(a)-af(b)}{b-a} + \frac{f(b)-f(a)}{b-a}x - c((a+b)x - ab - x^2)\right)g(x)\,dx.\end{aligned}$$

Now, using the integrals (21), we get

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\leq \frac{bf(a)-af(b)}{b-a} + \frac{f(b)-f(a)}{b-a}\,\frac{a+b}{2}\\ &\quad - c\left[\frac{(a+b)^2}{2} - ab - \int_a^b x^2\,g(x)\,dx\right]\\ &= \frac{f(a)+f(b)}{2} - c\left[\frac{a^2+b^2}{2} - \int_a^b x^2\,g(x)\,dx\right].\end{aligned}$$

This finishes the proof.

Remark 8 Using the probabilistic characterization of strong convexity given in Theorem 4 we can derive, alternatively, the left-hand side inequality of (20). Indeed, if X is a random variable with values in $[a,b]$ having a symmetric density function $g : [a,b] \to [0,\infty)$, then

$$\begin{aligned}E[X] &= \int_a^b xg(x)\,dx = \frac{a+b}{2},\\ E[X^2] &= \int_a^b x^2\,g(x)\,dx,\\ D^2[X] &= E[X^2] - (E[X])^2 = \int_a^b x^2\,g(x)\,dx - \left(\frac{a+b}{2}\right)^2,\end{aligned}$$

$$E[f(X)] = \int_a^b f(x)g(x)\,dx.$$

Thus, if a function $f : [a,b] \to \mathbb{R}$ is strongly convex with modulus c then, substituting the above values to (5), we obtain the left-hand side of (20).

6 Strongly h-Convex Functions

In this section we introduce the notion of strongly h-convex functions and present a Hermite–Hadamard-type inequality for such functions. Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to (0,\infty)$ be a given function. Following S. Varošanec [53], a function $f : I \to \mathbb{R}$ is said to be *h-convex* if

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \tag{22}$$

for all $x, y \in I$ and $t \in (0,1)$. This notion unifies and generalizes the known classes of convex functions, s - convex functions, Godunova–Levin functions, and P-functions, which are obtained by putting in (22) $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively. Many properties of such functions can be found, for instance, in [14].

We say that a function $f : I \to \mathbb{R}$ is *strongly h-convex with modulus c* if

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) - ct(1-t)(x-y)^2 \tag{23}$$

for all $x, y \in D$ and $t \in (0,1)$.

The following result is a counterpart of the Hermite–Hadamard inequality for strongly h-convex functions.

Theorem 18 [4] *Let $h : (0,1) \to (0,\infty)$ be a given function. If a function $f : I \to \mathbb{R}$ is Lebesgue integrable and strongly h-convex with modulus $c > 0$, then*

$$\frac{1}{2h(\frac{1}{2})}\left[f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2\right] \le \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\le (f(a) + f(b))\int_0^1 h(t)\,dt - \frac{c}{6}(b-a)^2$$

for all $a, b \in I$, $a < b$.

Proof Fix $a, b \in I$, $a < b$, and take $u = ta + (1-t)b$, $v = (1-t)a + tb$. Then, the strong h-convexity of f implies

$$\begin{aligned}
&f\big(\frac{a+b}{2}\big) = f\big(\frac{u+v}{2}\big)\\
&\le h\left(\frac{1}{2}\right) f(u) + h\left(\frac{1}{2}\right) f(v) - \frac{c}{4}(u-v)^2\\
&= h\left(\frac{1}{2}\right)[f(ta + (1-t)b) + f((1-t)a + tb)] - \frac{c}{4}\big((2t-1)a + (1-2t)b\big)^2.
\end{aligned}$$

Integrating the above inequality over the interval (0, 1), we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ \le & h\left(\frac{1}{2}\right)\Big[\int_0^1 f(ta+(1-t)b)dt+\int_0^1 f((1-t)a+tb)dt\Big] \\ - & \frac{c}{4}\int_0^1 \big((2t-1)a+(1-2t)b\big)^2 dt \\ = & h\left(\frac{1}{2}\right)\frac{2}{b-a}\int_a^b f(x)dx-\frac{c}{12}(b-a)^2, \end{aligned}$$

which gives the left-hand side inequality of (18).

For the proof of the right-hand side inequality of (18) we use inequality (23). Integrating over the interval (0, 1), we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &= \int_0^1 f((1-t)a+tb)dt \\ &\le f(a)\int_0^1 h(1-t)dt+f(b)\int_0^1 h(t)dt-c(b-a)^2\int_0^1 t(1-t)dt \\ &= \big(f(a)+f(b)\big)\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2, \end{aligned}$$

which gives the right-hand side inequality of (18).

Remark 9

1. In the case $c = 0$, the Hermite–Hadamard-type inequalities (18) coincide with the Hermite–Hadamard-type inequalities for h-convex functions proved by M. Z. Sarikaya, A. Saglam, and H. Yildirim in [51].
2. If $h(t) = t$, $t \in (0, 1)$, then the inequalities (18) reduce to the Hermite–Hadamard-type inequalities (13) for strongly convex functions. For $c = 0$ we get the classical Hermite–Hadamard inequalities.
3. If $h(t) = t^s$, $t \in (0, 1)$, then the inequalities (18) give

$$2^{s-1}\Big[f\big(\frac{a+b}{2}\big)+\frac{c}{12}(b-a)^2\Big] \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}-\frac{c}{6}(b-a)^2.$$

   For $c = 0$ it reduces to the Hermite–Hadamard-type inequalities for s-convex functions proved by S. S. Dragomir and S. Fitzpatrik [13].
4. If $h(t) = \frac{1}{t}$, $t \in (0, 1)$, then the inequalities (18) give

$$\frac{1}{4}f\big(\frac{a+b}{2}\big)+\frac{c}{48}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx \quad (\le +\infty).$$

   The case $c = 0$ corresponds to the Hermite–Hadamard-type inequalities for Godunova–Levin functions obtained by S. S. Dragomir, J. Pečarić, and L. E. Persson [15].

5. If $h(t) = 1$, $t \in (0, 1)$, then the inequalities (18) reduce to

$$\frac{1}{2} f\Big(\frac{a+b}{2}\Big) + \frac{c}{24}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx \le f(a) + f(b) - \frac{c}{6}(b-a)^2.$$

In the case $c = 0$ it gives the Hermite–Hadamard-type inequalities for P-convex functions proved by S. S. Dragomir, J. Pečarić, and L. E. Persson in [15].

7 Strongly Wright-Convex Functions

Let $(X, \|\cdot\|)$ be a normed space, D a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is called *strongly Wright-convex with modulus c* if

$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) - 2ct(1-t)\|x-y\|^2 \quad (24)$$

for all $x, y \in D$ and $t \in [0, 1]$.

We say that f is *strongly Wright-convex* if it satisfies condition (24) with some $c > 0$. The usual notion of Wright-convexity correspond to the case $c = 0$. Note that every strongly convex function is strongly Wright-convex, and every strongly Wright-convex function is strongly midconvex (with the same modulus c), but not the converse.

Example 3 Let $a : \mathbb{R} \to \mathbb{R}$ be an additive discontinuous function and $f_1(x) = a(x) + x^2,\ x \in \mathbb{R}$. By simple calculation one can check that f_1 is strongly Wright-convex with modulus 1. However, f_1 is not strongly convex (even it is not convex) because it is not continuous. Now, take the function $f_2(x) = |a(x)| + x^2,\ x \in \mathbb{R}$. Clearly, f_2 is strongly midconvex, but it is not strongly Wright-convex (even it is not Wright-convex) because it is discontinuous and bounded from below (see [37, Proposition 2]).

In [33] Ng proved that a function f defined on a convex subset of $\mathbb{R}^n$ is Wright-convex if and only if it can be represented in the form $f = f_1 + a$, where f_1 is a convex function and a is an additive function (see also [37]). Kominek [24] extended that result to functions defined on algebraically open subset of a vector space. In this section we present a similar representation theorem for strongly Wright-convex functions. We start with the following useful fact.

Lemma 4 [31] *Let D be a convex subset of a normed space and $c > 0$. If a function $f : D \to \mathbb{R}$ is convex and strongly midconvex with modulus c, then it is strongly convex with modulus c.*

Proof Fix arbitrary $x, y \in D,\ x \neq y,$ and $t \in (0, 1)$. Since f is strongly midconvex with modulus c, it satisfies the condition

$$f(qx + (1-q)y) \le qf(x) + (1-q)f(y) - cq(1-q)\|x-y\|^2, \quad (25)$$

for all dyadic $q \in (0, 1)$ (see Lemma 2). Consider the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(s) = f(sx + (1 - s)y), \quad s \in [0, 1].$$

By (25) we have

$$g(q) \le qg(1) + (1 - q)g(0) - cq(1 - q)\|x - y\|^2, \tag{26}$$

for all dyadic $q \in (0, 1)$. Since f is convex, also g is convex and hence it is continuous on the open interval $(0, 1)$. Take a sequence (q_n) of dyadic numbers in $(0, 1)$ tending to t. Using (26) for $q = q_n$ and the continuity of g at t, we obtain

$$g(t) \le tg(1) + (1 - t)g(0) - ct(1 - t)\|x - y\|^2.$$

Now, by the definition of g, we get

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y) - ct(1 - t)\|x - y\|^2,$$

which finishes the proof.

Theorem 19 [31] *Let D be an open convex subset of a normed space X and $c > 0$. A function $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c if and only if there exist a function $f_1 : D \to \mathbb{R}$ strongly convex with modulus c and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = f_1(x) + a(x), \; x \in D. \tag{27}$$

Proof Assume first that f is strongly Wright-convex with modulus c. Then f is also Wright-convex and hence, by the result of Kominek [24], f can be represented in the form $f = f_1 + a$, with some convex function f_1 and additive function a. Since f is strongly Wright-convex with modulus c, the function $f - a$ is also strongly Wright-convex with modulus c and, consequently, it is strongly midconvex with modulus c. Hence, by Lemma 1, $f_1 = f - a$ is strongly convex with modulus c, which proves that f has the representation (27). The converse implication is obvious.

Using the above theorem and the representation of strongly convex functions in inner product spaces given in Theorem 13, we obtain the following characterization of strongly Wright-convex functions in inner product spaces.

Corollary 3 [31] *Let $(X, \|\cdot\|)$ be a real inner product space, D be an open convex subset of X and $c > 0$. A function $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c if and only if there exist a convex function $g : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \; x \in D.$$

It is known that if a midconvex function f is bounded from above by a midconcave function g then f is Wright-convex and g is Wright-concave. Moreover, there exist

a convex function f_1, a concave function g_1, and an additive function a such that $f = f_1+a$ and $g = g_1+a$ (see [25, 34, 36]). In this section we present a counterpart of that result for strongly midconvex functions. We say that a function f is *strongly concave* (*strongly midconcave*) with modulus c if $-f$ is strongly convex (strongly midconvex) with modulus c. In the proof of the theorem below we adopt the method used in [25].

Theorem 20 [31] *Let D be an open convex subset of a normed space X and c be a positive constant. Assume that $f : D \to \mathbb{R}$ is strongly midconvex with modulus c, $g : D \to \mathbb{R}$ is strongly midconcave with modulus c and $f \le g$ on D. Then there exist an additive function $a : X \to \mathbb{R}$, a continuous function $f_1 : D \to \mathbb{R}$ strongly convex with modulus c and a continuous function $g_1 : D \to \mathbb{R}$ strongly concave with modulus c such that*

$$f(x) = f_1(x) + a(x) \text{ and } g(x) = g_1(x) + a(x) \tag{28}$$

for all $x \in D$.

Proof Since f is strongly midconvex, it is also midconvex. Therefore, by the theorem of Rodé [49], there exists a Jensen function $a_1 : D \to \mathbb{R}$ such that $a_1(x) \le f(x)$, $x \in D$. This function is of the form

$$a_1(x) = a(x) + b, \quad x \in D,$$

where $a : X \to \mathbb{R}$ is an additive function and b is a constant (see [27]). The function $g_1 = g - a$ is midconcave and

$$g_1(x) = g(x) - a(x) \ge f(x) - a(x) \ge b, \quad x \in D.$$

Therefore by the Bernstein–Doetsch theorem (see [27, 48]), g_1 is continuous and concave. On the other hand, the function $f_1 = f - a$ is midconvex and $f_1 \le g_1$ on D. Hence, applying the Bernstein–Doetsch theorem once more, we infer that f_1 is continuous and convex. Using Lemma 1 we obtain that f_1 is strongly convex with modulus c and g_1 is strongly concave with modulus c. Thus we get the representations (28), which completes the proof.

8 Strongly Schur-Convex Functions

In this session we present a relationship between strongly Wright-convex functions and the strong Schur-convexity.

Let $I \subset \mathbb{R}$ be an interval and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in I^n$, where $n \ge 2$. Following I. Schur (cf. e.g., [29, 50]) we say that x *is majorized by* y, and write $x \preceq y$, if there exists a doubly stochastic $n \times n$ matrix P (ie. matrix containing nonnegative elements with all rows and columns summing up to 1) such that $x = y \cdot P$. A function $F : I^n \to \mathbb{R}$ is said to be *Schur-convex* if $F(x) \le F(y)$ whenever $x \preceq y$, $x, y \in I^n$.

It is known, by the classical works of Schur [50], Hardy–Littlewood–Pólya [19] and Karamata [23] that if a function $f : I \to \mathbb{R}$ is convex then it *generates Schur-convex sums*, that is the function $F : I^n \to \mathbb{R}$ defined by

$$F(x) = F(x_1, \dots, x_n) = f(x_1) + \cdots + f(x_n)$$

is Schur-convex. It is also known that the convexity of f is a sufficient but not necessary condition under which F is Schur-convex. C. T. Ng [33] proved that a function generates Schur-convex sums if and only if it is Wright-convex. In this section we introduce the notion of strong Schur-convexity and we present a counterpart of the Ng representation theorem for functions generating strongly Schur-convex sums.

Let $(X, \|\cdot\|)$ be a (real) inner product space. We consider the space X^n $(n \geq 2)$ with the product norm

$$|||x||| = \sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}, \quad x = (x_1, \dots, x_n) \in X^n.$$

Similarly as in the classical case we define the majorization in X^n. Namely, given two n–tuples $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X^n$ we say that x *is majorized by* y, written $x \preccurlyeq y$, if

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \cdot P$$

for some doubly stochastic $n \times n$ matrix P.

Note that if $x \preccurlyeq y$ then $|||x|||^2 \leq |||y|||^2$. It follows, for instance, from the fact that the function $\|\cdot\|^2 : X \to \mathbb{R}$ is convex and so it generates Schur-convex sums (the proof is exactly the same as in the classical case of $X = \mathbb{R}$; cf. also the proof of Theorem 21 below, where we repeat the argument for the sake of completeness).

Motivated by the definition of strongly convex functions we propose a strengthening of the notion of Schur-convexity. Let D be a convex subset of X, $c > 0$ and $n \geq 2$. We say that a function $F : D^n \to \mathbb{R}$ is *strongly Schur-convex with modulus c* if

$$x \preccurlyeq y \implies F(x) \leq F(y) - c\left(|||y|||^2 - |||x|||^2\right)$$

for all $x, y \in D$. Note that the usual Schur-convexity corresponds to the case $c = 0$.

Now, we will prove that strongly convex functions generate strongly Schur-convex sums and functions generating strongly Schur-convex sums are strongly Jensen–convex.

Theorem 21 [41] *Let D be a convex subset of an inner product space $(X, \|\cdot\|)$ and $c > 0$. If a function $f : D \to \mathbb{R}$ is strongly convex with modulus c, then for every $n \geq 2$ the function $F : D^n \to \mathbb{R}$ given by*

$$F(x_1, \dots, x_n) = f(x_1) + \cdots + f(x_n), \quad (x_1, \dots, x_n) \in D^n,$$

is strongly Schur-convex with modulus c.

Proof Assume that $f : D \to \mathbb{R}$ is strongly convex with modulus c. Since X is an inner product space, the function $h : D \to \mathbb{R}$ given by $h(x) = f(x) - c\|x\|^2$, $x \in D$,

is convex (cf. Lemma 1). Let $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in D^n$ and $x \preccurlyeq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^{n} t_{ij} y_i, \quad j = 1, \dots, n,$$

and, by the convexity of h, we obtain

$$\begin{aligned} h(x_1) + \dots + h(x_n) &= \sum_{j=1}^{n} h\Big(\sum_{i=1}^{n} t_{ij} y_i\Big) \le \sum_{j=1}^{n} \sum_{i=1}^{n} t_{ij} h(y_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} h(y_i) = \sum_{i=1}^{n} h(y_i) \sum_{j=1}^{n} t_{ij} = h(y_1) + \dots + h(y_n). \end{aligned}$$

Consequently,

$$\begin{aligned} F(x) &= f(x_1) + \dots + f(x_n) \\ &= h(x_1) + \dots + h(x_n) + c\left(\|x_1\|^2 + \dots + \|x_n\|^2\right) \\ &\le h(y_1) + \dots + h(y_n) + c\left(\|x_1\|^2 + \dots + \|x_n\|^2\right) \\ &= f(y_1) + \dots + f(y_n) - c\left(\|y_1\|^2 + \dots + \|y_n\|^2\right) + c\left(\|x_1\|^2 + \dots + \|x_n\|^2\right) \\ &= F(y) - c\left(|||y|||^2 - |||x|||^2\right). \end{aligned}$$

This shows that F is strongly Schur-convex with modulus c, which was to be proved.

Remark 10 The converse theorem is not true. For instance, if $a : \mathbb{R} \to \mathbb{R}$ is an additive discontinuous function, then $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a(x) + x^2$, $x \in \mathbb{R}$, is not strongly convex with any $c > 0$ (because it is not continuous) but it generates strongly Schur-convex sums. To see this take $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$ $(n \ge 2)$ such that $x \preccurlyeq y$. Then $x = y \cdot P$ for some doubly stochastic $n \times n$ matrix $P = [t_{ij}]$. By the additivity of a we have

$$\begin{aligned} a(x_1) + \dots + a(x_n) &= a(x_1 + \dots + x_n) = a\Big(\sum_{j=1}^{n} \sum_{i=1}^{n} t_{ij} y_i\Big) \\ &= a\Big(\sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} y_i\Big) = a\Big(\sum_{i=1}^{n} y_i \sum_{j=1}^{n} t_{ij}\Big) = a(y_1) + \dots + a(y_n). \end{aligned}$$

Hence,

$$\begin{aligned} f(x_1) + \dots + f(x_n) &= a(x_1) + \dots + a(x_n) + x_1^2 + \dots + x_n^2 \\ &= a(y_1) + \dots + a(y_n) + y_1^2 + \dots + y_n^2 - (y_1^2 + \dots + y_n^2 - x_1^2 - \dots - x_n^2) \end{aligned}$$

$$= f(y_1) + \cdots + f(y_n) - \left(\|y\|^2 - \|x\|^2\right).$$

This proves that $F : \mathbb{R}^n \to \mathbb{R}$ defined by $F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$ is strongly Schur-convex with modulus 1.

Theorem 22 [41] *Let D be a convex subset of an inner product space $(X, \|\cdot\|)$, $c > 0$ and $f : D \to \mathbb{R}$. If for some $n \geq 2$ the function $F : D^n \to \mathbb{R}$ given by*

$$F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n), \quad (x_1, \ldots, x_n) \in D^n$$

is strongly Schur-convex with modulus c, then f is strongly Jensen-convex with modulus c.

Proof Take $y_1, y_2 \in D$ and put $x_1 = x_2 = \frac{1}{2}(y_1 + y_2)$. Consider the points

$$y = (y_1, y_2, y_2, \ldots, y_2), \quad x = (x_1, x_2, y_2, \ldots, y_2)$$

(if $n = 2$, then we take $y = (y_1, y_2)$, $x = (x_1, x_2)$). Now, if

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \end{bmatrix}$$

then $x = y \cdot P$ and $x \preccurlyeq y$. Therefore, by the strong Schur-convexity of F,

$$F(x) \leq F(y) - c\left(|||y|||^2 - |||x|||^2\right),$$

whence

$$2\, f\Big(\frac{y_1 + y_2}{2}\Big) \leq f(y_1) + f(y_2) - c\Big(\|y_1\|^2 + \|y_2\|^2 - 2\Big\|\frac{y_1 + y_2}{2}\Big\|^2\Big). \tag{29}$$

By the parallelogram law we have

$$\|y_1\|^2 + \|y_2\|^2 = \frac{1}{2}\|y_1 + y_2\|^2 + \frac{1}{2}\|y_1 - y_2\|^2.$$

Consequently, by (29),

$$f\left(\frac{y_1 + y_2}{2}\right) \leq \frac{f(y_1) + f(y_2)}{2} - \frac{c}{4}\|y_1 - y_2\|^2,$$

which means that f is strongly Jensen-convex with modulus c.

Remark 11 The converse theorem is not true. For instance, let $a : \mathbb{R} \to \mathbb{R}$ be an additive discontinuous function such that $a(1) = 0$ and let $t \in (0, 1)$ with $a(t) \neq 0$.

Then the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |a(x)| + x^2$, $x \in \mathbb{R}$, is strongly Jensen-convex with modulus 1 (because $x \mapsto f(x) - x^2 = |a(x)|$ is a Jensen-convex function, (cf. Lemma 3), but it does not generate strongly Schur-convex sums with modulus 1. Indeed, if $n = 2$, $x = (t, 1-t)$ and $y = (1, 0)$, then $x \preccurlyeq y$, but

$$F(x) = |a(t)| + |a(1-t)| + t^2 + (1-t)^2 > t^2 + (1-t)^2 = F(y) - \left(\|y\|^2 - \|x\|^2\right).$$

The following result is a counterpart of the theorem of Ng [33]. It characterizes the functions generating strongly Schur-convex sums.

Theorem 23 [41] *Let D be a convex subset of an inner product space $(X, \|\cdot\|)$, $f : D \to \mathbb{R}$ and $c > 0$. The following conditions are equivalent:*

(i) *For every $n \geq 2$ the function $F : D^n \to \mathbb{R}$ defined by*

$$F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n), \quad (x_1, \ldots, x_n) \in D^n, \tag{30}$$

is strongly Schur-convex with modulus c.

(ii) *For some $n \geq 2$ the function F given by (30) is strongly Schur-convex with modulus c.*

(iii) *The function f is strongly Wright-convex with modulus c.*

(iv) *There exist a convex function $g : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \quad x \in D. \tag{31}$$

Proof The implication (i) $\Longrightarrow$ (ii) is obvious.

To prove (ii) $\Longrightarrow$ (iii) fix $y_1, y_2 \in D$ and $t \in (0, 1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3, \ldots, n$. Then, by the similar argumentation as in the proof of Theorem 22, we have

$$x = (x_1, \ldots, x_n) \preccurlyeq y = (y_1, \ldots, y_n).$$

Therefore, using the strong convexity of F, we obtain

$$F(x) \leq F(y) - c\left(|||y|||^2 - |||x|||^2\right),$$

and hence

$$\begin{aligned} &f\left(ty_1 + (1-t)y_2\right) + f\left((1-t)y_1 + ty_2\right) \leq f(y_1) + f(y_2) \\ &- c\left(\|y_1\|^2 + \|y_2\|^2 - \|ty_1 + (1-t)y_2\|^2 - \|(1-t)y_1 + ty_2\|^2\right). \end{aligned} \tag{32}$$

Using elementary properties of the inner product we get

$$\|y_1\|^2 + \|y_2\|^2 - \|ty_1 + (1-t)y_2\|^2 - \|(1-t)y_1 + ty_2\|^2$$

$$
\begin{aligned}
&= \|y_1\|^2 + \|y_2\|^2 \\
&- \left(t^2\|y_1\|^2 + (1-t)^2\|y_2\|^2 + (1-t)^2\|y_1\|^2 + t^2\|y_2\|^2 + 4t(1-t)\langle y_1|y_2\rangle\right) \\
&= 2t(1-t)\left(\|y_1\|^2 - 2\langle y_1|y_2\rangle + \|y_2\|^2\right) = 2t(1-t)\|y_1 - y_2\|^2.
\end{aligned}
$$

Consequently, from (32) we get

$$
f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) \le f(y_1) + f(y_2) - 2ct(1-t)\|y_1 - y_2\|^2,
$$

which means that f is strongly Wright-convex with modulus c.

The implication (iii) $\Longrightarrow$ (iv) follows from Corollary 3.

To see that (iv) $\Longrightarrow$ (i) assume that f has the representation (31). Then the function $h = g + c\|\cdot\|^2$ is strongly convex with modulus c and hence, by Theorem 21, it generates strongly Schur-convex. Therefore, for any $x = (x_1, \dots, x_n) \preccurlyeq y = (y_1, \dots, y_n)$ we have

$$
h(x_1) + \cdots + h(x_n) \le h(y_1) + \cdots + h(y_n) - c\left(|||y|||^2 - |||x|||^2\right).
$$

Consequently, using the additivity of a (similarly as in Remark 10), we arrive at

$$
\begin{aligned}
F(x) &= f(x_1) + \cdots + f(x_n) = h(x_1) + \cdots + h(x_n) + a(x_1) + \cdots + a(x_n) \\
&\le h(y_1) + \cdots + h(y_n) - c\left(|||y|||^2 - |||x|||^2\right) + a(y_1) + \cdots + a(y_n) \\
&= f(y_1) + \cdots + f(y_n) - c\left(|||y|||^2 - |||x|||^2\right) = F(y) - c\left(|||y|||^2 - |||x|||^2\right),
\end{aligned}
$$

which shows that F is strongly Schur-convex with modulus c. This finishes the proof.

9 Strongly Convex Functions of Higher Order

In the classical theory of convex functions their natural generalization are convex functions of higher order. Let us recall the definition. Let $n \in \mathbb{N}$ and $x_0, \dots, x_n$ be distinct points in I. Denote by $[x_0, \dots, x_n; f]$ the divided difference of f at $x_0, \dots, x_n$ defined by the recurrence

$$
\begin{aligned}
[x_0; f] &= f(x_0), \\
[x_0, \dots, x_n; f] &= \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}, \quad n \in \mathbb{N}.
\end{aligned}
$$

Following Hopf and Popoviciu a function $f : I \to \mathbb{R}$ is called *convex of order n* (or *n-convex*) if

$$
[x_0, \dots, x_{n+1}; f] \ge 0
$$

for all $x_0 < \dots < x_{n+1}$ in I. It is well known (and easy to verify) that 1-convex functions are ordinary convex functions. Many results on n-convex functions one

can found, among others, in [45, 27, 48]. In this section we introduce the notion of strongly n-convex functions and investigate properties of this class of functions. Let c be a positive constant and $n \in \mathbb{N}$. We say that a function $f : I \to \mathbb{R}$ is *strongly convex of order n with modulus c* (or *strongly n-convex with modulus c*) if

$$[x_0, \ldots, x_{n+1}; f] \geq c, \tag{33}$$

for all $x_0 < \ldots < x_{n+1}$ in I. Note that for $n = 1$ condition (33) reduces to

$$\frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \geq c,$$

or

$$f(x_1) \leq \frac{x_2 - x_1}{x_2 - x_0} f(x_0) + \frac{x_1 - x_0}{x_2 - x_0} f(x_2) - c(x_2 - x_1)(x_1 - x_0), \quad x_0 < x_1 < x_2.$$

Hence, putting $t = \frac{x_2 - x_1}{x_2 - x_0}$ and, consequently, $1 - t = \frac{x_1 - x_0}{x_2 - x_0}$ and $x_1 = tx_0 + (1 - t)x_2$, we get

$$f(tx_0 + (1 - t)x_2) \leq tf(x_0) + (1 - t)f(x_2) - ct(1 - t)(x_2 - x_0)^2$$

for all $x_0, x_2 \in I$ and $t \in (0, 1)$, which means that f is strongly convex with modulus c.

The following theorem gives a relationship between strongly n-convex and n-convex functions. It plays a crucial role in proving results of this section. For $n = 1$ it reduces to Lemma 1.

Theorem 24 [18] *Let $I \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$ and $c > 0$. A function $f : I \to \mathbb{R}$ is strongly n-convex with modulus c if and only if the function $g(x) = f(x) - cx^{n+1}$, $x \in I$, is n-convex.*

The proof of this theorem is based on the following simple facts whose proofs are straightforward.

Lemma 5 *For each distinct $x_0, \ldots, x_n \in \mathbb{R}$ the operator $[x_0, \ldots, x_n; \cdot]$ is linear.*

Lemma 6 $[x_0, \ldots, x_n; x^n] = 1$ *for each $n \in \mathbb{N}$ and distinct $x_0, \ldots, x_n \in \mathbb{R}$.*

Proof of Theorem 24 If f is strongly n-convex with modulus c and $g(x) = f(x) - cx^{n+1}$, then, by Lemma 5 and Lemma 6, we get

$$[x_0, \ldots, x_{n+1}; g] = [x_0, \ldots, x_{n+1}; f] - [x_0, \ldots, x_{n+1}; cx^{n+1}] \geq c - c = 0,$$

which means that g is n-convex. Conversely, if g is n-convex then for $f(x) = g(x) + cx^{n+1}$ we have

$$[x_0, \ldots, x_{n+1}; f] = [x_0, \ldots, x_{n+1}; g] + [x_0, \ldots, x_{n+1}; cx^{n+1}] \geq 0 + c = c,$$

which proves that f is strongly n-convex with modulus c.

It is known that a function $f : I \to \mathbb{R}$ defined on an open interval I is n-convex with $n > 1$ if and only if it is of the class C^{n-1} in I and its $(n-1)$th derivative $f^{(n-1)}$ is convex (see [27, Thm. 15.8.4]). Moreover, if f is of the class C^n in I then it is n-convex if and only if $f^{(n)}$ is increasing in I, and also if f is of the class C^{n+1} in I then it is n-convex if and only if $f^{(n+1)}$ is nonnegative in I (see [27, Thm 15.8.5, Thm 15.8.6]). The following theorems are counterparts of these results for strongly n-convex functions.

Theorem 25 [18] *Let $I \subset \mathbb{R}$ be an open interval, $c > 0$, and $n > 1$. A function $f : I \to \mathbb{R}$ is strongly n-convex with modulus c if and only if it is of the class C^{n-1} in I and its $(n-1)$th derivative $f^{(n-1)}$ is strongly convex with modulus $\frac{c}{2}(n+1)!$.*

Proof ($\Rightarrow$) Assume that f is strongly n-convex with modulus c. By Theorem 24 f can be represented in the form $f(x) = g(x) + cx^{n+1}$, $x \in I$, where g is an n-convex function. Hence

$$f^{(n-1)}(x) = g^{(n-1)}(x) + \frac{c}{2}(n+1)!\, x^2,\ x \in I.$$

Since $g^{(n-1)}$ is convex, this representation means that $f^{(n-1)}$ is strongly convex with modulus $\frac{c}{2}(n+1)!$.
($\Leftarrow$) By the assumption and Theorem 24 $f^{(n-1)}$ is of the form $f^{(n-1)}(x) = g(x) + \frac{c}{2}(n+1)!\, x^2$, $x \in I$, with a convex function g. Integrating both sides $n-1$ times, we obtain

$$f(x) = G(x) + cx^{n+1},\ x \in I,$$

where G is an n-convex function. Thus, by Theorem 24, f is strongly n-convex with modules c.

The next theorem shows that f is strongly n-convex if and only if its nth derivative is strongly increasing in some sense.

Theorem 27 [18] *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be of the class C^n in I. Then f is strongly n-convex with modulus c if and only if $f^{(n)}$ satisfies the condition*

$$(f^{(n)}(x) - f^{(n)}(y))(x-y) \geq c(n+1)!(x-y)^2,\ x, y \in I. \tag{34}$$

Proof ($\Rightarrow$) By Theorem 24 f is of the form $f(x) = g(x) + cx^{n+1}$, $x \in I$, with an n-convex g. Hence

$$f^{(n)}(x) = g^{(n)}(x) + c(n+1)!x,\ x \in I.$$

Since $g^{(n)}$ is increasing, we have

$$(g^{(n)}(x) - g^{(n)}(y))(x-y) \geq 0,\ x, y \in I.$$

Thus, for all $x, y \in I$,

$$(f^{(n)}(x) - f^{(n)}(y))(x-y) = (g^{(n)}(x) - g^{(n)}(y))(x-y) + c(n+1)!(x-y)^2$$

$$\geq c(n+1)!(x-y)^2.$$

($\Leftarrow$) Assume (34) and put $g(x) = f(x) - cx^{n+1}$, $x \in I$. Then

$$(g^{(n)}(x) - g^{(n)}(y))(x-y) = (f^{(n)}(x) - f^{(n)}(y))(x-y) - c(n+1)!(x-y)^2 \geq 0,$$

which means that g is n-convex. Thus, by Theorem 24 again, f is strongly n-convex with modulus c.

Theorem 27 [18] *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be of the class C^{n+1} in I. Then f is strongly n-convex with modulus c if and only if $f^{(n+1)} \geq c(n+1)!$ $x \in I$.*

Proof ($\Rightarrow$) Since $f(x) = g(x) + cx^{n+1}$, $x \in I$, with an n-convex g, we have

$$f^{(n+1)}(x) = g^{(n+1)}(x) + c(n+1)! \geq c(n+1)! \ \ x \in I.$$

($\Leftarrow$) Put $g(x) = f(x) - cx^{n+1}$, $x \in I$. Then

$$g^{(n)}(x) = f^{(n)}(x) - c(n+1)! \geq 0, \ \ x \in I,$$

which means that g is n-convex. Hence f is strongly n-convex with modulus c.

Now, we recall the definition of Jensen n-convex functions and extend it to strongly Jensen n-convex functions.

Let $\triangle_h^n$ be the difference operator of nth order with increment $h > 0$ defined by the recurrence:

$$\triangle_h^0 f(x) = f(x), \ \ \triangle_h^n f(x) = \triangle_h^{n-1} f(x+h) - \triangle_h^{n-1} f(x), \ n \in \mathbb{N}.$$

A function $f : I \to \mathbb{R}$ is said to be *n-convex in the sense of Jensen* (or *Jensen n-convex*) if

$$\triangle_h^{n+1} f(x) \geq 0$$

for all $x \in I$ and $h > 0$ such that $x + (n+1)h \in I$ (cf. e.g., [27, 48]).

We say that a function $f : I \to \mathbb{R}$ is *strongly n-convex with modulus $c > 0$ in the sense of Jensen* (or *strongly Jensen n-convex with modulus c*) if

$$\triangle_h^{n+1} f(x) \geq c(n+1)! \, h^{n+1} \tag{35}$$

for all $x \in I$ and $h > 0$ such that $x + (n+1)h \in I$. Note that for $n = 1$ condition (35) reduces to

$$\triangle_h^2 f(x) \geq 2ch^2$$

or

$$f(x+2h) - 2f(x+h) + f(x) \geq 2ch^2.$$

Putting $u = x$ and $v = x + 2h$, we obtain

$$f\left(\frac{u+v}{2}\right) \leq \frac{f(u)+f(v)}{2} - \frac{c}{4}(u-v)^2, \quad u, v \in I,$$

which means that f is strongly Jensen convex with modulus c.

Remark 12 Every function $f : I \to \mathbb{R}$ strongly n-convex with modulus c is strongly Jensen n-convex with modulus c. It follows from the fact that if points $x_0 < \ldots < x_{n+1}$ are equally spaced, that is $x_i = x_0 + ih$, $i = 1, \ldots, n+1$, with some $h > 0$, then

$$[x_0, \ldots, x_{n+1}; f] = \frac{\Delta_h^{n+1} f(x_0)}{(n+1)!\, h^{n+1}}$$

(see Kuczma [27, Lem. 15.2.5]). If f is strongly n-convex with modulus c, then $[x_0, \ldots, x_{n+1}; f] \geq c$ for all $x_0 < \ldots < x_{n+1}$ in I. In particular, for equally spaced points we get

$$\Delta_h^{n+1} f(x_0) = [x_0, \ldots, x_{n+1}; f](n+1)!\, h^{n+1} \geq c(n+1)!\, h^{n+1},$$

which means that f is strongly Jensen n-convex with modulus c.

The next result is analogous to Theorem 24 and gives a relationship between strongly Jensen n-convex functions and Jensen n-convex functions.

Theorem 28 [18] *Let $I \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$ and $c > 0$. A function $f : I \to \mathbb{R}$ is strongly Jensen n-convex with modulus c if and only if the function $g(x) = f(x) - cx^{n+1}$, $x \in I$, is Jensen n-convex.*

The proof of this theorem is based on the following simple facts.

Lemma 7 [27, Lem. 15.1.1] *The operator Δ_h^n is linear.*

Lemma 8 $\Delta_h^n\, x^n = n!h^n$, *for every $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $h > 0$.*

Proof of Theorem 28 ($\Rightarrow$) Using the strong Jensen n-convexity of f and Lemmas 7 and 8 we get

$$\Delta_h^{n+1} g(x) = \Delta_h^{n+1} f(x) - \Delta_h^{n+1} x^{n+1} \geq c(n+1)!\, h^{n+1} - c(n+1)!\, h^{n+1} = 0,$$

which shows that g is Jensen n-convex.

($\Leftarrow$) By the Jensen n-convexity of g we have

$$\Delta_h^{n+1} f(x) = \Delta_h^{n+1} g(x) + \Delta_h^{n+1} x^{n+1} \geq c(n+1)!\, h^{n+1},$$

which proves that f is strongly Jensen n-convex with modulus c. □

It is known that Jensen n-convex functions need not be continuous (and hence they need not be n-convex). However, for continuous functions the concepts of n-convexity and Jensen n-convexity are equivalent. There are also many theorems giving relatively weak conditions under which Jensen n-convex functions are continuous (cf. e.g., [27, Chap. 15], [11, 17, 48] and the references therein). Similar

results hold for strongly Jensen n-convex functions. We present here, as an example, a counterpart of the classical theorem of Ciesielski [11] (cf. also Ger [17]).

Theorem 29 [18] *Let I be an open interval and $n \in \mathbb{N}$. If a function $f : I \to \mathbb{R}$ is strongly Jensen n-convex with modulus $c > 0$ and bounded on a set $A \subset I$ having positive Lebesgue measure (or of the second category and with the Baire property), then f is continuous on I and strongly n-convex with modulus c.*

Proof By Theorem 28, f is of the form $f(x) = g(x) + cx^{n+1}$, $x \in I$, where g is Jensen n-convex. If f is bounded on A, then g is also bounded on A (without loss of generality we may assume that A is bounded). Hence, by the theorem of Ciesielski, g is continuous and n-convex. Consequently, f is continuous and strongly n-convex with modulus c.

10 Connections with Beckenbach Generalized Convexity

The fact that a function $f : I \to \mathbb{R}$ is convex means, geometrically, that for any two distinct points on the graph of f the segment joining these points lies above the corresponding part of the graph. Beckenbach [8] generalized this idea by replacing the segments by graphs of continuous functions belonging to a two-parameter family $\mathcal{F}$ of functions. The *generalized convex functions* obtained in such a way have many properties known for the classical convex functions (cf. e.g., [8, 39, 48]). In this section we will show that strong convexity is equivalent to generalized convexity with respect to a certain two-parameter family.

Let $\mathcal{F}$ be a family of continuous real functions defined on an interval $I \subset \mathbb{R}$. Following Beckenbach [8] we say that $\mathcal{F}$ is a *two-parameter family* if for any two points $(x_1, y_1),\ (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = y_i \quad \text{for} \quad i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $(x_1, y_1),\ (x_2, y_2)$ will be denoted by $\varphi_{(x_1,y_1),\,(x_2,y_2)}$. A function $f : I \to \mathbb{R}$ is said to be *convex with respect to $\mathcal{F}$* (briefly, *$\mathcal{F}$–convex*) if for any $x_1, x_2 \in I$, $x_1 < x_2$

$$f(x) \leq \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(x) \ \text{ for all } \ x \in [x_1, x_2].$$

The definition above is motivated by consideration of the class

$$\mathcal{F} = \{ax + b : a, b \in \mathbb{R}\}.$$

It is clear that $\mathcal{F}$ is a two-parameter family and $\mathcal{F}$–convexity coincides with the classical convexity. In a similar way we can characterize the strong convexity.Given a fixed number $c > 0$ define

$$\mathcal{F}_c = \{cx^2 + ax + b : a, b \in \mathbb{R}\}.$$

Clearly, $\mathcal{F}_c$ is also a two-parameter family and the following theorem holds.

Theorem 30 [30] *A function $f : I \to \mathbb{R}$ is strongly convex with modulus c if and only if f is $\mathcal{F}_c$–convex.*

Proof Fix $x_1, x_2 \in I$ and take $\varphi = \varphi_{(x_1,f(x_1)),(x_2,f(x_2))} \in \mathcal{F}_c$. Then $\varphi(x) = cx^2 + ax + b$, where the coefficients a, b are uniquely determined by the conditions $\varphi(x_i) = f(x_i)$, $i = 1, 2$. Hence, for every $t \in [0, 1]$, we have

$$\begin{aligned}\varphi(tx_1 + (1-t)x_2) &= c\,(tx_1 + (1-t)x_2)^2 + a(tx_1 + (1-t)x_2) + b\\ &= c\left(t^2x_1^2 + 2t(1-t)x_1x_2 + (1-t)^2x_2^2\right) + a(tx_1 + (1-t)x_2) + b\\ &= t(cx_1^2 + ax_1 + b) + (1-t)(cx_2^2 + ax_2 + b) - ct(1-t)\left(x_1^2 - 2x_1x_2 + x_2^2\right)\\ &= tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2.\end{aligned}$$

From here, if f is $\mathcal{F}_c$-convex, then

$$\begin{aligned}f(tx_1 + (1-t)x_2) &\le \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(tx_1 + (1-t)x_2)\\ &= tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2,\end{aligned}$$

which means that f is strongly convex with modulus c.

Conversely, if f is strongly convex with modulus c, then

$$\begin{aligned}f(tx_1 + (1-t)x_2) &\le tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2\\ &= \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(tx_1 + (1-t)x_2),\end{aligned}$$

which shows that f is $\mathcal{F}_c$-convex.

Due to Tornheim [52], the idea of Beckenbach has been extended by taking n-parameter families. The so obtained generalized convex functions have many properties known for n-convex functions (see e.g., [8, 9, 39, 48, 52]). We will show that strong n-convexity is equivalent to generalized convexity with respect to a certain n-parameter family.

Let $n \ge 2$. A family $\mathcal{F}$ of continuous real functions defined on an interval $I \subset \mathbb{R}$ is called an *n-parameter family* if for any n points $(x_1, y_1), \ldots, (x_n, y_n) \in I \times \mathbb{R}$ with $x_1 < \ldots < x_n$ there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = y_i \quad \text{for} \quad i = 1, \ldots, n.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $(x_1, y_1), \ldots, (x_n, y_n)$ will be denoted by $\varphi_{(x_1,y_1),\ldots,(x_n,y_n)}$. A function $f : I \to \mathbb{R}$ is said to be *convex with respect to the n-parameter family $\mathcal{F}$* (briefly, *$\mathcal{F}$-convex*) if for any $x_1 < \ldots < x_n$ in I

$$f(x) \le \varphi_{(x_1,f(x_1)),\ldots,(x_n,f(x_n))}(x), \; x \in [x_{n-1}, x_n].$$

It is well known that if

$$\mathcal{F}_n = \{a_nx^n + \cdots + a_1x + a_0 : a_0, \ldots a_n \in \mathbb{R}\},$$

i.e., $\mathcal{F}_n$ is the set of all polynomials of degree at most n, then $\mathcal{F}_n$ is an $(n+1)$-parameter family and the generalized convexity with respect to $\mathcal{F}_n$ coincides with n-convexity (cf. [48, 52]). In a similar way we can characterize the strong n-convexity. Let $c > 0$ be a fixed number and

$$\mathcal{F}_{n,c} = \{cx^{n+1} + a_n x^n + \cdots + a_1 x + a_0 : a_0, \ldots a_n \in \mathbb{R}\}.$$

Clearly, $\mathcal{F}_{n,c}$ is also an $(n+1)$-parameter family and the following theorem holds.

Theorem 31 [18] *A function $f : I \to \mathbb{R}$ is strongly n-convex with modulus c if and only if f is $\mathcal{F}_{n,c}$-convex.*

Proof Fix arbitrarily points $x_1, \ldots, x_{n+1}$ in I. Let φ be the unique polynomial in $\mathcal{F}_{n,c}$ determined by $\varphi(x_i) = f(x_i),\ i = 1, \ldots, n+1$. Then ψ defined by

$$\psi(x) = \varphi(x) - cx^{n+1},\ x \in I,$$

belongs to $\mathcal{F}_n$ and is uniquely determined by $\psi(x_i) = f(x_i) - cx_i^{n+1},\ i = 1, \ldots, n+1$. Clearly,

$$f(x) \geq \varphi(x),\ x \in [x_n, x_{n+1}]$$

if and only if

$$f(x) - cx^{n+1} \geq \psi(x),\ x \in \left[x_n, x_{n+1}\right].$$

This means that f is $\mathcal{F}_{n,c}$-convex if and only if $f(x) - cx^{n+1}$ is $\mathcal{F}_n$-convex. Since the $\mathcal{F}_n$-convexity is equivalent to the n-convexity, we obtain, by Theorem 24, that $\mathcal{F}_{n,c}$-convexity is equivalent to the strong n-convexity with modulus c.

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Some New Algorithms for Solving General Equilibrium Problems

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Abstract In this chapter, we investigate some unified iterative methods for solving the general equilibrium problems using the auxiliary principle technique. The convergence of the proposed methods is analyzed under some suitable conditions. As special cases, we obtain a number of known and new classes of equilibrium and variational inequality problems. Results obtained in this chapter continue to hold for these new and previously known problems. The ideas and techniques of this chapter may inspire the interested readers to explore applications of the general equilibrium problems in pure and applied sciences.

Keywords Variational inequalities · Algorithms · Auxiliary principle · Convergence analysis · Fixed point problems

1 Introduction

Equilibrium problems theory provides us a natural, novel, and unified framework to study a wide class of problems arising in economics, finance, transportation, network, and structural analysis, elasticity and optimization. Equilibrium problems were introduced by Blum and Oettli [1] and Noor and Oettli [20] in 1994. Since then, the ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative; see [1, 2, 3–22]. Equilibrium problems also include variational inequalities and related optimization problems as special cases. Inspired and motivated by the recent research work going in this field, Noor and Rassias [19] considered and investigated a new class of equilibrium problems, which is called *mixed quasi general equilibrium problems.* There are several methods including projection and its variant forms, Wiener–Hopf equations, and auxiliary

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principle for solving variational inequalities. It is known that projection methods and variant forms including Wiener–Hopf equations can not be extended for equilibrium. This fact has motivated to use the auxiliary principle technique. Glowinski, Lions, and Tremolieres [5] used this technique to study the existence of a solution of the mixed variational inequalities, whereas Noor–Noor–Rassias [11] used this technique to suggest and analyze an iterative method for solving mixed quasi variational inequalities. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique; see [5, 13–15, 17–19]. We again use the auxiliary principle technique to suggest a class of new iterative methods for solving mixed quasi general equilibrium problems. The convergence of these methods requires only the jointly monotonicity of the trifunction in conjunction with skew symmetry of the bifunction. Since mixed quasi general equilibrium problems include equilibrium, general variational inequalities, and complementarity problems as special cases, results obtained in this chapter continue to hold for these problems. Our results can be considered an important and significant extension of the known results for solving equilibrium, variational inequalities, and complementarity problems.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let K be a nonempty and closed set in H. We recall the following concepts and notations, which are needed.

Definition 1 ([3, 21]). Let K be any set in H. The set K is said to be g-convex (relative convex), if there exists a function $g : K \longrightarrow K$ such that

$$g(u) + t(g(v) - g(u)) \in K, \forall u, v \in H : g(u), g(v) \in K, t \in [0,1].$$

Note that every convex set is a relative convex, but the converse is not true, see [3, 21]. In passing, we remark that the notion of the relative convex set was introduced by Noor [10] implicitly in 1988.

Definition 2 The function $f : K \longrightarrow H$ is said to be g-convex (relative convex), if there exists a function g such that

$$f(g(u) + t(g(v) - g(u))) \leq (1-t)f(g(u)) + tf(g(v)),$$
$$\forall u, v \in H : g(u), g(v) \in K, t \in [0,1].$$

Clearly every convex function is relative convex, but the converse is not true; see [3, 21]. For the properties, applications and other aspects of the relative convex functions and convex sets, see [1, 12, 16, 17] and the references therein.

For given continuous trifunction $F(.,.,.) : K \times K \times K \longrightarrow R$, continuous bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{\infty\}$ and nonlinear operators $T, g : H \longrightarrow H$, consider the problem of finding $u \in H : g(u) \in K$ such that

$$F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in H : g(v) \in K, \tag{1}$$

which is called the *mixed quasi general equilibrium problem with trifunction,* introduced and studied by Noor and Rassias [19].

We now discuss some special cases.

I. If $g \equiv I$, where I is the identity operator, then problem (1) is equivalent to finding $u \in K$ such that

$$F(u, T(u), v) + \varphi(v, u) - \varphi(u, u) \geq 0, \qquad \forall v \in K, \tag{2}$$

which is the mixed quasi equilibrium problem with trifunction, introduced and studied by Noor [15, 17].

II. We note that for $F(g(u), T(g(u)), g(v)) = \langle T(g(u)), g(v) - g(v)\rangle$, problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K. \tag{3}$$

Inequality (3) is known as the *mixed quasi general variational inequality,* which was introduced by Noor [15].

III. If $\varphi(.,.) = \varphi(.)$ is the indicator function of a closed and relative convex-valued set $K(u)$, then problem (1) reduces to finding $u \in H : g(u) \in K(u)$ such that

$$F(g(u), T(g(u)), g(v)) \geq 0, \forall v \in H : g(v) \in K(u), \tag{4}$$

which is called the general quasi equilibrium problem and appears to be a new one.

IV. If $F(g(u), T(g(u)), g(v) = \langle T(g(u)), g(v) - g(u)\rangle$, then problem (4) is equivalent to finding $u \in H : g(u) \in K(u)$ such that

$$\langle T(g(u)), g(v) - g(u)\rangle \geq 0, \forall v \in H : g(v) \in K(u), \tag{5}$$

which is known as the general quasi variational inequality introduced by Noor [15]. For the applications and numerical methods of general quasi variational inequalities; see [3–20] and the references therein.

V. If $g = I$, the identity operator, the general quasi variational inequalities (3) are equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \ \ \forall \ \ v \in K, \tag{6}$$

which are known as the mixed quasi variational inequalities; see [3–19].

VI. We note that for $F(g(u), T(g(u)), g(v)) = B(g(u)), T(g(u)), g(v) - g(v)\rangle$, problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$B(g(u), T(g(u)), g(v) - g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0,$$

$$\forall v \in H : g(v) \in K. \tag{7}$$

Inequality (7) is known as the *mixed quasi general trifunction variational inequality,* which appears to be new one.

It is clear that problems (2)–(7) are special cases of the general equilibrium problems (1). In brief, for a suitable and appropriate choice of the operators T, g, and the space H, one can obtain a wide class of equilibrium, variational inequalities, and complementarity problems. This clearly shows that problem (1) is quite general and unifying one. Furthermore, problem (1) has important applications in various branches of pure and applied sciences; see [1, 2, 3–22].

Definition 3 [19]. The trifunction $F(.,.,.) : K \times K \times K \to R$ with respect to the operators T, g, is said to be:

(i) *partially relaxed jointly strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$F(g(u), T(g(u))g(v)) + F(g(v), T(g(v)), g(z)) \leq \alpha \|g(z) - g(u)\|^2, \forall u, v, z \in K.$$

(ii) *jointly monotone*, if

$$F(g(u), T(g(u)), g(v)) + F(g(v), T(g(v)), g(u)) \leq 0, \forall u, v \in K.$$

(iii) *jointly pseudomonotone,* if

$$\begin{aligned} &F(g(u), T(g(u)), g(v)) + \varphi(g(v) - g(u)) - \varphi(g(u), g(u)) \geq 0 \\ &\qquad \Longrightarrow \\ &- F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall u, v \in K. \end{aligned}$$

(iv) *jointly hemicontinuous,* $\forall u, v \in K, t \in [0,1]$, if the mapping $F(g(u) + t(g(v) - g(u)), T(g(u) + t(g(v) - g(u)), g(v))$ is continuous.

We remark that if $z = u$, then partially relaxed jointly strongly monotonicity is exactly jointly monotonicity of the operator $F(.,.,.)$. For $g \equiv I$, the identity operator, Definition 2.1 reduces to the standard definition of partially relaxed jointly strongly monotonicity, jointly monotonicity, and jointly pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity, but not conversely. This implies that the concepts of partially relaxed strongly monotonicity and pseudomonotonicity are weaker than monotonicity.

Noor and Rassias [19] have proved that problem (1) is equivalent to its dual problem under some conditions. We include this result due to its importance. We include all the details for the sake of completeness and to convey the main idea of the technique involved.

Lemma 1 *Let $F(.,.,.)$ be jointly pseudomonotone, jointly hemicontinuous, and relative convex with respect to third argument. If the bifunction $\varphi(.,.)$ is relative convex with respect to first argument, then the general equilibrium problem* (1) *is equivalent to finding $u \in H : g(u) \in K$ such that*

$$-F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K. \tag{8}$$

Proof Let $u \in H : g(u) \in K$ be a solution of (1). Then

$$F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K$$

which implies

$$-F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in H : g(v) \in K, \tag{9}$$

since $F(.,.,.)$ is jointly monotone

Conversely, let $u \in K$ satisfy (8). Since K is a g-convex set, $\forall u, v \in H : g(u), g(v) \in K, t \in [0,1], g(v_t) = g(u) + t(g(v) - g(u)) \equiv (1-t)g(u) + tg(v) \in K$.

Taking $g(v) = g(v_t)$ in (9), we have

$$\begin{aligned} F(g(v_t), T(g(v_t)), g(u)) &\leq \varphi(g(v_t), g(u)) - \varphi(g(u), g(u)) \\ &\leq t\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\}. \end{aligned} \tag{10}$$

Now using (10) and relative convexity of $F(.,.)$ with respect to third argument, we have

$$\begin{aligned} 0 &\leq F(g(v_t), T(g(v_t)), g(v_t)) \\ &= F(g(v_t), T(g(v_t)), (1-t)g(u) + tg(v)) \\ &\leq tF(g(v_t), T(g(v_t)), g(v)) + (1-t)F(g(v_t), T(g(v_t)), g(u)) \\ &\leq tF(g(v_t), T(g(v_t)), g(v)) + t(1-t)\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\} \end{aligned} \tag{11}$$

Dividing (11) by t and letting $t \longrightarrow 0$, we have

$$F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \forall v \in K,$$

the required (1). □

Remark 1 Problem (8) is known as the *dual mixed quasi general equilibrium problem.* One can easily show that the solution set of problem (8) is closed and relative convex set. From Lemma 2.1, it follows that the solution set of problems (1) and (8) are the same. This inter relationship has played an important role in the study of well-posedness of equilibrium problems and variational inequalities. In fact, Lemma 2.1 can be viewed as a natural generalization and extension of a well-known Minty's Lemma in variational inequalities theory; see [5, 6, 8].

Definition 4 The bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is called *skew symmetric,* if and only if,

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) - \varphi(v, v) \geq 0, \forall u, v \in H.$$

Clearly if the skew-symmetric bifunction $\varphi(.,.)$ is bilinear, then

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \forall u, v \in H.$$

This shows that the bifunction $\varphi(.,.)$ is positive.

3 Main Results

In this section, we suggest and analyze some new iterative methods for solving the problem (1) by using the auxiliary principle technique [5] as developed by Noor [13, 15, 17] and Noor et al. [18] in recent years.

For a given $u \in H : g(u) \in K$ satisfying (1), consider the problem of finding a unique $w \in H : g(w) \in K$ such that

$$\begin{aligned}&\rho F(g(w), T(g(w)), g(v)) + \langle (1-\lambda)(g(w) - g(u)), g(v) - g(w)\rangle \\ &\geq \rho\{\varphi(g(w), g(w)) - \varphi(g(v), g(w))\}, \forall v \in H : g(v) \in K,\end{aligned} \tag{12}$$

which is called the auxiliary mixed quasi general equilibrium problem and where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the nonconvex equilibrium problems (1). This observation enables us to suggest the following method for solving (1).

Algorithm 1 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned}&\rho F(g(u_{n+1}), T(g(u_{n+1})), g(v)) + \langle (1-\lambda)(g(u_{n+1} - g(u_n)), g(v) - g(u_{n+1})\rangle \\ &\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall v \in H : g(v) \in K,\end{aligned} \tag{13}$$

where $\lambda > 0$ is a constant. Algorithm 1 is called the implicit method for solving (1).

We may write Algorithm 1 in the following equivalent form, which is useful to derive other iterative methods for solving (1) and related problems.

Algorithm 2 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned}&\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K \\ &\rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - g(u_n) - \lambda(g(y_n) - g(u_n)), g(v) - g(u_{n+1})\rangle \\ &\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K\end{aligned}$$

For $\lambda = 0$, Algorithm 2 collapses to:

Algorithm 3 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned}&\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K \\ &\rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K.\end{aligned}$$

Algorithm 3 is analogues of the extragradient method of Korpelevich, see [16] and appears to be a new one.

For $\lambda = 1$, Algorithm 3.2 reduces to the following two-step iterative method for solving (1). Such type of methods have been studied and investigated by Noor [16, 17] for general variational inequalities.

Algorithm 4 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K$$
$$\rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - g(y_n), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K$$

For $\lambda = \frac{1}{2}$, Algorithm 2 reduces to:

Algorithm 5 [17]. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_n), T(g(u_n)), g(v)) + \langle g(y_n - g(u_n), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(y_n), g(y_n) - \varphi(g(v), g(y_n))\}, \forall g(v) \in K$$
$$\rho F(g(y_n), T(g(y_n)), g(v)) + \langle g(u_{n+1} - \frac{1}{2}(g(y_n) + g(u_n)), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K$$

Note that if $g \equiv I$, the identity operator, Algorithm 1 reduces to a method for solving the equilibrium problems with trifunction (2), which are mainly due to Noor [17].

Algorithm 6 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, T(u_{n+1}, v) + (1 - \lambda)(u_{n+1} - u_n), v - u_{n+1}\rangle$$
$$\geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\} \geq 0, \forall v \in K.$$

For the convergence analysis of Al; Algorithm 6, see Noor [17].

For $F(g(u), T(g(u)), (v)) = \langle T(g(u)), g(v) - g(u)\rangle$, Algorithm 1 reduces to:

Algorithm 7 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(g(u_{n+1})) + (1 - \lambda)(g(u_{n+1} - (g(u_n)), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(v), g(u_{n+1}))\}, \forall v \in K,$$

for solving mixed quasi general variational inequalities [17].

For suitable and appropriate choice of the operators and the space H, one can obtain various new and known methods for solving general equilibrium, variational inequalities, and complementarity problems.

We now study the convergence analysis of Algorithm 1.

Theorem 1 Let the trifunction $F(.,.,.)$ be jointly pseudomonotone. If the bifunction $\varphi(.,.)$ is skew symmetric, then the approximate solution u_{n+1}obtained from Algorithm 1 satisfies the inequality

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_n) - g(u_{n+1})\|^2, \tag{14}$$

where u is the exact solution of (1).

Proof Let $u \in H : g(u) \in K$ be a solution of (1). Then

$$F(g(u), T(g(u)), g(v)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u)) \forall v \in H : g(v) \in K,$$

which implies that

$$-F(g(v), T(g(v), g(u)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u)), \forall v \in H : g(v) \in K, \tag{15}$$

since $F(.,.,.)$ is jointly pseudomonotone.

Taking $v = u_{n+1}$ in (15), we have

$$-F(g(u_{n+1}), T(g(u_{n+1})), g(u)) \geq \varphi(g(u), g(u)) - \varphi(g(u_{n+1}), g(u)) \tag{16}$$

Taking $v = u$ in (13), we have

$$\begin{aligned} &\rho F(g(u_{n+1}), T(g(u_{n+1})), g(u)) + \langle (1-\lambda)(g(u_{n+1}) - g(u_n)), g(u) - g(u_{n+1}) \rangle \\ &\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(u), g(u_{n+1}))\}. \end{aligned} \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} &(1-\lambda)\langle g(u_{n+1}) - g(u_n) \rangle \\ &\geq \rho\{\varphi(g(u_n), g(u_n)) - \varphi(g(_{n+1}), g(u)) - \varphi(g(u), g(u_{n+1}) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\geq 0, \end{aligned} \tag{18}$$

where we have used the fact that the bifunction $\varphi(.,.)$ is a skew symmetric.

From (18) and using the inequality

$$2\langle v, u \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \forall u, v \in H,$$

we obtain

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_)\|^2 - \|g(u_n) - g(u_{n+1})\|^2,$$

which is the required result. □

Theorem 2 Let H be a finite dimensional space. Let the trifunction $F(.,.,.)$ be jointly pseudomonotone and the bifunction $\varphi(.,.)$ be skew symmetric. If u_{n+1} is the approximate solution obtained from Algorithm 3.1, and g^{-1} exists, then

$$\lim_{n \longrightarrow \infty} u_n = u,$$

where $u \in H; g(u) \in K$ is a solution of (1).

Proof Let $u \in H : g(u) \in K$ be a solution of (1). From (14), we see that the sequences $\{\|g(u) - g(u_n)\|\}$ is nonincreasing under the assumptions of Theorem 2 and consequently $\{g(u_n)\}$ is bounded. Also from (14), we have

$$\sum_{n=0}^{\infty} \|g(u_{n+1} - g(u_n)\|^2 \leq \|g(u) - g(u_n)\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0, \tag{19}$$

since g^{-1} exists.

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_i}\}$ of this sequence converges to $\hat{u} \in H : g(\hat{u}) \in K$. Replacing u_n by u_{n_i} in (13) and taking the limit as $n_i \longrightarrow \infty$ and using (19), we have

$$F(g(\hat{u}), T(g(\hat{u})), g(v)) + \varphi(g(v), g(\hat{u})) - \varphi(g(\hat{u}), g(\hat{u})) \geq 0, \forall v \in H : g(v) \in K,$$

which shows that $\hat{u}$ solves (1) and

$$\|g(u_{n+1}) - g(\hat{u})\| \leq \|g(u_n) - g(\hat{u})\|^2.$$

Thus, it follows that from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and

$$\lim_{n \longrightarrow \infty} = \hat{u},$$

the required result. □

Algorithm 1 is an implicit method, which is its difficult to implement. In order to overcome this drawback, we again use the auxiliary principle technique to suggest an explicit iterative method for solving problem (1). This is the main motivation of next Algorithm.

For a given $u \in H : g(u) \in K$ satisfying (1), consider the problem of finding a unique $w \in H : g(w) \in K$ such that

$$\begin{aligned} &\rho F(g(u), T(g(u)), g(v)) + \langle (1 - \lambda)(g(w) - g(u)), g(v) - g(w) \rangle \\ &\geq \rho\{\varphi(g(w), g(w)) - \varphi(g(v), g(w))\}, \forall v \in H : g(v) \in K, \end{aligned} \tag{20}$$

which is called the auxiliary mixed quasi general equilibrium problem. we would like to emphasize that problems (12) and (20) are quite different from each other.

We note that if $w = u$, then clearly w is a solution of the nonconvex equilibrium problems (1). This observation enables us to suggest the following method for solving (1).

Algorithm 8 For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(g(u_{n+1}), T(g(u_{n+1})), g(v)) + \langle (1-\lambda)(g(u_{n+1} - g(u_n)), g(v) - g(u_{n+1})\rangle$$
$$\geq \rho\{\varphi(g(u_{n+1}), g(u_{n+1}) - \varphi(g(v), g(u_{n+1}))\}, \forall v \in H : g(v) \in K.$$

Algorithm 1 is called the explicit method for solving (1). Using the technique of Theorem 1 and Theorem 2, one can study the convergence analysis of Algorithm 8.

Conclusion In this chapter, we have suggested some new unified iterative methods for solving a class of mixed quasi general equilibrium problems, introduced and studied by Noor and Rassias [19]. The comparison of these methods with other methods is an interesting and fascinating problem for future research. One may find the novel and innovative applications of these general equilibrium problems in various branches of pure and applied sciences.

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Contractive Operators in Relational Metric Spaces

Mihai Turinici

Abstract In Sect. 1, some fixed point results for altering contractive maps on (amorphous) metric spaces are given, extending the one due to Khan, Swaleh and Sessa [Bull Aust Math Soc 30:1–9, 1984]. In Sect. 2, a class of monotone contractions is analyzed, via coupled fixed point techniques, in the realm of quasi-ordered metric spaces. Note that, a highly unusual feature of the related fixed point techniques is that, in many cases with a practical relevance, no coupled starting point hypothesis for these operators is needed. Finally, in Sect. 3, some fixed point results are given for contractive operators acting on relational metric spaces.

Keywords Metric space · Picard operator · Altering contractive map · Quasi-order · Monotone application · Ran–Reurings theorem · Coupled fixed point · Relation · Meir–Keeler contraction

1 Altering Contractive Maps

1.1 Introduction

Let X be a nonempty set; and $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over it. Call the subset Y of X, *almost singleton* (in short: *asingleton*) provided $y_1, y_2 \in Y$ implies $y_1 = y_2$; and *singleton*, if, in addition, Y is nonempty; note that, in this case, $Y = \{y\}$, for some $y \in X$. Further, let $T \in \mathcal{F}(X)$ be a selfmap of X. (Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ stands for the class of all functions from A to B; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$). Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. The determination of such elements is to be performed in the context below, comparable with the one in Rus [34, Chap. 2, Sect. 2.2]:

(1a) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent

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(1b) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent and $\lim_n (T^n x)$ belongs to $\mathrm{Fix}(T)$

(1c) We say that T is a *globally strong Picard operator* (modulo d) if it is a strong Picard operator (modulo d), and $\mathrm{Fix}(T)$ is an asingleton (hence, a singleton).

In this perspective, a basic result to the question we deal with is the 1922 one due to Banach [2]: it states that, whenever T is $(d; \alpha)$-*contractive*, i.e.,

$$\text{(a01)} \;\; d(Tx, Ty) \leq \alpha d(x, y), \;\; \forall x, y \in X,$$

for some $\alpha \in [0, 1[$, then T is a globally strong Picard operator (modulo d). This result found a multitude of applications in operator equations theory; so, it was the subject of many extensions. For example, a natural way of doing this is by considering "functional" contractive conditions of the form

$$\text{(a02)} \;\; d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),$$
$$\forall x, y \in X;$$

where $F : R_+^5 \to R_+$ is an appropriate function. For more details about the possible choices of F, we refer to the 1977 paper by Rhoades [33]; see also Turinici [40]. Here, we shall be concerned with a 2004 contribution in the area due to Berinde [4]. Given $\alpha, \lambda \geq 0$, let us say that T is a *weak* $(d; \alpha, \lambda)$-*contraction*, provided

$$\text{(a03)} \;\; d(Tx, Ty) \leq \alpha d(x, y) + \lambda d(Tx, y), \;\; \text{for all } x, y \in X.$$

Theorem 1 *Suppose that T is a weak $(d; \alpha, \lambda)$-contraction, where $\alpha \in [0, 1[$. In addition, let (X, d) be complete. Then, T is a strong Picard operator (modulo d).*

In a subsequent paper devoted to the same question, Berinde [3] claims that this class of contractions introduced by him is for the first time considered in the literature. Unfortunately, his assertion is not true: conclusions of Theorem 1 are "almost" covered by a related 1984 statement due to Khan et al. [20], in the context of altering distances. This, among others, motivated us to propose an appropriate extension of the quoted statement. Also, for completeness reasons, we provide a "functional" extension of Berinde's result.

1.2 Preliminaries

Let (X, d) be a metric space. We say that the sequence (x_n) in X, d-*converges* to $x \in X$ (and write this as: $x_n \xrightarrow{d} x$), iff $d(x_n, x) \to 0$; that is

$$\text{(b01)} \;\; \forall \varepsilon > 0, \exists p = p(\varepsilon): \;\; p \leq n \Longrightarrow d(x_n, x) \leq \varepsilon.$$

Denote $\lim_n (x_n) = \{x \in X; x_n \xrightarrow{d} x\}$; when the underlying set is nonempty, (x_n) is called d-*convergent*. Note that, in this case, $\lim_n (x_n)$ is a singleton, $\{z\}$; as usually, we write $\lim_n (x_n) = z$. Further, let us say that (x_n) is d-*Cauchy*, provided $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; that is

(b02) $\forall \varepsilon > 0, \exists q = q(\varepsilon)$: $q \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon$.

Clearly, any d-convergent sequence is d-Cauchy too; when the reciprocal holds too, (X, d) is called *complete*. Concerning this aspect, note that any d-Cauchy sequence $(x_n; n \geq 0)$ is *d-semi-Cauchy*, i.e.,

(b03) $\rho_n := d(x_n, x_{n+1}) \to 0$ (hence, $d(x_n, x_{n+i}) \to 0, \forall i \geq 1$), as $n \to \infty$.

The following result about such objects is useful in the sequel. Given the sequence $(r_n; n \geq 0)$ in R and the point $r \in R$, let us write

$r_n \to r+$ (respectively, $r_n \to r++$), if $r_n \to r$ and
$r_n \geq r$ (respectively, $r_n > r$), for all $n \geq 0$ large enough.

Proposition 1 *Suppose that $(x_n; n \geq 0)$ is d-semi-Cauchy, but not d-Cauchy. There exists then $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that*

$$j \leq m(j) < n(j), \ \ \alpha(j) := d(x_{m(j)}, x_{n(j)}) > \eta, \ \forall j \geq 0 \tag{1}$$

$$n(j) - m(j) \geq 2, \ \beta(j) := d(x_{m(j)}, x_{n(j)-1}) \leq \eta, \ \forall j \geq j(\eta) \tag{2}$$

$$\alpha(j) \to \eta++ \text{ (hence, } \alpha(j) \to \eta) \text{ as } j \to \infty \tag{3}$$

$$\alpha_{p,q}(j) := d(x_{m(j)+p}, x_{n(j)+q}) \to \eta, \text{ as } j \to \infty, \ \forall p, q \in \{0, 1\}. \tag{4}$$

A proof of this may be found in Khan et al. [20]. For completeness reasons, we supply an argument which differs, in part, from the original one.

Proof **(Proposition 1)** As $(x_n; n \geq 0)$ is not d-Cauchy, there exists $\eta > 0$ with

$$A(j) := \{(m, n) \in N \times N; j \leq m < n, d(x_m, x_n) > \eta\} \neq \emptyset, \ \forall j \geq 0.$$

Having this precise, denote, for each $j \geq 0$,

$m(j) = \min\mathrm{Dom}(A(j))$, $n(j) = \min A(m(j))$.

As a consequence, the couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ fulfills (1). On the other hand, letting the index $j(\eta) \geq 0$ be such that

$$d(x_k, x_{k+1}) < \eta, \ \ \forall k \geq j(\eta), \tag{5}$$

it is clear that (2) holds too. Finally, by the triangular property,

$$\eta < \alpha(j) \leq \beta(j) + \rho_{n(j)-1} \leq \eta + \rho_{n(j)-1}, \ \ \forall j \geq j(\eta);$$

and this yields (3); hence, the case $(p = 0, q = 0)$ of (4). Combining with

$$\alpha(j) - \rho_{n(j)} \leq d(x_{m(j)}, x_{n(j)+1}) \leq \alpha(j) + \rho_{n(j)}, \ \forall j \geq j(\eta)$$

establishes the case $(p = 0, q = 1)$ of the same. The remaining situations are deductible in a similar way.

1.3 Main Result

Let (X, d) be a metric space; and $\varphi \in \mathcal{F}(R_+)$ be an *altering function*; i.e.,

(c01) φ is continuous, increasing, and reflexive-sufficient [$\varphi(t) = 0$ iff $t = 0$].

The associated map (from $X \times X$ to R_+)

(c02) $e(x, y) = \varphi(d(x, y)),\ \ x, y \in X$

has the immediate properties

$$e(x, y) = e(y, x),\ \forall x, y \in X \quad (e \text{ is symmetric}) \tag{6}$$

$$e(x, y) = 0 \Longleftrightarrow x = y \quad (e \text{ is reflexive-sufficient}). \tag{7}$$

So, it is a (reflexive sufficient) *symmetric*, under the Hicks–Rhoades terminology [13]. In general, $e(.,.)$ is not endowed with the triangular property; but, in compensation to this, one has (as φ is increasing and continuous)

$$e(x, y) > e(u, v) \Longrightarrow d(x, y) > d(u, v) \tag{8}$$

$$x_n \xrightarrow{d} x, y_n \xrightarrow{d} y \ \text{ implies } \ e(x_n, y_n) \to e(x, y). \tag{9}$$

Let in the following, $T \in \mathcal{F}(X)$ be a selfmap of X. The formulation of the problem involving $\mathrm{Fix}(T)$ is the already sketched one. In the following, we are trying to solve it in the precise context. Denote, for $x, y \in X$,

(c03) $M_1(x, y) = e(x, y)$, $M_2(x, y) = (1/2)[e(x, Tx) + e(y, Ty)]$,
$M_3(x, y) = \min\{e(x, Ty), e(Tx, y)\}$,
$M(x, y) = \max\{M_1(x, y), M_2(x, y), M_3(x, y)\}$.

Further, given $\psi \in \mathcal{F}(R_+)$, we say that T is $(d, e; M, \psi)$*-contractive*, provided

(c04) $e(Tx, Ty) \leq \psi(d(x, y))M(x, y)$, $\forall x, y \in X, x \neq y$.

The properties of ψ to be used here write

(c05) ψ is strictly subunitary on $R_+^0 :=]0, \infty[$: $\psi(s) < 1$, $\forall s \in R_+^0$
(c06) ψ is right Boyd–Wong on R_+^0: $\limsup_{t \to s+} \psi(t) < 1$, $\forall s \in R_+^0$.

This is related to the developments in Boyd and Wong [10]; we do not give details. The main result of this exposition is as follows.

Theorem 2 *Suppose that T is $(d, e; M, \psi)$-contractive, where $\psi \in \mathcal{F}(R_+)$ is strictly subunitary and right Boyd–Wong on R_+^0. In addition, let (X, d) be complete. Then, T is a globally strong Picard operator (modulo d).*

Proof First, let us check the asingleton property for $\mathrm{Fix}(T)$. Let $z_1, z_2 \in \mathrm{Fix}(T)$ be such that $z_1 \neq z_2$; hence $\delta := d(z_1, z_2) > 0$, $\varepsilon := e(z_1, z_2) > 0$. By definition,

$$M_1(z_1, z_2) = \varepsilon,\ \ M_2(z_2, z_2) = 0,\ \ M_3(x, y) = \varepsilon;\ \text{hence } M(x, y) = \varepsilon.$$

By the contractive condition (written at (z_1, z_2))

$$\varepsilon = e(z_1, z_2) = e(Tz_1, Tz_2) \le \psi(\delta)M(z_1, z_2) = \psi(\delta)\varepsilon;$$

hence, $1 \le \psi(\delta) < 1$; contradiction; and the asingleton property follows. It remains now to verify the strong Picard property. Fix some $x_0 \in X$; and put $(x_n = T^n x_0;$ $n \ge 0)$. If $x_n = x_{n+1}$ for some $n \ge 0$, we are done; so, without loss, one may assume

(c07) $\rho_n := d(x_n, x_{n+1}) > 0$ (hence, $\sigma_n := e(x_n, x_{n+1}) > 0$), for all n.

There are several steps to be passed.

(I) For the arbitrary fixed $n \ge 0$, we have

$$M_1(x_n, x_{n+1}) = \sigma_n,$$
$$M_2(x_n, x_{n+1}) = (1/2)[\sigma_n + \sigma_{n+1}] \le \max\{\sigma_n, \sigma_{n+1}\},$$
$$M_3(x_n, x_{n+1}) = 0;\ \text{hence, } M(x_n, x_{n+1}) \le \max\{\sigma_n, \sigma_{n+1}\}.$$

By the contractive condition (written at (x_n, x_{n+1})),

$$\sigma_{n+1} \le \psi(\rho_n) \max\{\sigma_n, \sigma_{n+1}\},\ \forall n.$$

This, by the working condition, yields (as ψ is strictly subunitary on R_+^0)

$$\sigma_{n+1}/\sigma_n \le \psi(\rho_n) < 1,\ \forall n. \tag{10}$$

As a direct consequence,

$$\sigma_n > \sigma_{n+1} \text{ (hence, } \rho_n > \rho_{n+1}\text{), for all } n.$$

The sequence $(\rho_n; n \ge 0)$ is therefore strictly descending in R_+; hence, $\rho := \lim_n (\rho_n)$ exists in R_+ and $\rho_n > \rho, \forall n$. Likewise, the sequence $(\sigma_n = \varphi(\rho_n); n \ge 0)$ is strictly descending in R_+; hence, $\sigma := \lim_n (\sigma_n)$ exists; with, in addition, $\sigma = \varphi(\rho)$. We claim that $\rho = 0$. Assume by contradiction that $\rho > 0$; hence $\sigma > 0$. Passing to $\limsup$ as $n \to \infty$ in (10) yields

$$1 \le \limsup_n \psi(\rho_n) \le \limsup_{t \to \rho+} \psi(t) < 1;$$

contradiction. Hence, $\rho = 0$; i.e.,

$$\rho_n := d(x_n, x_{n+1}) \to 0,\ \text{as } n \to \infty. \tag{11}$$

(II) We now show that $(x_n; n \ge 0)$ is d-Cauchy. Suppose that this is not true. By Proposition 1, there exist $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \ge 0)$, $(n(j); j \ge 0)$, in such a way that (1)–(4) hold. Denote for simplicity $\zeta = \varphi(\eta)$; hence, $\zeta > 0$. By the notations used there, we may write as $j \to \infty$

$$\lambda_j := e(x_{m(j)+1}, x_{n(j)+1}) = \varphi(\alpha_{1,1}(j)) \to \zeta.$$

In addition, we have (again under $j \to \infty$)

$$M_1(x_{m(j)}, x_{n(j)}) = \varphi(\alpha(j)) \to \zeta$$
$$M_2(x_{m(j)}, x_{n(j)}) = (1/2)[\varphi(\rho_{m(j)}) + \varphi(\rho_{n(j)})] \to 0$$
$$M_3(x_{m(j)}, x_{n(j)}) = \min\{\varphi(\alpha_{0,1}(j)), \varphi(\alpha_{1,0}(j))\} \to \zeta;$$

and this, by definition, yields

$$\mu_j := M(x_{m(j)}, x_{n(j)}) \to \zeta \text{ as } j \to \infty.$$

From the contractive condition (written at $(x_{m(j)}, x_{n(j)})$)

$$\lambda_j/\mu_j \leq \psi(\alpha(j)), \ \forall j \geq j(\eta);$$

so that, passing to lim sup as $j \to \infty$

$$1 \leq \limsup_j \psi(\alpha(j)) \leq \limsup_{t \to \eta+} \psi(t) < 1;$$

contradiction. Hence, $(x_n; n \geq 0)$ is d-Cauchy, as claimed.

(III) As (X, d) is complete, there exists $z \in X$ with $x_n \xrightarrow{d} z$; hence, $\gamma_n := d(x_n, z) \to 0$ as $n \to \infty$.

Two alternatives are open before us:

(i) For each $h \in N$, there exists $k > h$ with $x_k = z$. In this case, there exists a sequence of ranks $(m(i); i \geq 0)$ with $m(i) \to \infty$ as $i \to \infty$ such that $x_{m(i)} = z$ (hence, $x_{m(i)+1} = Tz$), $\forall i$. Letting i tend to infinity and using the fact that $(y_i := x_{m(i)+1}; i \geq 0)$ is a subsequence of $(x_i; i \geq 0)$, we get $z = Tz$.

(ii) There exists $h \in N$ such that $n \geq h \Longrightarrow x_n \neq z$ (whence, $\gamma_n > 0$). Suppose that $z \neq Tz$; i.e., $\theta := d(z, Tz) > 0$; hence, $\omega := e(z, Tz) > 0$. Note that, in such a case, $\delta_n := d(x_n, Tz) \to \theta$. From our previous notations, we have (as $n \to \infty$)

$$\lambda_n := e(x_{n+1}, Tz) = \varphi(\delta_{n+1}) \to \varphi(\theta) = \omega.$$

In addition (again under $n \to \infty$),

$$M_1(x_n, z) = \varphi(\gamma_n) \to 0, \ M_2(x_n, z) = (1/2)[\sigma_n + \omega] \to \omega/2$$
$$M_3(x_n, z) = \min\{\varphi(\delta_n), \varphi(\gamma_{n+1})\} \to 0;$$

wherefrom,

$$(0 <)\mu_n := M(x_n, z) \to \omega/2, \text{ as } n \to \infty.$$

By the contractive condition (written at (x_n, z))

$$\lambda_n \leq \psi(\gamma_n)\mu_n < \mu_n, \ \forall n \geq h$$

we then have (passing to limit as $n \to \infty$), $\omega \leq \omega/2$; hence $\omega = 0$. This yields $\theta = 0$; contradiction. Hence, z is fixed under T and the proof is complete.

In particular, the right Boyd–Wong on R_+^0 property of ψ is assured when this function is strictly subunitary and decreasing on R_+^0. As a consequence, the following particular version of our main result is available:

Theorem 3 *Suppose that T is $(d, e; M, \psi)$-contractive, where $\psi \in \mathcal{F}(R_+)$ is strictly subunitary and decreasing on R_+^0. In addition, let (X, d) be complete. Then, T is globally strong Picard (modulo d).*

Let $a, b, c \in \mathcal{F}(R_+)$ be a triple of functions. We say that the selfmap T of X is $(d, e; a, b, c)$-*contractive* if

(c08) $e(Tx, Ty) \leq a(d(x, y))e(x, y) + b(d(x, y))[e(x, Tx) + e(y, Ty)]$
$+ c(d(x, y))\min\{e(x, Ty), e(Tx, y)\}, \ \forall x, y \in X, x \neq y.$

Denote for simplicity $\psi = a + 2b + c$; it is clear that, under such a condition, T is $(d, e; M; \psi)$-contractive. Consequently, the following statement is a particular case of Theorem 2 above:

Theorem 4 *Suppose that T is $(d, e; a, b, c)$-contractive, where the triple of functions $a, b, c \in \mathcal{F}(R_+)$ is such that the associated function $\psi = a + 2b + c$ is strictly subunitary and right Boyd–Wong on R_+^0. In addition, let (X, d) be complete. Then, conclusions of Theorem 2 hold.*

In particular, when a, b, c are all decreasing on R_+^0, the right Boyd–Wong property on R_+^0 of the function ψ is retainable; note that, in this case, Theorem 4 is also reducible to Theorem 3. This is just the 1984 fixed point result in Khan et al. [20].

Finally, it is worth mentioning that the nice contributions of these authors were the starting point for a series of results involving altering contractions, like the ones in Bhaumik et al. [9], Nashine and Samet [27], or Sastry and Babu [39]; see also Pathak and Shahzad [30]. However, according to the developments in Jachymski [17], most of these (including the Dutta-Choudhury's contribution [12]) are in fact reducible to standard techniques; we do not give details.

1.4 Further Aspects

Let again (X, d) be a metric space, and $T \in \mathcal{F}(X)$ be a selfmap of X. A basic particular case of Theorem 4 corresponds to the choices φ = identity and [a, b, c = constants]. The corresponding form of Theorem 4 is comparable with Theorem 1. However, the inclusion between these is not complete. This raises the question of determining proper extensions of Theorem 1, close enough to Theorem 4. A direct answer to this is provided as follows.

Theorem 5 *Let the numbers $a, b \in R_+$ and the function $K \in \mathcal{F}(R_+)$ be such that*

(d01) $d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + K(d(Tx, y)), \forall x, y \in X$
(d02) $a + 2b < 1$ *and* $K(t) \to 0 = K(0)$ *as* $t \to 0$.

In addition, let (X, d) be complete. Then, T is a strong Picard map (modulo d).

Proof Take an arbitrary fixed $u \in X$. By the very contractive condition (written at $(T^n u, T^{n+1}u)$), we have the evaluation

$$d(T^{n+1}u, T^{n+2}u) \le \lambda d(T^n u, T^{n+1}u), \ \ \forall n \ge 0. \tag{12}$$

where $\lambda := (a+b)/(1-b) < 1$. This yields

$$d(T^n u, T^{n+1}u) \le \lambda^n d(u, Tu), \ \ \forall n \ge 0. \tag{13}$$

Consequently, $(T^n u; n \ge 0)$ is d-Cauchy; whence (by completeness)

$$T^n u \xrightarrow{d} z := T^\infty u, \text{ for some } z \in X.$$

From the contractive condition (written at $(T^n u, z)$),

$$d(T^{n+1}u, Tz) \le ad(T^n u, z) + b[d(T^n u, T^{n+1}u) + d(z, Tz)] + K(d(T^{n+1}u, z)), \ \forall n.$$

Passing to limit as $n \to \infty$ gives (by the imposed conditions) $d(z, Tz) \le bd(z, Tz)$; so that (as $0 \le b < 1/2$), $d(z, Tz) = 0$; hence $z = Tz$. The proof is thereby complete.

In particular, when $b = 0$ and $K(.)$ is linear ($K(t) = \lambda t$, $t \in R_+$, for some $\lambda \ge 0$), this result is just Theorem 1. Note that, from (13), one has for these "limit" fixed points, the error approximation formula

$$d(T^n u, T^\infty u) \le [\lambda^n/(1-\lambda)]d(u, Tu), \ \ \forall n \in N. \tag{14}$$

However, the non-singleton property of Fix(T) makes this "local" evaluation to be without practical effect, by the highly unstable character of the map $u \mapsto T^\infty u$. In fact, assume for simplicity that T is continuous; and fix in the following $u_0 \in X$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X(u_0, \delta)$ implies $Tx \in X(Tu_0, \varepsilon)$; here, for each $x \in X$, $\rho > 0$, $X(x, \rho) = \{y \in X; d(x, y) < \rho\}$ (the open sphere with center x and radius ρ). The above evaluation (14) gives a "local-global" relation like

$$d(T^n u, T^\infty u) \le [\lambda^n/(1-\lambda)]\mu(u_0), \ \forall n \ge 0, \forall u \in X(u_0, \delta); \tag{15}$$

where, by definition, $\mu(u_0) = \sup\{d(x, Tx); x \in X(u_0, \delta)\}$. Now, in practice, the starting point u_0 is approximated by a certain $v_0 \in X(u_0, \delta)$; with, in general, $v_0 \ne u_0$. Suppose that the iterates $(T^n v_0; n \ge 0)$ are calculated in a complete (and exact) way. The approximation formula (15) gives, for the point in question,

$$d(T^n v_0, T^\infty v_0) \le [\lambda^n/(1-\lambda)]\mu(u_0), \ \forall n \ge 0. \tag{16}$$

This yields a good evaluation for the fixed point $T^\infty v_0$; but, it may have no impact upon the fixed point $T^\infty u_0$ (that we want to approximate), as long as it is distinct from the preceding fixed point.

Summing up, any such contraction T is Hyers–Ulam unstable, whenever Fix(T) is not a singleton. But, when Fix(T) is a singleton, T is Hyers–Ulam stable. Some related facts may be found in the 1998 monograph by Hyers, Isac and Rassias [14].

2 Monotone Contractive Maps

2.1 Introduction

Let X be a nonempty set. Take a metric $d(.,.)$ on it; as well as a *quasi-order* $(\le)$ (i.e.: reflexive and transitive relation) over X. Call the subset Y of X, $(\le)$-*almost-singleton* (in short: $(\le)$-*asingleton*) provided $y_1, y_2 \in Y$ and $y_1 \le y_2$ imply $y_1 = y_2$; and $(\le)$-*singleton*, if, in addition, Y is nonempty. Further, let $T \in \mathcal{F}(X)$ be a selfmap of X; endowed with the properties

(a01) T is $(\le)$-increasing: $x \le y$ implies $Tx \le Ty$
(a02) T is almost $(\le)$-progressive: $X(T, \le) := \{x \in X; x \le Tx\} \neq \emptyset$.

The determination of elements in $\mathrm{Fix}(T)$ is to be performed in the context below, comparable with the one in Turinici [42]:

(1a) We say that T is a *Picard operator* (modulo $(d, \le)$) if, for each $x \in X(T, \le)$, $(T^n x; n \ge 0)$ is d-convergent

(1b) We say that T is a *strong Picard operator* (modulo $(d, \le)$) if, for each $x \in X(T, \le)$, $(T^n x; n \ge 0)$ is d-convergent and $\lim_n (T^n x)$ belongs to $\mathrm{Fix}(T)$

(1c) We say that T is a *globally strong Picard operator* (modulo $(d, \le)$) if it is a strong Picard operator (modulo $(d, \le)$), and $\mathrm{Fix}(T)$ is $(\le)$-asingleton (hence, necessarily, $(\le)$-singleton).

A useful result in the area is the 2008 one obtained by Agarwal, El-Gebeily and O'Regan [1]. This needs some conventions and specific requirements. Given $\varphi \in \mathcal{F}(R_+)$, let us introduce the condition

(a03) T is $(d, \le; \varphi)$-contractive: $d(Tx, Ty) \le \varphi(d(x, y))$, for all $x, y \in X, x \le y$.

The functions to be taken into consideration here are as follows. Call $\varphi \in \mathcal{F}(R_+)$ *(strongly) regressive*, provided: $\varphi(0) = 0$ and $\varphi(t) < t$, $\forall t \in R_+^0$. The class of all these will be denoted as $\mathcal{F}(re)(R_+)$; and the subclass of all increasing $\varphi \in \mathcal{F}(re)(R_+)$ is indicated as $\mathcal{F}(re, in)(R_+)$. Given $\varphi \in \mathcal{F}(re, in)(R_+)$, let us say that it is *Matkowski admissible*, provided

(a04) $\varphi^n(t) \xrightarrow{d} 0$ as $n \to \infty$, for all $t \in R_+^0$;

here, for each $n \ge 0$, φ^n denotes the nth iterate of φ. This concept is related to the developments in Matkowski [24]; we do not give details.

Theorem 6 *Suppose that T is $(d, \le; \varphi)$-contractive, for some Matkowski admissible function $\varphi \in \mathcal{F}(re, in)(R_+)$. In addition, let (X, d) be complete and one of the assumptions below hold:*

(i) *T is continuous: $x_n \xrightarrow{d} x$ implies $Tx_n \xrightarrow{d} Tx$*

(ii) *$(\le)$ is d-selfclosed: the d-limit of each ascending sequence is an upper bound of it (with respect to $(\le)$).*

Then, T is a strong Picard operator (modulo $(d, \le)$).

Now, for technical reasons (to be explained a bit further) it would be useful for us to determine under which conditions upon these data, T is a globally strong Picard

operator (modulo d). A basic contribution in the area is the 2004 one obtained by Ran and Reurings [32].

(A) Let $(X, d, \leq)$ be an ordered metric space. Define a relation $(<>)$ on X, as

(a05) $x <> y$ iff either $x \leq y$ or $y \leq x$ (i.e.: x and y are *comparable*).

This relation is reflexive and symmetric; but not in general transitive. Further, let T be a selfmap of X. The following conditions are to be used here:

(a06) $(X, \leq)$ is (up/down)-directed: $\forall x, y \in X$, $\{x, y\}$ has upper and lower bounds
(a07) T is almost progressive (regressive): $x \leq Tx$ ($x \geq Tx$), for at least one $x \in X$
(a08) T is almost progressive/regressive: $x <> Tx$, for at least one $x \in X$
(a09) T is monotone (increasing or decreasing).

Finally, given $\alpha > 0$, let us say that T is $(d, \leq; \alpha)$*-contractive*, if

(a10) $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall x, y \in X, x \leq y$;

note that, by the preceding convention, this may be also expressed as:

(a11) $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall x, y \in X, x <> y$.

The announced answer may now be written as below:

Theorem 7 *Assume that T is $(d, \leq; \alpha)$-contractive, for some $\alpha \in]0, 1[$. In addition, let (X, d) be complete, $(X, \leq)$ be (up/down)-directed, and T be almost progressive/regressive, monotone, d-continuous. Then, T is a globally strong Picard operator (modulo d).*

According to many authors (cf. [1, 28, 29] and the references therein), this result (referred to as: Ran–Reurings theorem) is credited to be the first extension of the 1922 Banach theorem [2] to the realm of (partially) ordered metric spaces. Unfortunately, the assertion is not true; some early statements of this type have been obtained two decades ago by Turinici [41], in the context of ordered metrizable uniform spaces.

Now, as Ran–Reurings theorem (expressed in a quasi-order setting) extends Banach's, it is natural to discuss its position within the classification scheme proposed by Rhoades [33]. The conclusion to be derived reads: the Ran–Reurings theorem is but a particular case of the 1968 fixed point statement in Maia [23]. Further, an application of this result is given to functional type coupled fixed point statements. The obtained facts are then applied to fixed point problems involving component-wise monotone operators acting on product quasi-ordered metric spaces.

2.2 Ran–Reurings Results

In the following, some extended variants are given for the Ran–Reurings result above.

(A) Let X be a nonempty set. Take a metric $d(.,.)$ on it; and let $(\leq)$ be a *quasi-order* (i.e., reflexive and transitive relation) over X; the triple $(X, d, \leq)$ will

be referred to as a *quasi-ordered metric space*. Further, let T be a selfmap of X. As before, we are interested to determine sufficient conditions involving these data so as T be globally strong Picard (modulo d). Technically speaking, we have: **(I)** conditions upon $(X, d, \leq)$, and **(II)** conditions upon T.

The first category of conditions refers to completeness and chain properties.

(I-a) The following completeness properties of our structure are to be used here:

(b01) (X, d) is complete: each d-Cauchy sequence in X is d-convergent
(b02) (X, d) is $(\leq)$-complete: each ascending d-Cauchy sequence in X is d-convergent.

Clearly, the former of these implies the latter; the reciprocal is not in general valid.

(I-b) The next condition upon the same structure needs a lot of conventions. For each $x, y \in X$, denote: $x <> y$ iff either $x \leq y$ or $y \leq x$ (i.e., x and y are comparable). This relation is reflexive and symmetric; but not in general transitive. Given $x, y \in X$ and $k \geq 2$, any element $A = (z_1, \ldots, z_k) \in X^k$ with $z_1 = x$, $z_k = y$, and $(z_i <> z_{i+1}, i \in \{1, \ldots, k-1\})$, will be referred to as a *k-dimensional $(<>)$-chain* between x and y; in this case, $k = \dim(A)$ (the *dimension* of A) and $\Lambda(A) = d(z_1, z_2) + \ldots + d(z_{k-1}, z_k)$ is the *length* of A; the class of all these chains will be denoted as $C_k(x, y; <>)$. Further, put $C(x, y; <>) = \cup\{C_k(x, y; <>); k \geq 2\}$; any element of it will be referred to as a *$(<>)$-chain* in X joining x and y. Let $(\sim)$ stand for the relation over X

$x \sim y$ iff $C(x, y; <>)$ is nonempty.

Clearly, $(\sim)$ is reflexive and symmetric; so is $(<>)$. Moreover, $(\sim)$ is transitive; hence, it is an equivalence over X. Assume in the following that

(b03) $(\sim)$ is *total*: $x \sim y$, for each $x, y \in X$.

The second category of conditions has four basic components.

(II-a) Concerning the monotone type properties of T, the following conditions enter into our discussion:

(b04) T is $(\leq)$-increasing: $x \leq y$ implies $Tx \leq Ty$
(b05) T is $(<>)$-increasing: $x <> y$ implies $Tx <> Ty$.

Clearly, the former of these implies the latter; but, the reciprocal is not in general valid.

(II-b) Further, the starting type properties of T are being expressed as:

(b06) T is almost $(\leq)$-progressive: $X(T, \leq) := \{x \in X; x \leq Tx\}$ is nonempty.

(II-c) Passing to the contractive properties of T, the following condition is to be used:

(b07) T is $(d, \leq; \varphi)$-contractive: $d(Tx, Ty) \leq \varphi(d(x, y))$, $\forall x, y \in X, x \leq y$;

here, $\varphi \in \mathcal{F}(R_+)$ is a function. Note that, by the symmetry of $d(.,.)$, this may also be written as

(b08) T is $(d, <>; \varphi)$-contractive: $d(Tx, Ty) \le \varphi(d(x, y)), \forall x, y \in X, x <> y$.

The functions to be taken into consideration here are as follows. Remember that $\varphi \in \mathcal{F}(R_+)$ is *(strongly) regressive*, provided [$\varphi(0) = 0$ and $\varphi(t) < t, \forall t \in R_+^0$]. The class of all these will be denoted as $\mathcal{F}(re)(R_+)$; and the subclass of all increasing $\varphi(re)(R_+)$ is indicated as $\mathcal{F}(re, in)(R_+)$. Given $\varphi \in \mathcal{F}(re, in)(R_+)$, let us say that it is *Matkowski (respectively, strongly Matkowski) admissible*, provided

(b09) $\lim_n \varphi^n(t) = 0$ (respectively, $\sum_n \varphi^n(t) < \infty$), $\forall t \in R_+^0$.

Note that a strongly Matkowski admissible function is Matkowski admissible as well; but the reciprocal is not in general true. These concepts are related to the developments in Matkowski [24]; we do not give details.

(II-d) Finally, the continuity properties of T are to be considered in the perspective of conditions below:

(b10) T is d-continuous: $x_n \xrightarrow{d} x$ implies $Tx_n \xrightarrow{d} Tx$

(b11) T is $(d, \le)$-continuous: (x_n) is ascending and $x_n \xrightarrow{d} x$ implies $Tx_n \xrightarrow{d} Tx$.

Note that, the former of these implies the latter; but the reciprocal is not in general true.

(B) Having these precise, we may now pass to the question we just formulated. Our first main result is as follows.

Theorem 8 *Assume that T is $(d, \le; \varphi)$-contractive, for some Matkowski admissible $\varphi \in \mathcal{F}(re, in)(R_+)$. In addition, let (X, d) be $(\le)$-complete, $(\sim)$ be total, and T be $(\le)$-increasing, almost $(\le)$-progressive, $(d, \le)$-continuous. Then, T is a globally strong Picard operator (modulo d); precisely,*

(i) $\mathrm{Fix}(T) = \{z\}$*, for some (uniquely determined) $z \in X$,*

(ii) $T^n x \xrightarrow{d} z$ *as $n \to \infty$, for each $x \in X$.*

Proof Let $x, y \in X$ be arbitrary fixed. As $(\sim)$ is total, there exists a k-dimensional $(<>)$-chain $A = (z_1, \ldots, z_k) \in X^k$ (where $k \ge 2$) joining x and y. This, along with T being $(\le)$-increasing, yields for all $n \ge 0$

$$T^n z_i <> T^n z_{i+1}, \ \forall i \in \{1, \ldots, k-1\};$$

so that, $T^n(A) = (T^n z_1, \ldots, T^n z_k) \in X^k$ is a k-dimensional $(<>)$-chain joining $T^n x$ and $T^n y$. Moreover, by the contractive property, one gets (for the same n)

$$d(T^n z_i, T^n z_{i+1}) \le \varphi^n(d(z_i, z_{i+1})), \ \forall i \in \{1, \ldots, k-1\}.$$

Taking the triangular inequality into account, gives

$$d(T^n x, T^n y) \le \sum_{i=1}^{k-1} \varphi^n(d(z_i, z_{i+1})), \ \forall n \ge 0.$$

As a direct consequence of this, one has, as φ is Matkowski admissible,

$$\lim_n d(T^n x, T^n y) = 0, \text{ for each couple } x, y \in X; \tag{17}$$

referred to as: T is *asymptotic constant*. In particular, this tells us that $Fix(T)$ is a singleton; for, if $z_1, z_2 \in \mathrm{Fix}(T)$, we have (by the above relation) $d(z_1, z_2) = 0$; whence, $z_1 = z_2$. It remains to establish the strong Picard property (modulo d). The argument will be divided into several parts.

Part 1 As T is almost $(\leq)$-progressive, $X(T, \leq)$ is nonempty. Let x_0 be an element of it; and put $(x_n = T^n x_0; n \geq 0)$; note that, as T is $(\leq)$-increasing, $(x_n; n \geq 0)$ is ascending. By the contractive property,

$$d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1})),\ \forall n;$$

so that, inductively, we get (as φ is increasing)

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)),\ \forall n.$$

Combining this with the Matkowski property of φ gives

$$d(x_n, x_{n+1}) \to 0,\ \text{as } n \to \infty, \tag{18}$$

which means (cf. a previous convention): $(x_n; n \geq 0)$ is d*-semi-Cauchy*.

Part 2 Let us now establish that $(x_n; n \geq 0)$ is d-Cauchy. Fix in the following $\varepsilon > 0$. By the d-semi-Cauchy property above, there exists a rank $j(\varepsilon)$ such that

$$d(x_n, x_{n+1}) < \varepsilon - \varphi(\varepsilon),\ \forall n \geq j(\varepsilon). \tag{19}$$

We now claim that

$$(\forall p \geq 1):\ [d(x_n, x_{n+p}) < \varepsilon,\ \forall n \geq j(\varepsilon)]; \tag{20}$$

and, from this, the required property is clear. To verify the assertion, an induction argument is to be used with respect to p. The case $p = 1$ is clear, by the d-semi-Cauchy property of our sequence. Assume that the property in question holds for some $p \geq 1$; we show that it holds as well for $p + 1$. From the inductive hypothesis and contractive condition (applied to (x_n, x_{n+p})), one gets (as φ is increasing)

$$d(x_{n+1}, x_{n+p+1}) \leq \varphi(d(x_n, x_{n+p})) \leq \varphi(\varepsilon).$$

This, along with the triangular inequality, gives

$$d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1}) \leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon;$$

and establishes our assertion.

Part 3 As (X, d) is $(\leq)$-complete, $x_n \xrightarrow{d} z$ for some (uniquely determined) $z \in X$. This, along with T being $(d, \leq)$-continuous, tells us that $(y_n := Tx_n;\ n \geq 0)$, d-converges towards Tz. On the other hand, $(y_n = x_{n+1};\ n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; whence, $y_n \xrightarrow{d} z$. Combining these, gives $z = Tz$; wherefrom (by the singleton property we just derived) $\mathrm{Fix}(T) = \{z\}$.

Part 4 Finally, let $x \in X$ be arbitrary fixed. By a preceding step, we have

$$\lim_n d(T^n x, z) = \lim_n d(T^n x, T^n z) = 0;$$

wherefrom $T^n x \xrightarrow{d} z$. The proof is complete.

Remark 1 In particular, $(\sim)$ is total whenever $(X, \leq)$ is up-directed. For, let $x, y \in X$ be arbitrary fixed. As $(X, \leq)$ is up-directed, there exists $u \in X$ such that $x \leq u$, $y \leq u$. This yields $x <> u$, $u <> y$; wherefrom $x \sim y$; so that, $(\sim)$ is total, as claimed.

Concerning the imposed conditions, it is to be noted that the almost $(\leq)$-progressive property of T is a pretty hard one; so, we may ask whether it may be removed. An affirmative answer to this is possible; but with the price of the function φ (appearing in the contractive assumption) being strongly Matkowski.

Precisely, the following variant of the statement above is available, as our second main result:

Theorem 9 *Assume that T is $(d, <>; \varphi)$-contractive, for some strongly Matkowski admissible $\varphi \in \mathcal{F}(re, in)(R_+)$. In addition, let (X, d) be complete, $(\sim)$ be total, and T be $(<>)$-increasing, d-continuous. Then, T is a globally strong Picard operator (modulo d).*

Proof Let $x, y \in X$ be arbitrary fixed. As $(\sim)$ is total, there exists a k-dimensional $(<>)$-chain $A = (z_1, \ldots, z_k) \in X^k$ (where $k \geq 2$) joining x and y. As T is $(<>)$-increasing, we have, for all $n \geq 0$

$$T^n z_i <> T^n z_{i+1}, \ \forall i \in \{1, \ldots, k-1\};$$

so that, $T^n(A) = (T^n z_1, \ldots, T^n z_k) \in X^k$ is a k-dimensional $(<>)$-chain joining $T^n x$ and $T^n y$. Moreover, by the contractive property, one gets (for the same n)

$$d(T^n z_i, T^n z_{i+1}) \leq \varphi^n(d(z_i, z_{i+1})), \ \forall i \in \{1, \ldots, k-1\}.$$

This, by the triangular inequality, yields

$$d(T^n x, T^n y) \leq \sum_{i=1}^{k-1} \varphi^n(d(z_i, z_{i+1})), \ \forall n \geq 0;$$

As a direct consequence of this, one has, as φ is strongly Matkowski admissible,

$$\textstyle\sum_n d(T^n x, T^n y) < \infty, \text{ for each couple } x, y \in X; \tag{21}$$

referred to as: T is *strongly asymptotic constant*. In particular, T is asymptotic constant (see above); wherefrom, by the same way as the one used in our first main result, $Fix(T)$ is a singleton. It then remains for us to establish that T is a strong Picard operator (modulo d).

Let $x \in X$ be arbitrary fixed. From the strong asymptotic constant property of T, we have (with $y = Tx$)

$$\sum_n d(T^n x, T^{n+1}x) < \infty;$$

wherefrom, the sequence $(T^n x; n \geq 0)$ is d-Cauchy. As (X, d) is complete, $T^n x \xrightarrow{d} z$, for some $z \in X$; and since T is d-continuous, $(T^{n+1}x = T(T^n x); n \geq 0)$, d-converges to Tz. On the other hand the sequence $(T^{n+1}x; n \geq 0)$ d-converges to z; because, it is a subsequence of $(T^n x; n \geq 0)$; and this yields (as d is sufficient) $z = Tz$; i.e. (see above) $\mathrm{Fix}(T) = \{z\}$. Finally, let $y \in X$ be arbitrary fixed. From the asymptotic constant property of T we then have $T^n y \xrightarrow{d} z$; and this ends the argument.

Finally, the following combination of these is our third main result (useful in applications):

Theorem 10 *Assume that T is $(d, \leq; \varphi)$-contractive, for some $\varphi \in \mathcal{F}(re, in)(R_+)$. In addition, let (X, d) be complete, $(X, \leq)$ be up-directed, and T be $(\leq)$-increasing, d-continuous. Finally, assume that one of the extra conditions below holds:*

(i) *T is almost $(\leq)$-progressive and φ is Matkowski admissible*

(ii) *φ is strongly Matkowski admissible.*

Then, T is a globally strong Picard operator (modulo d).

In particular, when φ is *linear* ($\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in]0, 1[$), these results are directly comparable with the related ones in Turinici [42], established by means of the Maia theorem [23].

2.3 Coupled Fixed Points

In the following, a basic application of these facts to coupled fixed point theorems is discussed.

Let $(X, d; \leq)$ be a quasi-ordered metric space. Denote, for simplicity, $X^2 = X \times X$; define a quasi-ordered metric structure and a conjugate map over it as: for the pair $z = (x, y)$, $w = (u, v)$ in X^2,

(c01) $\Delta(z, w) = \max\{d(x, u), d(y, v)\}$; $z \preceq w$ iff $x \leq u$, $y \geq v$; $z^* = (y, x)$.

The basic relationships between these are: for each $z = (x, y)$ and $w = (u, v)$ in X^2,

$$\Delta(z, w) = \Delta(z^*, w^*);\ z \preceq w \text{ if and only if } w^* \preceq z^*;\ (z^*)^* = z. \tag{22}$$

Having these precise, let $F : X^2 \to X$ be a map; and $\Phi : X^2 \to X^2$ be the associated coupled operator

(c02) $\Phi(z) = (F(z), F(z^*))$, for $z := (x, y) \in X^2$;

note that it is *compatible* with the conjugation

$$\Phi(z^*) = (\Phi(z))^*, \text{ for each } z \in X^2. \tag{23}$$

Further, let $T : X \to X$ be the *diagonal* operator generated by F, in the sense: $T(x) = F(x, x)$, $x \in X$. Denote, as usually, $\mathrm{Fix}(\Phi) = \{z \in X^2; z = \Phi(z)\}$; note that

$$z = (u, v) \in \mathrm{Fix}(\Phi) \text{ whenever } u = F(u, v),\ v = F(v, u);$$

we then say that (u, v) is a *coupled fixed point* of F. As we shall see below, there exists a very strong connection between the fixed points of T and the ones of Φ. This, ultimately, allows us to determine $\mathrm{Fix}(T)$ as long as we have information about $\mathrm{Fix}(\Phi)$.

Lemma 1 *Under these conventions, we have*

(i) $\mathrm{Fix}(\Phi)$ *is conjugation-invariant:* $c := (a, b) \in \mathrm{Fix}(\Phi)$ *if and only if* $c^* := (b, a) \in \mathrm{Fix}(\Phi)$

(ii) *if* $\mathrm{Fix}(\Phi)$ *is a singleton,* $\{c = (a, b)\}$, *then* $a = b$; *hence,* $c = (a, a)$; *moreover,* $\mathrm{Fix}(T) = \{a\}$.

Proof (i) Evident, by the compatible property.

(ii) From the previous part, $c^* = (b, a) \in \mathrm{Fix}(\Phi)$; and then, $c = c^*$; wherefrom, $a = b$ and $\mathrm{Fix}(\Phi) = \{(a, a)\}$. In this case, by definition, $a \in \mathrm{Fix}(T)$. Suppose that $b \in \mathrm{Fix}(T)$. Then, again by definition, $(b, b) \in \mathrm{Fix}(\Phi)$; so, by the above representation of $\mathrm{Fix}(\Phi)$, $(a, a) = (b, b)$; wherefrom $a = b$. The proof is complete.

In the following, we list conditions under which an existence and uniqueness property for the fixed points of Φ is to be reached. These, by the auxiliary fact above, yield an existence and uniqueness property for the associated to F diagonal operator T. We distinguish between **(I)** conditions about $(X^2, \Delta, \preceq)$ (expressed in terms of $(X, d, \leq)$), and **(II)** conditions about Φ (expressed in terms of F).

(I-a) Suppose that (X, d) is complete. Then, evidently, (X^2, Δ) is complete too.

(I-b) Suppose that

(c03) $(X, \leq)$ is (up/down)-directed: for each $x, y \in X$, the subset $\{x, y\}$ has upper and lower bounds.

Note that, in this case $(X^2, \preceq)$ is up-directed. In fact, given $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ in X^2, an upper bound (modulo $(\preceq)$) of $\{z_1, z_2\}$ is $w = (u, v)$; where u is an upper bound of $\{x_1, x_2\}$ and v is a lower bound of $\{y_1, y_2\}$; hence, the assertion.

(II-a) A basic condition about F is to be written as

(c04) F is *mixed monotone*: $(x, y) \preceq (u, v)$ implies $F(x, y) \leq F(u, v)$.

Note that, in such a situation,

$$\Phi \text{ is } (\preceq)\text{-increasing} : (x, y) \preceq (u, v) \text{ implies } \Phi(x, y) \preceq \Phi(u, v). \tag{24}$$

In fact, let (x, y) and (u, v) in X^2 be such that $(x, y) \preceq (u, v)$; i.e., $x \leq u$, $y \geq v$. Then (by the mixed monotone property)

$$F(x, y) \leq F(u, v),\ F(v, u) \leq F(y, x) \text{ (hence, } F(y, x) \geq F(v, u));$$

and the claim follows.

A simpler way of expressing this is the following. Let us say that the function F is *(1-increasing,2-decreasing)*, if it is increasing in the first variable and decreasing in the second one:

$\forall (a,b) \in X^2$: $F(.,b) =$ increasing, $F(a,.) =$ decreasing.

Lemma 2 *The mapping F is mixed monotone iff it is (1-increasing,2-decreasing).*

Proof (i) Assume that F is mixed monotone; and let $(a,b) \in X^2$ be arbitrary fixed. If $x_1 \leq x_2$ then, as $(x_1,b) \preceq (x_2,b)$, we must have, by hypothesis $F(x_1,b) \leq F(x_2,b)$. Likewise, take $y_1, y_2 \in X$ with $y_1 \geq y_2$; then, as $(a,y_1) \preceq (a,y_2)$, one gets $F(a,y_1) \leq F(a,y_2)$.

(ii) Assume that F is (1-increasing,2-decreasing); and let (x_1,y_1), (x_2,y_2) in X^2 be such that $(x_1,y_1) \preceq (x_2,y_2)$; i.e., $x_1 \leq x_2$, $y_1 \geq y_2$. Then (by the admitted property), $F(x_1,y_1) \leq F(x_2,y_1) \leq F(x_2,y_2)$; and this ends the argument.

(II-b) Another basic condition imposed upon F may be written as

(c05) F has coupled starting points (u,v), in the sense: $u \leq F(u,v)$, $v \geq F(v,u)$.

Then, evidently, $w = (u,v)$ is ($\preceq$)-starting for Φ, in the sense: $w \preceq \Phi(w)$.

(II-c) Further, given $\varphi \in \mathcal{F}(re,in)(R_+)$, call F, *$(d, \preceq; \varphi)$-contractive*, provided

(c06) $d(F(x,y),F(u,v)) \leq \varphi(\Delta((x,y),(u,v)))$, when $(x,y) \preceq (u,v)$.

A direct consequence of this is

$$\begin{aligned} &\Phi \text{ is } (\Delta, \preceq; \varphi) - \text{contractive}: \\ &\Delta(\Phi(x,y),\Phi(u,v)) \leq \varphi(\Delta((x,y),(u,v))), \text{when } (x,y) \preceq (u,v). \end{aligned} \tag{25}$$

(II-d) Finally, suppose that F is (Δ, d)-continuous: $z_n \xrightarrow{e} z$ implies $F(z_n) \xrightarrow{d} F(z)$. Then, Φ is Δ-continuous: $z_n \xrightarrow{e} z$ implies $\Phi(z_n) \xrightarrow{e} \Phi(z)$.

Putting these together we have, by the third main result above (applied to $(X^2, \Delta, \preceq)$ and Φ):

Theorem 11 *Assume that F is $(d, \preceq; \varphi)$-contractive, for some $\varphi \in \mathcal{F}(re,in)(R_+)$. In addition, let (X,d) be complete, $(X, \leq)$ be (up/down)-directed, F be mixed monotone, d-continuous. Finally, assume that one of the extra conditions below holds:*

(i) *φ is Matkowski admissible and F admits coupled starting points*

(ii) *φ is strongly Matkowski admissible.*

Then, the following conclusions are available:

(a) *F has a unique coupled fixed point (a,a), with $a \in X$*

(b) *the associated to F diagonal operator T fulfills* $\mathrm{Fix}(T) = \{a\}$*; where $a \in X$ is the above one*

(c) *for each $(x_0,y_0) \in X^2$, the iterative process*

$$(x_{n+1} = F(x_n,y_n), y_{n+1} = F(y_n,x_n); n \geq 0)$$

converges towards (a,a); whence, $x_n \xrightarrow{d} a$, $y_n \xrightarrow{d} a$.

In particular, when the second extra condition above is taken as

(iii) φ is strongly Matkowski admissible and F admits coupled starting points,

this result is just the one in Bhaskar and Lakshmikantham [8]; if, in addition, φ is linear. So, according to the authors, only mappings F with coupled starting points may have coupled fixed points. However, as explicitly stated above, existence of coupled starting points is superfluous when φ is strongly Matkowski admissible; hence, all the more linear. Further aspects may be found in Berinde [5].

2.4 Monotone Operators

Let $\{(X_i, d_i; \leq_i); 1 \leq i \leq r\}$ be a system of quasi-ordered metric spaces. Denote $X = \prod\{X_i; 1 \leq i \leq r\}$ (the Cartesian product of the ambient sets); and put, for $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_r)$ in X

(d01) $d(x, y) = \max\{d_1(x_1, y_1), \ldots, d_r(x_r, y_r)\}$,
(d02) $x \leq y$ iff $x_i \leq_i y_i$, $i \in \{1, \ldots, r\}$.

Clearly, $d(.,.)$ is a (standard) metric on X; and $(\leq)$ acts as a quasi-ordering over the same. As a consequence of this, we may now introduce all previous conventions. Note that, by the very definitions above, we have, for the sequence $(x^n = (x_1^n, \ldots, x_r^n); n \geq 0)$ in X and the point $x = (x_1, \ldots, x_r)$ in X,

$$x^n \xrightarrow{d} x \text{ iff } d_i(x_i^n, x_i) \to 0 \text{ as } n \to \infty, \text{ for all } i \in \{1, \ldots, r\} \tag{26}$$

$$(x^n; n \geq 0) \text{ is } d\text{-Cauchy iff } (x_i^n; n \geq 0) \text{ is } d_i\text{-Cauchy}, \forall i \in \{1, \ldots, r\}. \tag{27}$$

Let $(T_i : X \to X_i; 1 \leq i \leq r)$ be a system of maps; it generates an *associated* selfmap T of X, according to the convention

(d03) $Tx = (T_1x, \ldots, T_rx)$, $x = (x_1, \ldots, x_r) \in X$.

In the following, some basic monotone conditions upon this map are discussed.

(I) Let P be a subset of $\{1, \ldots, r\}$; note that, the case of $P = \emptyset$ or $P = \{1, \ldots, r\}$ is not excluded; this is also true for its *complement* $P^c := \{1, \ldots, r\} \setminus P$. For each couple $u = (u_1, \ldots, u_r)$, $v = (v_1, \ldots, v_r)$ in X, let $(u, v; P)$ be the point $w = (w_1, \ldots, w_r) \in X$, introduced as

(d04) $w_h = u_h$, $h \in P$; $w_k = v_k$, $k \in P^c$.

The following property is almost immediate; so, we do not give details.

Lemma 3 *The mapping $(u, v) \mapsto (u, v; P)$ is continuous in the sense*

$$\begin{aligned} &(x^n = (x_1^n, \ldots, x_r^n)), (y^n = (y_1^n, \ldots, y_r^n)) \subseteq X,\ x, y \in X, \\ &x^n \xrightarrow{d} x, y^n \xrightarrow{d} y \textit{ imply } (x^n, y^n; P) \xrightarrow{d} (x, y; P). \end{aligned} \tag{28}$$

(B) Let $i \in \{1, \ldots, r\}$ be arbitrary fixed; and P_i be a subset of $\{1, \ldots, r\}$ (under the general meaning above). Call T_i, *P_i-monotone*, provided:

(d05) for each pair $(x, y), (u, v) \in X^2$ with $x \le u$, $y \ge v$,
we have $T_i(x, y; P_i) \le_i T_i(u, v; P_i)$.

A characterization of this concept may be given along the lines below. Define the quasi-order ($\sqsubseteq_i$) over X, as: for each $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r)$ in X,

(d06) $x \sqsubseteq_i y$ iff: $(x_h \le_h y_h, h \in P_i), (x_k \ge_k y_k, k \in P_i^c)$.

Call T_i, *P_i-coupled-monotone*, in case

(d07) $x, y \in X, x \sqsubseteq_i y$ implies $T_i(x) \le_i T_i(y)$.

Lemma 4 *We have: T_i is P_i-monotone iff T_i is P_i-coupled-monotone.*

Proof (i) Suppose that T_i is P_i-monotone; and let $x, y \in X$ be such that $x \sqsubseteq_i y$. This yields $a := (x, y; P_i) \le b := (y, x; P_i)$; hence, $b = (y, x; P_i) \ge a = (x, y; P_i)$. By the imposed condition, we have $T_i(a, b; P_i) \le_i T_i(b, a; P_i)$; or, equivalently, $T_i(x) \le_i T_i(y)$; i.e., T_i is P_i-coupled-monotone.

(ii) Suppose that T_i is P_i-coupled-monotone; and let the pair $(x, y), (u, v) \in X^2$ be such that $x \le u$, $y \ge v$. By definition, $(x, y; P_i) \sqsubseteq_i (u, v; P_i)$; so that, by hypothesis, $T_i(x, y; P_i) \le_i T_i(u, v; P_i)$; wherefrom, T_i is P_i-monotone.

Another characterization of this property is by means of the component variables. For each $j \in \{1, \ldots, r\}$, let us say that T_i is *j-increasing* (resp., *j-decreasing*) provided, for each $a = (a_1, \ldots, a_r) \in X$,

(d08) $x, y \in X, x \le y$ imply
$T_i(x, a; \{j\}) \le_i T_i(y, a; \{j\})$ (resp., $T_i(x, a; \{j\}) \ge_i T_i(y, a; \{j\})$;

or, equivalently, T_i is increasing (resp., decreasing) with respect to the jth variable. If one of these properties holds, then T_i is called *j-monotone*; and if this is valid for all $j \in \{1, \ldots, r\}$, we say that T_i is *component-wise monotone*. Denote, in this last case (for each $i \in \{1, \ldots, r\}$)

(d09) inc(T_i)=$\{j \in \{1, \ldots, r\}; T_i$ is j-increasing$\}$,
dec(T_i)=$\{j \in \{1, \ldots, r\}; T_i$ is j-decreasing$\}$.

Proposition 2 *The following are valid:*

(i) *If T_i is P_i-coupled monotone, then it is component-wise monotone, with $P_i =$* inc(T_i)*, $P_i^c =$* dec(T_i)

(ii) *If T_i is component-wise monotone, then it is P_i-monotone, where $P_i =$* inc(T_i).

Proof (i) Suppose that T_i is P_i-coupled-monotone; and let $a \in X$ be fixed in the sequel. Further, take some pair $x, y \in X$ with $x \le y$. Given $j \in P_i$, the pair $u = (x, a; \{j\}), v = (y, a; \{j\})$ in X fulfills $u \sqsubseteq_i v$; so that, by hypothesis, $T_i(u) \le_i T_i(v)$; wherefrom, T_i is j-increasing. Likewise, given $j \in P_i^c$, the pair $u = (x, a; \{j\})$, $v = (y, a; \{j\})$ in X fulfills $v \sqsubseteq_i u$; so that, by hypothesis, $T_i(v) \le_i T_i(u)$; wherefrom, T_i is j-decreasing.

(ii) Suppose that T_i is component-wise monotone; and denote $P_i = \mathrm{inc}(T_i)$. We show that, for each $a \in X$,

$$\begin{aligned} x \leq y &\Longrightarrow T_i(x, a; P_i) \leq_i T_i(y, a; P_i) \\ x \geq y &\Longrightarrow T_i(a, x; P_i) \leq_i T_i(a, y; P_i); \end{aligned} \tag{29}$$

and, from this, conclusion follows as: for each pair $(x, y), (u, v) \in X^2$,

$$(x \leq u, y \geq v) \Longrightarrow T_i(x, y; P_i) \leq T_i(u, y; P_i) \leq T_i(u, v; P_i).$$

By duality reasons, it will suffice verifying its first half. Let $P_i = \{m_1, \ldots, m_q\}$ be the representation of this index set, where $m_1 < \ldots < m_q$; and let $x, y \in X$ be such that $x \leq y$. Denote $u^0 = (x, a; P_i)$; clearly, $u^0 = (x, u^0; \{m_1\})$. So, if we put $u^1 = (y, u^0; \{m_1\})$, the component-wise property above gives (by the definition of P_i) $T_i(u^0) \leq_i T_i(u^1)$. Further, $u^1 = (x, u^1; \{m_2\})$; so, if we put $u^2 = (y, u^1; \{m_2\})$, the same component-wise property above gives (by the definition of P_i) $T_i(u^1) \leq_i T_i(u^2)$. By a finite induction it is clear that, after q steps, one gets the desired fact.

2.5 *Main Result*

Let $\{(X_i, d_i; \leq_i); 1 \leq i \leq r\}$ be a system of quasi-ordered metric spaces. Denoting $X = \prod\{X_i; 1 \leq i \leq r\}$, let us introduce a "product" metric $d(.,.)$ over X and a "product" quasi-order $(\leq)$ over the same under the lines we already sketched. Further, put $X^2 = X \times X$; remember that a quasi-ordered metrical structure and a conjugate operator over it are to be introduced as: for $z = (x, y)$, $w = (u, v)$ in X^2

$$\Delta(z, w) = \max\{d(x, u), d(y, v)\};\ z \preceq w \text{ iff } x \leq u,\ y \geq v;\ z^* = (y, x).$$

Further, let $(T_i : X \to X_i; 1 \leq i \leq r)$ be a system of maps; it generates an *associated* selfmap T of X, under the convention

$$Tx = (T_1 x, \ldots, T_r x),\ x = (x_1, \ldots, x_r) \in X.$$

In the following, we list the conditions to be imposed upon our data. These, roughly speaking, are **(I)** conditions/properties involving the ambient spaces, and **(II)** conditions/properties imposed upon the introduced operators.

The first group of conditions involves the ambient quasi-ordered metric spaces.

(I-a) Assume in the following that (X_i, d_i) is complete, $\forall i \in \{1, \ldots, r\}$. Note that, in such a case, (X, d) and (X^2, Δ) are complete too.

(I-b) Suppose that

(e01) for each $i \in \{1, \ldots, r\}$, $(X_i, \leq_i)$ is (up/down)-directed: for each $x_i, y_i \in X$, $\{x_i, y_i\}$ has upper and lower bounds (modulo $(\leq_i)$).

Then, by definition, $(X, \leq)$ is (up/down)-directed; wherefrom, $(X^2, \preceq)$ is up-directed.

The second group of conditions refers to the system $T = (T_1, \ldots, T_r)$.

(II-a) A basic one is related to the monotonicity of our underlying system:

(e02) for each $i \in \{1, \ldots, r\}$, there exists a subset P_i of $\{1, \ldots, r\}$, such that T_i is P_i-monotone;

the system of all such properties will be referred to as: T is $(P_1, \ldots, P_r)$-monotone. Remember that this holds whenever T_i is component-wise monotone, for $i \in \{1, \ldots, r\}$; it will suffice taking $P_i = \mathrm{inc}(T_i)$, $i \in \{1, \ldots, r\}$. An important consequence of the described fact is as follows. For each $i \in \{1, \ldots, r\}$, denote

(e03) $F_i(x, y) = T_i(x, y; P_i)$, $x, y \in X$.

This is a mapping in $\mathcal{F}(X^2, X_i)$, endowed with the property (cf. the preceding part)

$$(x, y), (u, v) \in X^2, x \leq u, y \geq v \Longrightarrow F_i(x, y) \leq_i F_i(u, v). \tag{30}$$

Note that, as a consequence, the mapping in $\mathcal{F}(X^2, X)$ introduced via

(e04) $F(x, y) = (F_1(x, y), \ldots, F_r(x, y))$, $x, y \in X$

is mixed monotone; i.e., (see above)

$$(x, y), (u, v) \in X^2, x \leq u, y \geq v \Longrightarrow F(x, y) \leq F(u, v). \tag{31}$$

(II-b) Another basic condition to be considered upon $T = (T_1, \ldots, T_r)$ writes

(e05) T has $(P_1, \ldots, P_r)$-coupled starting points $(u = (u_1, \ldots, u_r), v = (v_1, \ldots, v_r))$, in the sense: $u_i \leq_i T_i(u, v; P_i)$, $v_i \geq_i T_i(v, u; P_i)$, for all $i \in \{1, \ldots, r\}$.

Note that, in such a case, the associated map F admits (u, v) as coupled starting point: $u \leq F(u, v)$, $v \geq F(v, u)$.

(II-c) A special condition upon $T = (T_1, \ldots, T_r)$ is of contractive type; there exists $\varphi \in \mathcal{F}(re, in)(R_+)$, such that

(e06) $\forall i \in \{1, \ldots, r\}$: $d_i(T_i(x), T_i(y)) \leq \varphi(d(x, y))$, $\forall x, y \in X$, $x \sqsubseteq_i y$;

referred to as: T is $(P_1, \ldots, P_r; \varphi)$-*contractive*. Note that, as a direct consequence, one has an evaluation like:

$$\forall i \in \{1, \ldots, r\} : \ d_i(F_i(z), F_i(w)) \leq \varphi(\Delta(z, w)), \ \forall z, w \in X^2, z \preceq w. \tag{32}$$

Indeed, take some $i \in \{1, \ldots, r\}$; and let $(x, y), (u, v) \in X^2$ be such that $(x, y) \preceq (u, v)$. Then, $(x, y; P_i) \sqsubseteq_i (u, v; P_i)$; so that, by the contractive hypothesis,

$$d_i(F_i(x, y), F_i(u, v)) \leq \varphi(d((x, y; P_i), (u, v; P_i))) \leq \varphi(\Delta((x, y), (u, v)));$$

hence, the claim. Passing to the "vectorial" map F, it results from this that

$$d(F(z), F(w)) \leq \varphi(\Delta(z, w)), \ \forall z, w \in X^2, z \preceq w; \tag{33}$$

or, equivalently (see above): F is $(d, \preceq; \varphi)$-contractive.

(II-d) Suppose that

(e07) for each $i \in \{1, \ldots, r\}$, T_i is (d, d_i)-continuous (on X).

Note that, in such a case, by the continuity properties of the maps $(x, y) \mapsto (x.y; P)$ discussed in a previous place, one has

$$\forall i \in \{1, \ldots, r\}\text{: } F_i \text{ is } (\Delta, d_i)\text{-continuous; so, } F \text{ is } (\Delta, d)\text{-continuous.} \tag{34}$$

(II-e) Finally, as a distinct consequence of these conventions, one has

$$T_i(x) = F_i(x, x), \forall i \in \{1, \ldots, r\}\text{; hence, } T(x) = F(x, x); \tag{35}$$

or, in other words: T is the diagonal operator attached to F.

Putting these together, we have (by the coupled fixed point result above):

Theorem 12 *Suppose that, there exists a system of subsets* $(P_1, \ldots, P_r)$ *in* $\{1, \ldots, r\}$, *such that: T is $(P_1, \ldots, P_r)$-monotone and $(P_1, \ldots, P_r; \varphi)$-contractive. for some $\varphi \in \mathcal{F}(re, in)(R_+)$. In addition, let (X_i, d_i) be complete, $(X_i, \leq_i)$ be (up/down)-directed, and T_i be (d, d_i)-continuous, for each $i \in \{1, \ldots, r\}$. Finally, assume that one of the extra conditions below holds:*

(i) *φ is Matkowski admissible and T has $(P_1, \ldots, P_r)$-coupled starting points*

(ii) *φ is strongly Matkowski admissible.*

Then, the following conclusions hold

(a) *F has a unique coupled fixed point, (a, a) with $a = (a_1, \ldots, a_r) \in X$*

(b) *the vectorial operator T fulfills* $\mathrm{Fix}(T) = \{a\}$, *where $a \in X$ is as before*

(c) *for any couple $x^0 = (x_1^0, \ldots, x_r^0)$ and $y^0 = (y_1^0, \ldots, y_r^0)$ in X, the iterative process $(x^{n+1} = F(x_n, y_n), y^{n+1} = F(y_n, x_n); n \geq 0)$ converges towards (a, a); whence, $x^n \xrightarrow{d} a$, $y^n \xrightarrow{d} a$.*

In particular, when the second extra condition is taken as

(iii) φ is strongly Matkowski admissible and T has $(P_1, \ldots, P_r)$-coupled starting points,

this result is comparable with the one in Rus [36]. Precisely, according to the author, the only mappings F for which a couple fixed point is to be reached are those admitting at least one $(P_1, \ldots, P_r)$-coupled starting point. However, as explicitly stated above, the existence of such points is superfluous when φ is strongly Matkowski admissible; hence, all the more linear (like in his example of boundary value problem). Further aspects may be found in Rus [35].

2.6 An Application

Let $(M, e, \leq)$ be a quasi-ordered metric space. For technical reasons, the following notations will be introduced:

$$(X_1, d_1, \leq_1) = (X_2, d_2, \leq_2) = (X_3, d_3, \leq_3) = (M, e, \leq);$$
$$X = X_1 \times X_2 \times X_3 = M^3, X^2 = X \times X.$$

According to these notations, let d be the "product" metric of (d_1, d_2, d_3), and $(\le)$ be the "product" quasi-order of $(\le_1, \le_2, \le_3)$. Also, let us endow X^2 with the metric $\Delta(.,.)$ and the quasi-order $(\preceq)$ we just introduced.

Further, let $J : M^3 \to M$ be a mapping. In the following, we are intending to establish sufficient conditions under which the system of equations

$$x_1 = J(x_1, x_2, x_3),\ x_2 = J(x_2, x_1, x_3),\ x_3 = J(x_3, x_2, x_1) \tag{36}$$

should have a (unique) solution $a = (a_1, a_2, a_3) \in X = M^3$; referred to as a *tripled fixed point* of J. Clearly, this is nothing else than a fixed point of the vectorial operator $T = (T_1, T_2, T_3)$ in $\mathcal{F}(X)$, introduced as: for each $x = (x_1, x_2, x_3) \in X$,

(f01) $T_1(x) = J(x_1, x_2, x_3),\ T_2(x) = J(x_2, x_1, x_3),\ T_3(x) = J(x_3, x_2, x_1)$.

To solve this problem, it will suffice applying the previous developments.

In the following, we list the conditions to be imposed upon our data; as well as the associated properties. These, roughly speaking, are **(I)** conditions/properties regarding the ambient spaces (in fact: conditions imposed upon $(M, e, \le)$), and **(II)** conditions/properties involving the introduced operators. (in fact: conditions imposed upon J).

Concerning the first group, we have two basic conditions.

(I-a) Suppose that

(f02) (M, e) is complete (each e-Cauchy sequence is e-convergent).

Note that, in this case, (X_i, d_i) is complete, for each $i \in \{1, 2, 3\}$. In addition, the metric spaces (X, d) and (X^2, Δ) are complete too.

(I-b) Suppose that

(f03) $(M, \le)$ is (up/down)-directed.

This yields, in a formal way: for each $i \in \{1, 2, 3\}$, $(X_i, \le_i)$ is (up/down)-directed. Consequently, by the very definitions above, $(X, \le)$ is (up/down)-directed and $(X^2, \preceq)$ is up-directed.

We are now passing to the second group of conditions, related to the map J.

(II-a) The basic one involves the monotonicity of our underlying map:

(f04) J is 1-increasing, 2-decreasing, 3-increasing.

By the very definition of the associated maps (T_1, T_2, T_3), one gets directly

$$\begin{aligned} &T_1 \text{ is 1-increasing, 2-decreasing, 3-increasing} \\ &T_2 \text{ is 1-decreasing, 2-increasing, 3-increasing} \\ &T_3 \text{ is 1-increasing, 2-decreasing, 3-increasing.} \end{aligned} \tag{37}$$

An important consequence of this is the following. Define the mappings F_1, F_2, F_3 in $\mathcal{F}(X^2, M)$, according to: for each $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ in X,

(f05) $F_1(x, y) = T_1(x_1, y_2, x_3) = J(x_1, y_2, x_3)$,
$F_2(x, y) = T_2(y_1, x_2, x_3) = J(x_2, y_1, x_3)$,
$F_3(x, y) = T_3(x_1, y_2, x_3) = J(x_3, y_2, x_1)$.

By the imposed properties, these maps fulfill

$$(x, y), (u, v) \in X^2, x \le u, y \ge v \Longrightarrow F_i(x, y) \le_i F_i(u, v),\ i \in \{1, 2, 3\}. \tag{38}$$

This, in turn, tells us that the mapping in $F \in \mathcal{F}(X^2, X)$ introduced as

(f06) $F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y)), x, y \in X$

is mixed monotone. in the sense

$$(x, y), (u, v) \in X^2, x \le u, y \ge v \Longrightarrow F(x, y) \le F(u, v). \tag{39}$$

(II-b) The second condition upon J is of (linear) contractive type: there exists $\alpha \in]0, 1[$ such that

(f07) $e(J(x), J(y)) \le \alpha d(x, y)$, for each
$x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ in X with $x_1 \le y_1, x_2 \ge y_2, x_3 \le y_3$.

Then, we have contractive properties for the maps T_1,T_2, T_3, expressed as: for each $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ in X,

$$\begin{aligned} &e(T_1(x), T_1(y)) \le \alpha d(x, y),\ \text{whenever}\, x_1 \le y_1, x_2 \ge y_2, x_3 \le y_3 \\ &e(T_2(x), T_2(y)) \le \alpha d(x, y),\ \text{whenever}\, x_1 \ge y_1, x_2 \le y_2, x_3 \le y_3 \\ &e(T_3(x), T_3(y)) \le \alpha d(x, y),\ \text{whenever}\, x_1 \le y_1, x_2 \ge y_2, x_3 \le y_3. \end{aligned} \tag{40}$$

This yields corresponding contractive properties for the mappings F_1, F_2, F_3, and $F = (F_1, F_2, F_3)$; we do not give details.

(II-c) The third condition is continuity:

(f08) J is continuous from $X = M^3$ to M.

Note that, in such a case, the maps T_1, T_2, T_3 are continuous; in addition, the maps F_1, F_2, F_3 and $F = (F_1, F_2, F_3)$ are continuous too.

(II-d) Finally, as a distinct consequence of these, one has the diagonal property:

$$T_i(x) = F_i(x, x), \forall i \in \{1, 2, 3\}; \text{hence, } T(x) = F(x, x); \tag{41}$$

or, in other words: T is the diagonal operator attached to F.

Putting these together, we have (via Theorem 12 above):

Theorem 13 *Assume that conditions (f02)–(f04) and (f07)–(f08) hold. Then,*

(i) *F has a unique coupled fixed point, (a, a) with $a = (a_1, a_2, a_3) \in X$*

(ii) *the vectorial operator T fulfills $\mathrm{Fix}(T) = \{a\}$, where $a \in X$ is as before*

(iii) *for each couple $x^0 = (x_1^0, x_2^0, x_3^0)$ and $y^0 = (y_1^0, y_2^0, y_3^0)$ in X, the iterative process $(x^{n+1} = F(x^n, y^n), y^{n+1} = F(y^n, x^n); n \ge 0)$ converges towards (a, a); so that, necessarily, $x^n \xrightarrow{d} a$, $y^n \xrightarrow{d} a$.*

In particular, when in addition,

(f09) there exists a couple $x^0 = (x_1^0, x_2^0, x_3^0)$ and $y^0 = (y_1^0, y_2^0, t_3^0)$ in X, with $x^0 \le F(x^0, y^0)$, $y^0 \ge F(y^0, x^0)$,

this result is deductible from the one in Rus [36]. Further aspects may be found in Turinici [43].

3 Relational Metric Spaces

3.1 Introduction

Let X be a nonempty set. Remember that the subset Y of X is *almost-singleton* (in short: *asingleton*) provided $y_1, y_2 \in Y$ implies $y_1 = y_2$; and *singleton*, if, in addition, Y is nonempty; note that, in this case, $Y = \{y\}$, for some $y \in X$. Take a metric $d : X \times X \to R_+$ over X; as well as a selfmap $T \in \mathcal{F}(X)$. Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. Concerning the existence and uniqueness of such points, a basic result is the 1922 one due to Banach [2]. Call the selfmap T, $(d;\alpha)$-*contractive* (where $\alpha \ge 0$), if

(a01) $d(Tx, Ty) \le \alpha d(x, y)$, for all $x, y \in X$.

Theorem 14 *Assume that T is $(d;\alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let (X, d) be complete. Then,*

(i) $\mathrm{Fix}(T)$ *is a singleton,* $\{z\}$

(ii) $T^n x \xrightarrow{d} z$ *as $n \to \infty$, for each $x \in X$.*

This result (referred to as: Banach's fixed point theorem) found some basic applications to the operator equations theory. Consequently, a multitude of extensions for it were proposed. Here, we shall be interested in the relational way of enlarging Theorem 14, based on contractive conditions like

(a02) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \le 0$, for all $x, y \in X$ with $x\mathcal{R}y$;

where $F : R_+^6 \to R$ is a function and $\mathcal{R}$ is a relation over X. Note that, when $\mathcal{R} = X \times X$ (the *trivial* relation over X), a large list of such contractive maps is provided in Rhoades [33]. Further, when $\mathcal{R}$ is an *order* on X, an early 1986 result was obtained by Turinici [41], in the realm of ordered metrizable uniform spaces. Two decades after, this fixed point statement was rediscovered (in the ordered metrical setting) by Ran and Reurings [32]; see also Nieto and Rodriguez-Lopez [28]; and, since then, the number of such results increased rapidly. On the other hand, when $\mathcal{R}$ is an *amorphous* relation over X, an appropriate statement of this type is the 2012 one due to Samet and Turinici [37]. The "intermediary" particular case of $\mathcal{R}$ being *finitely transitive* was recently obtained by Karapinar and Berzig [18], under a class of $(\alpha\psi, \beta\varphi)$-contractive conditions suggested by Popescu [31]. It is our aim in the following to give further extensions of these results, when

(i) the contractive conditions are taken after the model in Meir and Keeler [26]
(ii) the finite transitivity of $\mathcal{R}$ is being assured in a "local" way.
Further aspects occasioned by these developments will be also discussed.

3.2 Preliminaries

Throughout this exposition, the ambient axiomatic system is Zermelo–Fraenkel's (abbreviated: (ZF)), as described by Cohen [11, Chap. 2, Sect. 3]. In fact, the *reduced* system (ZF–AC) will suffice; here, (AC) stands for the *Axiom of Choice*. The notations and basic facts about these are more or less usual. Some important ones are described below.

(A) Let X be a nonempty set. By a *relation* over X, we mean any nonempty part $\mathcal{R} \subseteq X \times X$. For simplicity, we sometimes write $(x, y) \in \mathcal{R}$ as $x\mathcal{R}y$. Note that $\mathcal{R}$ may be regarded as a mapping between X and $\mathcal{P}(X)$ (=the class of all subsets in X). Precisely, denote for $x \in X$: $X(x, \mathcal{R}) = \{y \in X; x\mathcal{R}y\}$ (the *section* of $\mathcal{R}$ through x); then, the desired mapping representation is $[\mathcal{R}(x) = X(x, \mathcal{R}), x \in X]$.

Among the classes of relations to be used, the following ones (listed in a "decreasing" scale) are important for us:

(P0) $\mathcal{R}$ is *trivial*; i.e., $\mathcal{R} = X \times X$; note that, in this case, $x\mathcal{R}y$, $\forall x, y \in X$
(P1) $\mathcal{R}$ is an *order*; i.e., it is *reflexive* $[x\mathcal{R}x, \forall x \in X]$, *transitive* [$x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$] and *antisymmetric* [$x\mathcal{R}y$ and $y\mathcal{R}x$ imply $x = y$]
(P2) $\mathcal{R}$ is a *quasi-order*; i.e., it is reflexive and transitive
(P3) $\mathcal{R}$ is transitive (see above).

A basic ordered structure is $(N, \leq)$; here, $N = \{0, 1, \ldots\}$ is the set of natural numbers and $(\leq)$ is defined as: $m \leq n$ iff $m + p = n$, for some $p \in N$. For each natural number $n \geq 1$, let $N(n, >) := \{0, \ldots, n-1\}$ stand for the *initial interval* (in N) induced by n. Any set P with $P \sim N$ (in the sense: there exists a bijection from P to N) will be referred to as *effectively denumerable*. In addition, given some natural number $n \geq 1$, any set Q with $Q \sim N(n, >)$ will be said to be *n-finite*; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable* iff it is either effectively denumerable or finite.

Given the relations $\mathcal{R}$, $\mathcal{S}$ over X, define their *product* $\mathcal{R} \circ \mathcal{S}$ as

(b01) $(x, z) \in \mathcal{R} \circ \mathcal{S}$ if, there exists $y \in X$ with $(x, y) \in \mathcal{R}$, $(y, z) \in \mathcal{S}$.

This allows us to introduce the *powers* of a relation $\mathcal{R}$ as

(b02) $\mathcal{R}^0 = \mathcal{I}$, $\mathcal{R}^{n+1} = \mathcal{R}^n \circ \mathcal{R}$, $n \in N$.

(Here, $\mathcal{I} = \{(x, x); x \in X\}$ is the *identical relation* over X). The following properties of these will be useful in the sequel:

$$\mathcal{R}^{m+n} = \mathcal{R}^m \circ \mathcal{R}^n,\ (\mathcal{R}^m)^n = \mathcal{R}^{mn},\ \forall m, n \in N. \tag{42}$$

Given $k \geq 2$, let us say that $\mathcal{R}$ is *k-transitive* provided $\mathcal{R}^k \subseteq \mathcal{R}$; clearly, *transitive* is identical with *2-transitive*. We may now complete the decreasing scale above as

(P4) $\mathcal{R}$ is *finitely transitive*; i.e., $\mathcal{R}$ is k-transitive for some $k \geq 2$
(P5) $\mathcal{R}$ is *locally finitely transitive*; i.e., for each (effectively) denumerable subset Y of X, there exists $k = k(Y) \geq 2$, such that the restriction to Y of $\mathcal{R}$ is k-transitive.
(P6) $\mathcal{R}$ is *amorphous*; i.e., it has no specific properties at all.

(B) Let (X, d) be a metric space. We introduce a d-convergence and d-Cauchy structure on X as follows. By a *sequence* in X, we mean any mapping $x : N \to X$. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with $i(n) \to \infty$ as $n \to \infty$ will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), d-converges to x (written as: $x_n \xrightarrow{d} x$) provided $d(x_n, x) \to 0$ as $n \to \infty$; i.e.,

$$\forall \varepsilon > 0, \exists i = i(\varepsilon) : \quad i \leq n \Longrightarrow d(x_n, x) < \varepsilon.$$

The set of all such points x will be denoted $\lim_n (x_n)$; note that, it is an asingleton, because d is triangular. If $\lim_n (x_n)$ is nonempty, then (x_n) is called *d-convergent*. We stress that the introduced convergence concept ($\xrightarrow{d}$) does match the standard requirements in Kasahara [19]. Further, call the sequence (x_n), *d-Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; i.e.,

$$\forall \varepsilon > 0, \exists j = j(\varepsilon) : \quad j \leq m < n \Longrightarrow d(x_m, x_n) < \varepsilon.$$

As d is triangular, any d-convergent sequence is d-Cauchy too; but, the reciprocal is not in general true.

The introduced concepts allow us to give a useful property.

Lemma 5 *The mapping $(x, y) \mapsto d(x, y)$ is d-Lipschitz, in the sense*

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v), \ \forall (x, y), (u, v) \in X \times X. \tag{43}$$

As a consequence, this map is d-continuous; i.e.,

$$x_n \xrightarrow{d} x, \ y_n \xrightarrow{d} y \text{ imply } d(x_n, y_n) \to d(x, y). \tag{44}$$

The verification is by using the triangular property of d; we do not give details.

(C) Let (X, d) be a metric space; and $\mathcal{R} \subseteq X \times X$ be a (nonempty) relation over X; the triple $(X, d, \mathcal{R})$ will be referred to as a *relational metric space*. Further, take some $T \in \mathcal{F}(X)$. Call the subset Y of X, *$\mathcal{R}$-almost-singleton* (in short: *$\mathcal{R}$-asingleton*) provided $y_1, y_2 \in Y$, $y_1 \mathcal{R} y_2 \Longrightarrow y_1 = y_2$; and *$\mathcal{R}$-singleton* when, in addition, Y is nonempty. We have to determine circumstances under which Fix(T) be nonempty; and, if this holds, to establish whether T is *fix-$\mathcal{R}$-asingleton* (i.e., Fix(T) is $\mathcal{R}$-asingleton); or, equivalently T is *fix-$\mathcal{R}$-singleton* (in the sense, Fix(T) is $\mathcal{R}$-singleton); To do this, we start from the basic hypotheses

(b03) T is $\mathcal{R}$-semi-progressive: $X(T,\mathcal{R}) := \{x \in X; x\mathcal{R}Tx\} \neq \emptyset$
(b04) T is $\mathcal{R}$-increasing: $x\mathcal{R}y$ implies $Tx\mathcal{R}Ty$.

In this setting, the basic directions under which the investigations to be conducted are described in the list below, comparable with the one in Turinici [42] (see also Rus [34, Chap. 2, Sect. 2.2]):

(2a) We say that T is a *Picard operator* (modulo $(d,\mathcal{R})$) if, for each $x \in X(T,\mathcal{R})$, $(T^n x; n \geq 0)$ is d-convergent

(2b) We say that T is a *strong Picard operator* (modulo $(d,\mathcal{R})$) when, for each $x \in X(T,\mathcal{R})$, $(T^n x; n \geq 0)$ is d-convergent; and $\lim_n (T^n x)$ belongs to $\mathrm{Fix}(T)$

(2c) We say that T is a *globally strong Picard operator* (modulo $(d,\mathcal{R})$) when it is a strong Picard operator (modulo $(d,\mathcal{R})$) and T is fix-$\mathcal{R}$-asingleton (hence, fix-$\mathcal{R}$-singleton).

The sufficient (regularity) conditions for such properties are being founded on *ascending orbital* concepts (in short: (a-o)-concepts). Namely, call the sequence $(z_n; n \geq 0)$ in X, *$\mathcal{R}$-ascending*, if $z_i\mathcal{R}z_{i+1}$ for all $i \geq 0$; and *T-orbital*, when it is a subsequence of $(T^n x; n \geq 0)$, for some $x \in X$; the intersection of these notions is just the precise one.

(2d) Call (X,d), *(a-o)-complete*, provided (for each (a-o)-sequence) d-Cauchy $\Longrightarrow$ d-convergent

(2e) We say that T is *$(a-o,d)$-continuous*, if ((z_n)=(a-o)-sequence and $z_n \xrightarrow{d} z$) imply $Tz_n \xrightarrow{d} Tz$

(2f) Call $\mathcal{R}$, *$(a-o,d)$-almost-selfclosed*, if: whenever the (a-o)-sequence $(z_n; n \geq 0)$ in X and the point $z \in X$ fulfill $z_n \xrightarrow{d} z$, there exists a subsequence $(w_n := z_{i(n)}; n \geq 0)$ of $(z_n; n \geq 0)$ with $w_n\mathcal{R}z$, for all $n \geq 0$.

When the orbital properties are ignored, these conventions give us ascending notions (in short, a-notions). Precisely, call (X,d), *a-complete*, provided (for each a-sequence) d-Cauchy $\Longrightarrow$ d-convergent. Further, let us say that T is *(a,d)-continuous*, if ((z_n)= a-sequence and $z_n \xrightarrow{d} z$) imply $Tz_n \xrightarrow{d} Tz$. Finally, call $\mathcal{R}$, *(a,d)-almost-self-closed*, if: whenever the a-sequence $(z_n; n \geq 0)$ in X and the point $z \in X$ fulfill $z_n \xrightarrow{d} z$, there exists a subsequence $(w_n; n \geq 0)$ of $(z_n; n \geq 0)$ with $w_n\mathcal{R}z$, for all $n \geq 0$.

Concerning these properties, the following auxiliary fact is useful for us.

Lemma 6 *Let the $\mathcal{R}$-ascending sequence $(z_n; n \geq 0)$ in X, and the natural number $k \geq 2$, be such that*

(b05) $\mathcal{R}$ is k-transitive on the subset $Z := \{z_n; n \geq 0\}$.

Then, necessarily,

$$(\forall r \geq 0): \ [(z_i, z_{i+1+r(k-1)}) \in \mathcal{R}, \ \forall i \geq 0]. \tag{45}$$

Proof We make use of an induction argument with respect to r. First, by the $\mathcal{R}$-ascending property, $(z_i, z_{i+1}) \in \mathcal{R}$, $\forall_i \geq 0$; whence, the case of $r = 0$ holds. Moreover, again from our choice, $(z_i, z_{i+k}) \in \mathcal{R}^k$; and this, along with the k-transitive

property, gives $(z_i, z_{i+k}) \in \mathcal{R}$; hence, the case of $r = 1$ holds too. Suppose that this property holds for some $r \geq 1$; we claim that it holds as well for $r + 1$. In fact, given $i \geq 0$, the $\mathcal{R}$-ascending property gives $(z_{i+1+r(k-1)}, z_{i+1+(r+1)(k-1)}) \in \mathcal{R}^{k-1}$; so that, by the inductive hypothesis (and properties of relational product)

$$(z_i, z_{i+1+(r+1)(k-1)}) \in \mathcal{R} \circ \mathcal{R}^{k-1} = \mathcal{R}^k;$$

and this, along with the k-transitive condition, yields $(z_i, z_{i+1+(r+1)(k-1)}) \in \mathcal{R}$. The proof is thereby complete.

3.3 Meir–Keeler Contractions

Let $(X, d, \mathcal{R})$ be a relational metric space; and T be a selfmap of X; supposed to be $\mathcal{R}$-semi-progressive and $\mathcal{R}$-increasing. The basic directions and sufficient regularity conditions under which the problem of determining the fixed points of T be solved were already listed. As a completion of them, we must formulate the metrical contractive type conditions upon our data. These, essentially, consist in a "relational" variant of the Meir–Keeler condition [26]. Denote, for $x, y \in X$:

$H(x, y) = \max\{d(x, Tx), d(y, Ty)\}$, $L(x, y) = (1/2)[d(x, Ty) + d(Tx, y)]$,
$G_1(x, y) = d(x, y)$, $G_2(x, y) = \max\{G_1(x, y), H(x, y)\}$,
$G_3(x, y) = \max\{G_2(x, y), L(x, y)\} = \max\{G_1(x, y), H(x, y), L(x, y)\}$.

Given $G \in \{G_1, G_2, G_3\}$, we say that T is *Meir–Keeler* $(d, \mathcal{R}; G)$*-contractive*, if

(c01) $[x\mathcal{R}y, G(x, y) > 0]$ implies $d(Tx, Ty) < G(x, y)$
(T is strictly $(d, \mathcal{R}; G)$-nonexpansive)
(c02) $\forall\varepsilon > 0, \exists\delta > 0$: $[x\mathcal{R}y, \varepsilon < G(x, y) < \varepsilon + \delta] \Longrightarrow d(Tx, Ty) \leq \varepsilon$
(T has the Meir–Keeler property).

Note that, by the former of these, the Meir–Keeler property may be written as

(c03) $\forall\varepsilon > 0, \exists\delta > 0$: $[x\mathcal{R}y, 0 < G(x, y) < \varepsilon + \delta] \Longrightarrow d(Tx, Ty) \leq \varepsilon$.

In the following, two basic examples of such contractions will be given.

(A) Let $\mathcal{F}(re)(R_+)$ stand for the class of all $\varphi \in \mathcal{F}(R_+)$ with the (strong) *regressive* property: $[\varphi(0) = 0; \varphi(t) < t, \forall t > 0]$. We say that $\varphi \in \mathcal{F}(re)(R_+)$ is *Meir–Keeler admissible*, if

(c04) $\forall\gamma > 0, \exists\beta \in]0, \gamma[, (\forall t)$: $\gamma \leq t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma$;
or, equivalently: $\forall\gamma > 0, \exists\beta \in]0, \gamma[, (\forall t)$: $0 \leq t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma$.

Now, given $G \in \{G_1, G_2, G_3\}$, $\varphi \in \mathcal{F}(R_+)$, call T, $(d, \mathcal{R}; G, \varphi)$*-contractive*, if

(c05) $d(Tx, Ty) \leq \varphi(G(x, y)), \forall x, y \in X, x\mathcal{R}y$.

Lemma 7 *Assume that T is $(d, \mathcal{R}; G, \varphi)$-contractive, where $\varphi \in \mathcal{F}(re)(R_+)$ is Meir–Keeler admissible. Then, T is Meir–Keeler $(d, \mathcal{R}; G)$-contractive.*

Proof (i) Let $x, y \in X$ be such that $x\mathcal{R}y$ and $G(x, y) > 0$. The contractive condition, and (φ = regressive), yield $d(Tx, Ty) < G(x, y)$; so that, the first part of the Meir–Keeler contractive condition holds.

(ii) Let $\varepsilon > 0$ be arbitrary fixed; and $\delta \in]0, \varepsilon[$ be the number assured by the Meir–Keeler admissible property of φ. Further, let $x, y \in X$ be such that $x\mathcal{R}y$ and $\varepsilon < G(x, y) < \varepsilon + \delta$. By the contractive condition and admissible property,

$$d(Tx, Ty) \leq \varphi(G(x, y)) \leq \varepsilon;$$

so that, the second part of the Meir–Keeler contractive condition holds too.

Some important classes of such functions are given below.

(I) For any $\varphi \in \mathcal{F}(re)(R_+)$ and any $s \in R_+^0$, put

(c06) $\Lambda_+\varphi(s) = \inf_{\varepsilon>0} \Phi(s+)(\varepsilon)$; where $\Phi(s+)(\varepsilon) = \sup \varphi(]s, s+\varepsilon[)$;
(c07) $\Lambda^+\varphi(s) = \sup\{\varphi(s), \Lambda_+\varphi(s)\}$.

By this very definition, we have the representation (for all $s \in R_+^0$)

$$\Lambda^+\varphi(s) = \inf_{\varepsilon>0} \Phi[s+](\varepsilon);\ \text{where } \Phi[s+](\varepsilon) = \sup\{\varphi([s, s+\varepsilon[). \tag{46}$$

From the regressive property of φ, these limit quantities are finite; precisely,

$$0 \leq \varphi(s) \leq \Lambda^+\varphi(s) \leq s,\ \ \forall s \in R_+^0. \tag{47}$$

The following consequence of this will be useful. Remember that, given the sequence $(r_n; n \geq 0)$ in R and the point $r \in R$, we denoted

$r_n \to r+$ (respectively, $r_n \to r++$), if $r_n \to r$ and
$r_n \geq r$ (respectively, $r_n > r$), for all $n \geq 0$ large enough.

Lemma 8 *Let $\varphi \in \mathcal{F}(re)(R_+)$ and $s \in R_+^0$ be arbitrary fixed. Then,*

(i) $\limsup_n (\varphi(t_n)) \leq \Lambda^+\varphi(s)$, *for each sequence (t_n) in R_+^0 with $t_n \to s+$; hence, in particular, for each sequence (t_n) in R_+^0 with $t_n \to s++$*

(ii) *there exists a sequence (r_n) in R_+^0 with $r_n \to s+$ and $\varphi(r_n) \to \Lambda^+\varphi(s)$.*

Proof (i) Given $\varepsilon > 0$, there exists a rank $p(\varepsilon) \geq 0$ such that $s \leq t_n < s + \varepsilon$, for all $n \geq p(\varepsilon)$; hence

$$\limsup_n (\varphi(t_n)) \leq \sup\{\varphi(t_n); n \geq p(\varepsilon)\} \leq \Phi[s+](\varepsilon).$$

It suffices taking the infimum over $\varepsilon > 0$ in this relation to get the desired fact.

(ii) When $\Lambda^+\varphi(s) = 0$, the written conclusion is clear, with $(r_n = s; n \geq 0)$; for, in this case, $\varphi(s) = 0$. Suppose now that $\Lambda^+\varphi(s) > 0$. By definition,

$$\forall \varepsilon \in]0, \Lambda^+\varphi(s)[,\ \exists \delta \in]0, \varepsilon[:\ \Lambda^+\varphi(s) - \varepsilon < \Lambda^+\varphi(s) \leq \Phi[s+](\delta) < \Lambda^+\varphi(s) + \varepsilon.$$

This tells us that there must be some r in $[s, s+\delta[$ with

$$\Lambda^+\varphi(s) - \varepsilon < \varphi(r) < \Lambda^+\varphi(s) + \varepsilon.$$

Taking a sequence (ε_n) in $]0, \Lambda^+\varphi(s)[$ with $\varepsilon_n \to 0$, there exists a corresponding sequence (r_n) in R_+^0 with $r_n \to s+$ and $\varphi(r_n) \to \Lambda^+\varphi(s)$.

Call $\varphi \in \mathcal{F}(re)(R_+)$, *Boyd–Wong admissible*, if

(c08) $\Lambda^+\varphi(s) < s$ (or, equivalently: $\Lambda_+\varphi(s) < s$), for all $s > 0$.

(This convention is related to the developments in Boyd and Wong [10]; we do not give details). In particular, $\varphi \in \mathcal{F}(re)(R_+)$ is Boyd–Wong admissible provided it is upper semicontinuous at the right on R_+^0:

$\Lambda^+\varphi(s) = \varphi(s)$, (or, equivalently: $\Lambda_+\varphi(s) \le \varphi(s)$), $\forall s \in R_+^0$.

Note that this is fulfilled when φ is continuous at the right on R_+^0; for, in such a case, $\Lambda_+\varphi(s) = \varphi(s)$, $\forall s \in R_+^0$.

(II) Call $\varphi \in \mathcal{F}(re)(R_+)$, *Matkowski admissible* [24], provided

(c09) φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$.

(Here, φ^n stands for the nth iterate of φ). Note that the obtained class of functions is distinct from the above introduced one, as simple examples show.

Now, let us say that $\varphi \in \mathcal{F}(re)(R_+)$ is *Boyd–Wong–Matkowski admissible* (abbreviated: BWM-admissible) if it is either Boyd–Wong admissible or Matkowski admissible. The following auxiliary fact will be useful (cf. Jachymski [16]):

Lemma 9 *Let $\varphi \in \mathcal{F}(re)(R_+)$ be a BWM-admissible function. Then, φ is Meir–Keeler admissible (see above).*

Proof (i) Suppose that $\varphi \in \mathcal{F}(re)(R_+)$ is Boyd–Wong admissible; and let $\gamma > 0$; hence, $\Lambda^+\varphi(\gamma) < \gamma$. Let the number $\eta > 0$ be such that $\Lambda^+\varphi(\gamma) < \eta < \gamma$. By definition, there exists $\beta = \beta(\eta) > 0$ such that $\gamma \le t < \gamma + \beta$ implies $\varphi(t) < \eta < \gamma$. On the other hand, if $t < \gamma$, then $\varphi(t) \le t < \gamma$; and conclusion follows.

(ii) Assume that $\varphi \in \mathcal{F}(re)(R_+)$ is Matkowski admissible. If the underlying property fails, then (for some $\gamma > 0$):

$\forall \beta > 0, \exists t \in [0, \gamma + \beta[$, such that $\varphi(t) > \gamma$ (hence, $\gamma < t < \gamma + \beta$).

As φ is increasing, this yields $\varphi(t) > \gamma$, $\forall t > \gamma$. By induction, we get $[\varphi^n(t) > \gamma, \forall n, \forall t > \gamma]$; hence, taking some $t > \gamma$ and passing to limit as $n \to \infty$, one gets $0 \ge \gamma$; contradiction. This ends the argument.

(B) Let us say that (ψ, φ) is a *pair of weak generalized altering functions* in $\mathcal{F}(R_+)$, if it fulfills the following conditions

(c10) ψ is increasing and $\varphi(0) = 0$
(c11) $(\forall \varepsilon > 0)$: $\limsup_n \varphi(t_n) > \psi(\varepsilon + 0) - \psi(\varepsilon)$, whenever $t_n \to \varepsilon++$
(c12) $(\forall \varepsilon > 0)$: $\varphi(\varepsilon) > \psi(\varepsilon) - \psi(\varepsilon - 0)$.

A basic example of such couples is the following. Let us say that (ψ, φ) is a *pair of generalized altering functions* in $\mathcal{F}(R_+)$, if

(c13) ψ is increasing continuous, $\varphi(0) = 0$, and $[\varphi(t) > 0, \forall t > 0]$
(c14) $(\forall \varepsilon > 0)$: $\limsup_n \varphi(t_n) > 0$, whenever $t_n \to \varepsilon++$.

Lemma 10 *Suppose that (ψ, φ) is a pair of generalized altering functions in $\mathcal{F}(R_+)$. Then, (ψ, φ) is a pair of weak generalized altering functions in $\mathcal{F}(R_+)$.*

Proof Assume that (ψ, φ) is as in the premise above. By the continuity of ψ, (c11) is just (c14). On the other hand, by the same reason, (c12) means: $\varphi(\varepsilon) > 0, \forall \varepsilon > 0$; which is assured via (c13), and then, the conclusion follows.

Given $G \in \{G_1, G_2, G_3\}$ and the couple (ψ, φ) of functions in $\mathcal{F}(R_+)$, let us say that T is $(d, \mathcal{R}; G, (\psi, \varphi))$*-contractive*, provided

(c15) $\psi(d(Tx, Ty)) \le \psi(G(x, y)) - \varphi(G(x, y)), \forall x, y \in X, x\mathcal{R}y.$

Lemma 11 *Suppose that T is $(d, \mathcal{R}; G, (\psi, \varphi))$-contractive, for a pair (ψ, φ) of weak generalized altering functions in $\mathcal{F}(R_+)$. Then, T is Meir–Keeler $(d, \mathcal{R}; G)$-contractive (see above).*

Proof (i) Let $x, y \in X$ be such that $x\mathcal{R}y$ and $G(x, y) > 0$. Then, $\varphi(G(x, y)) > 0$; wherefrom $\psi(d(Tx, Ty)) < \psi(G(x, y))$. This, via ($\psi$ = increasing), yields $d(Tx, Ty) < G(x, y)$; so, the first part of the Meir–Keeler contractive condition holds.

(ii) Assume by contradiction that the second part of the Meir–Keeler contractive condition fails, i.e., for some $\varepsilon > 0$,

$$\forall \delta > 0, \exists x_\delta, y_\delta \in X : \ [x_\delta \mathcal{R} y_\delta,\ \varepsilon < G(x_\delta, y_\delta) < \varepsilon + \delta,\ d(Tx_\delta, Ty_\delta) > \varepsilon].$$

Taking a zero converging sequence (δ_n) in R_+^0, we get a couple of sequences $(x_n; n \ge 0)$ and $(y_n; n \ge 0)$ in X, so as

$$(\forall n) : \ \ x_n \mathcal{R} y_n,\ \varepsilon < G(x_n, y_n) < \varepsilon + \delta_n,\ d(Tx_n, Ty_n) > \varepsilon. \tag{48}$$

By the contractive condition (and ψ = increasing), we get

$$\psi(\varepsilon) \le \psi(G(x_n, y_n)) - \varphi(G(x_n, y_n)), \ \ \forall n;$$

or, equivalently,

$$\varphi(G(x_n, y_n)) \le \psi(G(x_n, y_n)) - \psi(\varepsilon), \ \ \forall n. \tag{49}$$

By (48), $G(x_n, y_n) \to \varepsilon + +$; so that, passing to lim sup as $n \to \infty$,

$$\limsup_n \varphi(G(x_n, y_n)) \le \psi(\varepsilon + 0) - \varphi(\varepsilon).$$

But, from the hypothesis about (ψ, φ), these relations are contradictory. This ends the argument.

3.4 Main Result

Let $(X, d, \mathcal{R})$ be a relational metric space. Further, let T be a selfmap of X; supposed to be $\mathcal{R}$-semi-progressive and $\mathcal{R}$-increasing. The basic directions and sufficient regularity conditions under which the problem of determining the fixed points of T is to be solved were already listed.

The main result of this exposition is as follows.

Theorem 15 *Assume that T is Meir–Keeler $(d, \mathcal{R}; G)$-contractive, for some $G \in \{G_1, G_2, G_3\}$. In addition, let $\mathcal{R}$ be locally finitely transitive, (X, d) be (a-o)-complete, and one of the following conditions hold:*

(i) *T is $(a - o, d)$-continuous*

(ii) *$\mathcal{R}$ is $(a - o, d)$-almost-selfclosed and $G = G_1$*

(iii) *$\mathcal{R}$ is $(a - o, d)$-almost-selfclosed and T is $(d, \mathcal{R}; G, \varphi)$-contractive, for a certain Meir–Keeler admissible function $\varphi \in \mathcal{F}(re)(R_+)$*

(iv) *$\mathcal{R}$ is $(a - o, d)$-almost-selfclosed and T is $(d, \mathcal{R}; G, (\psi, \varphi))$-contractive, for a certain pair (ψ, φ) of weak generalized altering functions in $\mathcal{F}(R_+)$.*

Then T is a globally strong Picard operator (modulo $(d, \mathcal{R})$).

Proof First, we check the fix-$\mathcal{R}$-asingleton property. Let $z_1, z_2 \in \mathrm{Fix}(T)$ be such that $z_1 \mathcal{R} z_2$; and assume by contradiction that $z_1 \neq z_2$; whence (by sufficiency), $d(z_1, z_2) > 0$. From the very definitions above,

$$G_1(z_1, z_2) = G_2(z_1, z_2) = G_3(z_1, z_2) = d(z_1, z_2).$$

This, along with the strict $(d, \mathcal{R}; G)$-nonexpansive condition, yields

$$d(z_1, z_2) = d(Tz_1, Tz_2) < d(z_1, z_2);$$

contradiction; hence, the claim. It remains now to establish the strong Picard property (modulo $(d, \mathcal{R})$). The argument will be divided into several steps.

Part 1 We first assert that

$$G(x, Tx) = d(x, Tx), \text{ whenever } x\mathcal{R}Tx,\ x \neq Tx. \tag{50}$$

The case $G = G_1$ is clear; so, it remains to discuss the case $G \in \{G_2, G_3\}$. Let $x \in X$ be such that $x\mathcal{R}Tx$, $x \neq Tx$. By the strict $(d, \mathcal{R}; G)$-nonexpansive property of the selfmap T, we must have $d(Tx, T^2x) < G(x, Tx)$. On the other hand, as

$$L(x, Tx) = (1/2)[d(x, T^2x) + d(Tx, Tx)] \le (1/2)[d(x, Tx) + d(Tx, T^2x)] \le \max\{d(x, Tx), d(Tx, T^2x)\} = H(x, Tx),$$

it results that $G_2(x, Tx) = G_3(x, Tx) = H(x, Tx)$. This, along with

$$d(Tx, T^2x) < H(x, Tx) \Longrightarrow d(Tx, T^2x) < d(x, Tx) \Longrightarrow H(x, Tx) = d(x, Tx),$$

gives the desired fact.

Part 2 Take some $x_0 \in X$; and put $(x_n = T^n x_0; n \ge 0)$. If $x_n = x_{n+1}$ for some $n \ge 0$, we are done; so, without loss, one may assume that

(d02) $x_n \neq x_{n+1}$ (hence, $\rho_n := d(x_n, x_{n+1}) > 0$), $\forall n$.

From the preceding part, we derive

$$\rho_{n+1} = d(Tx_n, Tx_{n+1}) < G(x_n, x_{n+1}) = \rho_n, \ \forall n;$$

so that, the sequence $(\rho_n; n \geq 0)$ is strictly descending. As a consequence, $\rho := \lim_n \rho_n$ exists as an element of R_+. Assume by contradiction that $\rho > 0$; and let $\delta > 0$ be the number given by the Meir–Keeler $(d, \mathcal{R}; G)$-contractive condition upon T. By definition, there exists a rank $n(\delta)$ such that $n \geq n(\delta)$ implies $\rho < \rho_n < \rho + \delta$; hence (by a previous representation), $\rho < G(x_n, x_{n+1}) = \rho_n < \rho + \delta$. This, by the Meir–Keeler contractive condition we just quoted, yields (for the same n), $\rho_{n+1} = d(Tx_n, Tx_{n+1}) \leq \rho$; contradiction. Hence, $\rho = 0$; so that,

$$d(x_n, Tx_n) = d(x_n, x_{n+1}) \to 0, \ \text{as } n \to \infty. \tag{51}$$

Part 3 Suppose that

(d03) there exist $i, j \in N$ such that $i < j$, $x_i = x_j$.

Denoting $p = j - i$, we thus have $p > 0$ and $x_i = x_{i+p}$; so that

$$x_i = x_{i+np}, x_{i+1} = x_{i+np+1}, \ \text{for all } n \geq 0.$$

By the introduced notations, we then have $\rho_i = \rho_{i+np}$, for all $n \geq 0$. This, along with $\rho_{i+np} \to 0$ as $n \to \infty$, yields $\rho_i = 0$; in contradiction with the initial choice of $(\rho_n; n \geq 0)$. Hence, our working hypothesis cannot hold; wherefrom

$$\text{for all } i, j \in N\colon \ i \neq j \text{ implies } x_i \neq x_j. \tag{52}$$

Part 4 As a consequence of this, the map $n \mapsto x_n$ is injective; hence, $Y := \{x_n; n \geq 0\}$ is effectively denumerable. Denote by $k := k(Y) \geq 2$ the transitivity constant of $\mathcal{R}$ over Y (assured by the choice of this relation). Further, let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number associated by the Meir–Keeler $(d, \mathcal{R}; G)$-contractive property; without loss, one may assume that $\delta < \varepsilon$. By a previous part, there exists some rank $n(\delta) \geq 0$, such that

$$\begin{aligned} &(\forall n \geq n(\delta)): \ d(x_n, x_{n+1}) < \delta/4k; \ \text{whence} \\ &d(x_n, x_{n+h}) < h\delta/4k \leq \delta/4, \ \forall h \in \{1, \ldots, k\}. \end{aligned} \tag{53}$$

(The second evaluation above follows at once by the triangular property). We claim that the following relation holds

$$(\forall s \geq 1): \ [d(x_n, x_{n+s}) < \varepsilon + \delta/2, \ \forall n \geq n(\delta)]; \tag{54}$$

wherefrom, $(x_n; n \geq 0)$ is d-Cauchy. To do this, an induction argument upon s will be used. The case $s \in \{1, \ldots, k\}$ is evident, by the preceding evaluation. Assume that it holds for all $s \in \{1, \ldots, p\}$, where $p \geq k$; we must establish its validity for $s = p + 1$; or, in other words,

$$d(x_n, x_{n+p+1}) < \varepsilon + \delta/2, \ \forall n \geq n(\delta). \tag{55}$$

As $p \geq k$ (hence, $p - 1 \geq k - 1$), we have

$$p - 1 = i(k - 1) + j, \text{ for some } i \geq 1, \ j \in \{0, \ldots, k - 2\}.$$

Denote for simplicity $q = 1 + i(k - 1)$; hence, $2 \leq k \leq q \leq p = q + j$; in addition, by Lemma 6, $x_n \mathcal{R} x_{n+q}$. From the inductive hypothesis, (53), and the preceding part,

$$0 < d(x_n, x_{n+q}) < \varepsilon + \delta/2 < \varepsilon + \delta,$$
$$d(x_n, x_{n+1}), d(x_{n+q}, x_{n+q+1}) < \delta/4k < \varepsilon + \delta;$$

wherefrom (by definition), $H(x_n, x_{n+q}) < \varepsilon + \delta$. On the other hand, from the same premises (and the triangular inequality),

$$d(x_n, x_{n+q+1}) \leq d(x_n, x_{n+q}) + d(x_{n+q}, x_{n+q+1}) < \varepsilon + \delta/2 + \delta/4k,$$
$$d(x_{n+1}, x_{n+q}) = d(x_{n+1}, x_{n+1+q-1}) < \varepsilon + \delta/2;$$

wherefrom (again by definition), $L(x_n, x_{n+q}) < \varepsilon + \delta$; and, from this, one gets (in any case) $0 < G(x_n, x_{n+q}) < \varepsilon + \delta$. Taking the Meir–Keeler $(d, \mathcal{R}; G)$-contractive property of T into account, gives

$$d(x_{n+1}, x_{n+q+1}) = d(Tx_n, Tx_{n+q}) \leq \varepsilon;$$

so that, by the triangular inequality (and (53) again)

$$d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+q+1}) + d(x_{n+q+1}, x_{n+p+1})$$
$$\leq \varepsilon + \delta/4k + j\delta/4k < \varepsilon + \delta/8 + \delta/4 = \varepsilon + 3\delta/8 < \varepsilon + \delta/2;$$

and our claim follows.

Part 5 As (X, d) is (a-o)-complete, $x_n \xrightarrow{d} z$, for some (uniquely determined) $z \in X$. If there exists a sequence of ranks $(i(n); n \geq 0)$ with $i(n) \to \infty$ as $n \to \infty$ such that $x_{i(n)} = z$ (hence, $x_{i(n)+1} = Tz$) for all n, then, as $(x_{i(n)+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, one gets $z = Tz$; i.e., $z \in \text{Fix}(T)$. So, in the following, we may assume that the opposite alternative is true:

(d04) $\exists h \geq 0$: $n \geq h \Longrightarrow x_n \neq z$.

There are several cases to discuss.

Case 5a Suppose that T is $(a - o, d)$-continuous. Then, $y_n := Tx_n \xrightarrow{d} Tz$ as $n \to \infty$. On the other hand, $(y_n = x_{n+1};\ n \geq 0)$ is a subsequence of (x_n); and this yields (as d is sufficient), $z = Tz$.

Case 5b Suppose that $\mathcal{R}$ is $(a-o,d)$-almost-selfclosed. By definition, there exists a subsequence $(u_n := x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$, such that $u_n \mathcal{R} z$, $\forall n$. As $\lim_n i(n) = \infty$, one may arrange for $i(n) \geq n$, $\forall n$; so, from the accepted condition,

$$i(n) \geq h, \forall n \geq h; \text{ whence } u_n \neq z, \forall n \geq h. \tag{56}$$

This, along with $(Tu_n = x_{i(n)+1}; n \geq 0)$ being as well a subsequence of $(x_n; n \geq 0)$, gives (via (53) and Lemma 5)

$$\begin{aligned} &d(u_n, z), d(Tu_n, z) \to 0,\ d(u_n, Tu_n) \to 0, \\ &d(u_n, Tz) \to d(z, Tz),\ d(Tu_n, Tz) \to d(z, Tz); \\ &\text{whence, } H(u_n, z) \to d(z, Tz), L(x_n, z) \to (1/2)d(z, Tz). \end{aligned} \tag{57}$$

Two alternatives must now be treated.

Alter 1 Suppose that $G = G_1$. By the Meir–Keeler contractive condition,

$$d(Tu_n, Tz) < d(u_n, z),\ \forall n \geq h;$$

hence, $Tu_n \xrightarrow{d} Tz$. On the other hand, as $(Tu_n = x_{i(n)+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, we have $Tu_n \xrightarrow{d} z$. Combining these, gives (as d is sufficient), $z = Tz$; i.e., $z \in \mathrm{Fix}(T)$.

Alter 2 Suppose that $G \in \{G_2, G_3\}$. If $z \neq Tz$, we must have $b := d(z, Tz) > 0$. The above convergence properties of $(u_n; n \geq 0)$ tell us that, for a certain rank $n(b) \geq h$, we must have

$$d(u_n, Tu_n), d(u_n, z), d(Tu_n, z) < b/2,\ \forall n \geq n(b).$$

This, by the d-Lipschitz property of $d(.,.)$, gives

$$|d(u_n, Tz) - b| \leq d(u_n, z) < b/2, \forall n \geq n(b),$$

wherefrom: $b/2 < d(u_n, Tz) < 3b/2$, $\forall n \geq n(b)$. Combining these, yields

$$G(u_n, z) = b,\ \forall n \geq n(b). \tag{58}$$

Two sub-cases are now under discussion.

Alter 2a Suppose that T is $(d, \mathcal{R}; G, \varphi)$-contractive, for a certain Meir–Keeler admissible function $\varphi \in \mathcal{F}(re)(R_+)$. The case $G = G_1$ was already clarified; so, assume that $G \in \{G_2, G_3\}$. By (58) and this contractive property,

$$d(Tu_n, Tz) \leq \varphi(b),\ \forall n \geq n(b).$$

Passing to limit as $n \to \infty$ gives (by (57) above) $b \leq \varphi(b)$; contradiction; hence, $z = Tz$; i.e., $z \in \mathrm{Fix}(T)$.

Alter 2b Suppose that T is $(d, \mathcal{R}; G, (\psi, \varphi))$-contractive, for a certain pair (ψ, φ) of weak generalized altering functions in $\mathcal{F}(R_+)$. As before, the case $G = G_1$ is clear; so, assume that $G \in \{G_2, G_3\}$. By this contractive condition,

$$\psi(d(Tu_n, Tz)) \leq \psi(G(u_n, z)) - \varphi(G(u_n, z)), \forall n \geq n(b);$$

or, equivalently (combining with (58) above)

$$0 < \varphi(b) \leq \psi(b) - \psi(d(Tu_n, Tz)), \ \forall n \geq n(b). \tag{59}$$

Note that, as a consequence, $d(Tu_n, Tz) < b$, $\forall n \geq n(b)$. Passing to limit as $n \to \infty$ and taking (57) into account, yields $\varphi(b) \leq \psi(b) - \psi(b - 0)$. This, however, contradicts the choice of the pair (ψ, φ); so that, $z = Tz$. The proof is complete.

In particular, when T is $(d, \mathcal{R}; G_1, \varphi)$-contractive and $\varphi \in \mathcal{F}(re)(R_+)$ is Boyd–Wong admissible, our main result includes the cyclical fixed point theorem due to Kirk et al. [21]. On the other hand, when $\mathcal{R}$ is transitive, this result is comparable with the one in Turinici [42]. Note that, further extensions of these developments are possible, in the realm of triangular symmetric spaces, taken as in Hicks and Rhoades [13]; or, in the setting of partial metric spaces, introduced under the lines in Matthews [25]; we do not give details.

3.5 Further Aspects

In the following, some basic particular cases of the main result are discussed. Technically speaking, there are three categories of such statements; according to the alternatives of Theorem 15 we already listed.

Case 1 Let $(X, d, \mathcal{R})$ be a relational metric space; and T be a selfmap of X. By Theorem 15, we then get

Theorem 16 *Assume that T is $\mathcal{R}$-semi-progressive, $\mathcal{R}$-increasing, and Meir–Keeler $(d, \mathcal{R}; G)$-contractive, for some $G \in \{G_1, G_2, G_3\}$. In addition, let $\mathcal{R}$ be finitely transitive, (X, d) be (a-o)-complete and T be $(a - o, d)$-continuous. Then, T is a globally strong Picard operator (modulo $(d, \mathcal{R})$).*

In particular, let γ be a function in $\mathcal{F}(X \times X, R_+)$; and $\mathcal{C}$ stand for the associated relation: $[x\mathcal{C}y$ iff $\gamma(x, y) \geq 1]$. Then, if we take $\mathcal{R} := \mathcal{C}$ and $G = G_1$, this result includes the one in Berzig and Rus [7].

Case 2 Let $(X, d, \mathcal{R})$ be a relational metric space. Remember that $\varphi \in \mathcal{F}(re)(R_+)$ is *BWM-admissible*, when it is either Boyd–Wong admissible or Matkowski admissible. Further, let T be a selfmap of X. As another consequence of Theorem 15, we have the following statement (with practical value):

Theorem 17 *Assume that T is $\mathcal{R}$-semi-progressive, $\mathcal{R}$-increasing, and $(d, \mathcal{R}; G, \varphi)$-contractive, for some $G \in \{G_1, G_2, G_3\}$ and a certain BWM-admissible*

function $\varphi \in \mathcal{F}(re)(R_+)$. In addition, let $\mathcal{R}$ be finitely transitive, (X, d) be (a-o)-complete (each d-Cauchy $\mathcal{R}$-ascending T-orbital sequence in X is d-convergent), and one of the conditions below holds:

(i1) *T is $(a-o, d)$-continuous: for each $\mathcal{R}$-ascending T-orbital sequence $(x_n; n \geq 0)$ in X with $x_n \xrightarrow{d} x$, we have $Tx_n \xrightarrow{d} Tx$*

(i2) *$\mathcal{R}$ is $(a-o, d)$-almost-selfclosed: whenever the $\mathcal{R}$-ascending T-orbital sequence $(z_n; n \geq 0)$ in X and the point $z \in X$ fulfill $z_n \xrightarrow{d} z$, there exists a subsequence $(w_n; n \geq 0)$ of $(z_n; n \geq 0)$ with $w_n \mathcal{R} z$, for all $n \geq 0$.*

Then T is a globally strong Picard operator (modulo $(d, \mathcal{R})$).

The following particular cases of this result are to be noted.

(a1) Suppose that $\mathcal{R} = X \times X$ (= the trivial relation over X). Then, if $G = G_1$, Theorem 16 includes the Boyd–Wong's result [10] when φ is Boyd–Wong admissible; and, respectively, the Matkowski's result [24] when φ is Matkowski admissible. Moreover, when $G = G_3$, Theorem 16 includes the result in Leader [22]; see also Jachymski [15].

(a2) Suppose that $\mathcal{R}$ is an order on X. Then, if $G \in \{G_1, G_3\}$, Theorem 16 includes the results in Agarwal et al. [1]; see also O'Regan and Petruşel [29].

Case 3 Let again $(X, d, \mathcal{R})$ be a relational metric space; and T be a selfmap of X. As a final consequence of Theorem 15, we have the following

Theorem 18 *Assume in the following that T is $\mathcal{R}$-semi-progressive, $\mathcal{R}$-increasing, and $(d, \mathcal{R}; G, (\psi, \varphi))$-contractive, for a certain $G \in \{G_1, G_2, G_3\}$ and some pair (ψ, φ) of generalized altering functions in $\mathcal{F}(R_+)$. In addition, let $\mathcal{R}$ be finitely transitive, (X, d) be a-complete (each d-Cauchy $\mathcal{R}$-ascending sequence in X is d-convergent), and one of the conditions below holds:*

(j1) *T is (a, d)-continuous: for each $\mathcal{R}$-ascending sequence, $(x_n; n \geq 0)$ with $x_n \xrightarrow{d} x$, we have $Tx_n \xrightarrow{d} Tx$.*

(j2) *$\mathcal{R}$ is (a, d)-almost-selfclosed: whenever the $\mathcal{R}$-ascending sequence $(z_n; n \geq 0)$ in X and the point $z \in X$ fulfill $z_n \xrightarrow{d} z$, there exists a subsequence $(w_n; n \geq 0)$ of $(z_n; n \geq 0)$ with $w_n \mathcal{R} z$, for all $n \geq 0$.*

Then T is a globally strong Picard operator (modulo $(d, \mathcal{R})$).

In particular, let α, β be a couple of functions in $\mathcal{F}(X \times X, R_+)$; and $\mathcal{A}, \mathcal{B}$ stand for the associated relations

$x \mathcal{A} y$ iff $\alpha(x, y) \leq 1$; $x \mathcal{B} y$ iff $\beta(x, y) \geq 1$.

Then, if we take $\mathcal{R} := \mathcal{A} \cap \mathcal{B}$ and $G = G_1$, this result includes the one in Karapinar and Berzig [18], based on global contractive conditions like

(e02) $\psi(d(Tx, Ty)) \leq \alpha(x, y)\psi(d(x, y)) - \beta(x, y)\varphi(d(x, y)), \forall x, y \in X$;

referred to as: T is *$(\alpha\psi, \beta\varphi)$-contractive*. In fact, the quoted result (stated in terms of $\psi \in \mathcal{F}(R_+, R)$) is not in general correct; because, a relation like

$$\alpha(x, y) \leq 1 \Longrightarrow \alpha(x, y)\psi(d(x, y)) \leq \psi(d(x, y))$$

is not true, as long as $\psi(d(x, y)) < 0$. But, when one assumes that $\psi \in \mathcal{F}(R_+)$, the reasoning above is retainable. In this perspective, note that the quoted statement is an extension of the one in Samet et al. [38]; hence, so is Theorem 18 above. It is to be stressed that none of these corollaries may be viewed as a genuine extension for the fixed point statement in Samet and Turinici [37]; because, in the quoted result, the ambient relation $\mathcal{R}$ is not subjected to any kind of transitive type requirements. Further aspects (involving the same general setting) may be found in Berzig [6].

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Half-Discrete Hilbert-Type Inequalities, Operators and Compositions

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Abstract In this chapter, using the methods of weight functions and technique of real analysis, a half-discrete Hilbert-type inequality with a homogeneous kernel and a best possible constant factor is provided. Some equivalent representations, two types of reverses, the operator expressions as well as some particular examples are obtained. Furthermore, we also consider some strengthened versions of half-discrete Hilbert's inequality relating to Euler constant, the related inequalities and operators with the non-homogeneous kernel, and two kinds of compositions of two operators in certain conditions.

Keywords Half-discrete Hilbert-type inequality · Weight function · Equivalent form · Hilbert-type operator · Composition

1 Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+), ||f||_p = \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} > 0, ||g||_q > 0$. We have the following Hardy–Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q$, $||a||_p = \left\{\sum_{m=1}^\infty a_m^p\right\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we have the

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following discrete Hardy–Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$(cf. [1]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1–6]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [7] gave an extension of (1) for $p = q = 2$. In 2009 and 2011, Yang [3, 4] gave some extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2, \lambda \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \left\{\int_0^\infty \phi(x)|f(x)|^p dx\right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}\left(k_\lambda(x,y)y^{\lambda_2-1}\right)$ is decreasing with respect to $x > 0 (y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := \left\{\sum_{n=1}^{\infty} \phi(n)|a_n|^p\right\}^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1, k_1(x, y) = \frac{1}{x+y}, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including multi-dimensional Hilbert-type integral inequalities are provided by [8–21].

About the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. However, Yang [22]

gave a result with the kernel $\frac{1}{(1+nx)^{\lambda}}$ by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [23] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} dx < B(\lambda_1, \lambda_2) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{5}$$

where, $\lambda_1 \lambda_2 > 0$, $0 \leq \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$,

$$B(u, v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt (u, v > 0)$$

is the beta function. Zhong et al. [24–30] investigated several half-discrete Hilbert-type inequalities with particular kernels.

Applying the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_{\lambda}(x, n) a_n dx < k(\lambda_1) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{6}$$

which is an extension of (5) (see Yang and Chen [31]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [32] .

Remark 1 (1) Many different kinds of Hilbert-type discrete, half-discrete and integral inequalities with applications are presented in recent 20 years. Special attention is given to new results proved during 2009–2012. Included are many generalizations, extensions and refinements of Hilbert-type discrete, half-discrete and integral inequalities involving many special functions such as Riemann zeta, beta, gamma, hypergeometric, trigonometric, hyperbolic, zeta, Bernoulli functions, Bernoulli numbers and Euler constant et al. The following references [33–41] provide an extensive theory and applications of Analytic Number Theory that will provide a source study for further research on Hilbert-type inequalities.

(2) In his five books, Yang [3–6, 42] presented many new results on Hilbert-type operators with general homogeneous kernels of degree of real numbers and two pairs of conjugate exponents as well as the related inequalities. These research monographs contained recent developments of discrete, multiple half-discrete and integral types of operators and inequalities with proofs, examples and applications.

In this chapter, using the methods of weight functions and technique of real analysis, a half-discrete Hilbert-type inequality with a homogeneous kernel and a best possible constant factor is provided. Some equivalent representations, two types of reverses, the operator expressions as well as some particular examples are obtained. Furthermore, we also consider some strengthened versions of half-discrete Hilbert's inequality relating to Euler constant, the related inequalities and operators with the non-homogeneous kernel, and two kinds of compositions of two operators in certain conditions.

2 Half-Discrete Hilbert-Type Inequalities with the General Homogeneous Kernel and Operator Expressions

In this section, we agree that $p \in \mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda, \lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1+\lambda_2=\lambda$, $k_\lambda(x,y)(\geq 0)$ is a finite homogeneous function of degree $-\lambda$ in $\mathbf{R}_+^2$, satisfying for any $u,x,y \in \mathbf{R}_+$, $k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y)$.

2.1 Lemmas and Some Equivalent Inequalities

Definition 1 For $x \in \mathbf{R}_+, n \in \mathbf{N}$, define two weight functions $\omega_\lambda(\lambda_2,n)$ and $\varpi_\lambda(\lambda_1,x)$ as follows:

$$\omega_\lambda\,(\lambda_2,n) := n^{\lambda_2}\int_0^\infty k_\lambda(x,n)\frac{1}{x^{1-\lambda_1}}dx, \tag{7}$$

$$\varpi_\lambda(\lambda_1,x) := x^{\lambda_1}\sum_{n=1}^\infty k_\lambda(x,n)\frac{1}{n^{1-\lambda_2}}. \tag{8}$$

Setting $u=x/n$, we find

$$\begin{aligned}\omega_\lambda(\lambda_2,n) &= n^{\lambda_2}\int_0^\infty k_\lambda(nu,n)\frac{ndu}{(nu)^{1-\lambda_1}}\\ &= \int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du. \end{aligned}\tag{9}$$

Lemma 1 *If $\varpi_\lambda(\lambda_1,x)$ is finite for $x \in \boldsymbol{R}_+$, $f(x), a_n \geq 0$, and*

$$k(\lambda_1) := \int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du \in \mathbf{R}_+, \tag{10}$$

then (i) *for $p>1$, we have the following inequality:*

$$\begin{aligned}J_1 &:= \left\{\sum_{n=1}^\infty n^{p\lambda_2-1}\left(\int_0^\infty k_\lambda(x,n)f(x)dx\right)^p\right\}^{\frac{1}{p}}\\ &\leq [k(\lambda_1)]^{\frac{1}{q}}\left\{\int_0^\infty \varpi_\lambda(\lambda_1,x)x^{p(1-\lambda_1)-1}f^p(x)dx\right\}^{\frac{1}{p}}, \end{aligned}\tag{11}$$

$$\begin{aligned}\widetilde{J}_2 &:= \left\{\int_0^\infty \frac{x^{q\lambda_1-1}}{[\varpi_\lambda(\lambda_1,x)]^{q-1}}\left(\sum_{n=1}^\infty k_\lambda(x,n)a_n\right)^q dx\right\}^{\frac{1}{q}}\\ &\leq \left\{k(\lambda_1)\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right\}^{\frac{1}{q}}; \end{aligned}\tag{12}$$

(ii) for $p < 0$, or $0 < p < 1$, we have the reverses of (11) and (12).

Proof (i) For $p > 1$, by Hölder's inequality with weight (cf. [47]), it follows

$$\int_0^\infty k_\lambda(x,n)f(x)dx$$

$$= \int_0^\infty k_\lambda(x,n)\left[\frac{x^{(1-\lambda_1)/q}}{n^{(1-\lambda_2)/p}}f(x)\right]\left[\frac{n^{(1-\lambda_2)/p}}{x^{(1-\lambda_1)/q}}\right]dx$$

$$\leq \left\{\int_0^\infty k_\lambda(x,n)\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_0^\infty k_\lambda(x,n)\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}}\right\}^{\frac{1}{q}}$$

$$= [\omega_\lambda(\lambda_2,n)]^{\frac{1}{q}}\, n^{\frac{1}{p}-\lambda_2}\left\{\int_0^\infty k_\lambda(x,n)\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}. \tag{13}$$

Then by Lebesgue term, by term integration theorem (cf. [43]), in view of (9), we have

$$J_1 \leq [k(\lambda_1)]^{\frac{1}{q}}\left\{\sum_{n=1}^\infty\int_0^\infty k_\lambda(x,n)\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$

$$= [k(\lambda_1)]^{\frac{1}{q}}\left\{\int_0^\infty\sum_{n=1}^\infty k_\lambda(x,n)\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$

$$= [k(\lambda_1)]^{\frac{1}{q}}\left\{\int_0^\infty \varpi_\lambda(\lambda_1,x)x^{p(1-\lambda_1)-1}f^p(x)dx\right\}^{\frac{1}{p}}. \tag{14}$$

Hence, (11) follows.

By the same way as in obtaining (13), we obtain

$$\sum_{n=1}^\infty k_\lambda(x,n)a_n \leq [\varpi_\lambda(\lambda_1,x)]^{\frac{1}{p}}x^{\frac{1}{q}-\lambda_1}$$

$$\times\left\{\sum_{n=1}^\infty k_\lambda(x,n)\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}}a_n^q\right\}^{\frac{1}{q}}, \tag{15}$$

then by Lebesgue term, by term integration theorem and the same way as in obtaining (14), we have (12).

(ii) For $p < 0$, or $0 < p < 1$, by the reverse Hölder's inequality with weight (cf. [47]), we obtain the reverses of (13) and (14). Then by Lebesgue term by term

integration theorem, we still can obtain the reverses of (11) and (12). The lemma is proved.

Lemma 2 *As the assumptions of Lemma 1, then* (i) *for* $p > 1$, *we have the following inequality equivalent to* (11) *and* (12)*:*

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} k_\lambda(x, n) a_n f(x) dx$$
$$\leq \left\{ \int_0^{\infty} \varpi_\lambda(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ k(\lambda_1) \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}; \tag{16}$$

(ii) for $p < 0$ or $0 < p < 1$, we have the reverse of (16) equivalent to the reverses of (11) and (12).

Proof (i) For $p > 1$, by Hölder's inequality (cf. [47]), it follows

$$I = \sum_{n=1}^{\infty} n^{\frac{1}{q}-(1-\lambda_2)} \left[\int_0^{\infty} k_\lambda(x, n) f(x) dx \right] \left[n^{(1-\lambda_2)-\frac{1}{q}} a_n \right]$$
$$\leq J_1 \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}. \tag{17}$$

Then by (11), we have (16). On the other hand, assuming that (16) is valid, we set

$$b_n := n^{p\lambda_2 - 1} \left(\int_{\mathbf{0}}^{\infty} k_\lambda(x, n) f(x) dx \right)^{p-1}, n \in \mathbf{N}.$$

Then it follows $J_1^p = \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q$. If $J_1 = 0$, then (11) is trivially valid; if $J_1 = \infty$, then by (14), (11) keeps the form of equality ($= \infty$). Suppose that $0 < J_1 < \infty$. By (16), we have

$$0 < \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q = J_1^p = I$$
$$\leq \left\{ \int_0^{\infty} \varpi_\lambda(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ k(\lambda_1) \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}.$$

It follows

$$J_1 = \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{p}}$$

$$\leq [k(\lambda_1)]^{\frac{1}{q}} \left\{ \int_0^\infty \varpi_\lambda(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}},$$

and then (11) follows. Hence, (11) and (16) are equivalent.

By Hölder's inequality and the same way, we can obtain

$$I \leq \left\{ \int_0^\infty \varpi_\lambda(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}} \widetilde{J}_2. \tag{18}$$

Then by (12), we have (16). On the other hand, assuming that (16) is valid, we set

$$f(x) = \frac{x^{q\lambda_1 - 1}}{[\varpi_\lambda(\lambda_1, x)]^{q-1}} \left(\sum_{n=1}^\infty k_\lambda(x, n) a_n \right)^{q-1} \quad (x \in \mathbf{R}_+).$$

Then it follows $\widetilde{J}_2^q = \int_0^\infty \varpi_\lambda(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx$. By (16) and the same way, we can obtain

$$\widetilde{J}_2 = \left\{ \int_0^\infty \varpi_\lambda(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{q}}$$
$$\leq \left\{ k(\lambda_1) \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}},$$

and then (12) is equivalent to (16).

Hence inequalities (11), (12) and (16) are equivalent.

(ii) For $p < 0$ or $0 < p < 1$, by the same way, we can obtain the reverse of (16) equivalent to the reverses of (11) and (12). The lemma is proved.

By Lemma 2, we still have

Theorem 1 *As the assumptions of Lemma 1, there exists a function $\theta_{\lambda_1}(x) \in (0, 1)$, such that*

$$k(\lambda_1)(1 - \theta_{\lambda_1}(x)) < \varpi_\lambda(\lambda_1, x) < k(\lambda_1)(x \in \mathbf{R}_+). \tag{19}$$

If $0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx < \infty$, *and* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q < \infty$, *then* (i) *for* $p > 1$, *we have the following equivalent inequalities:*

$$I = \sum_{n=1}^\infty \int_0^\infty k_\lambda(x, n) a_n f(x) dx$$

$$< k(\lambda_1) \left\{ \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{20}$$

$$J_1 = \left\{ \sum_{n=1}^{\infty} n^{p\lambda_2 - 1} \left(\int_0^{\infty} k_\lambda(x,n) f(x) dx \right)^p \right\}^{\frac{1}{p}}$$

$$< k(\lambda_1) \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{21}$$

$$J_2 := \left\{ \int_0^{\infty} x^{q\lambda_1 - 1} \left(\sum_{n=1}^{\infty} k_\lambda(x,n) a_n \right)^q dx \right\}^{\frac{1}{q}}$$

$$< k(\lambda_1) \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}; \tag{22}$$

(ii) for $p < 0 (0 < q < 1)$, we have the equivalent reverses of (20), (21) and (22);
(iii) for $0 < p < 1 (q < 0)$, we have the following equivalent inequalities:

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} k_\lambda(x,n) a_n f(x) dx$$

$$> k(\lambda_1) \left\{ \int_0^{\infty} (1 - \theta_{\lambda_1}(x)) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{23}$$

$$J_1 = \left\{ \sum_{n=1}^{\infty} n^{p\lambda_2 - 1} \left(\int_0^{\infty} k_\lambda(x,n) f(x) dx \right)^p \right\}^{\frac{1}{p}}$$

$$> k(\lambda_1) \left\{ \int_0^{\infty} (1 - \theta_{\lambda_1}(x)) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{24}$$

$$\widehat{J_2} := \left\{ \int_0^{\infty} \frac{x^{q\lambda_1 - 1}}{(1 - \theta_{\lambda_1}(x))^{q-1}} \left(\sum_{n=1}^{\infty} k_\lambda(x,n) a_n \right)^q dx \right\}^{\frac{1}{q}}$$

$$> k(\lambda_1) \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{p}}. \tag{25}$$

Lemma 3 *Suppose that $h(t)$ is a non-negative measurable function in $\mathbf{R}_+, a \in \mathbf{R}$, and there exists a constant $\delta_0 > 0$, such that for any $\delta \in [0, \delta_0)$,*

$$k(a \pm \delta) := \int_0^{\infty} h(t) t^{(a \pm \delta)-1} dt \in \mathbf{R}.$$

Then we have

$$k(a \pm \delta) = k(a) + o(1)(\delta \to 0^+). \tag{26}$$

Proof For any $\delta \in \left[0, \frac{\delta_0}{2}\right)$, it follows

$$h(t)t^{(a\pm\delta)-1} \leq g(t) := \begin{cases} h(t)t^{\left(a-\frac{\delta_0}{2}\right)-1}, t \in (0,1], \\ h(t)t^{\left(a+\frac{\delta_0}{2}\right)-1}, t \in (1,\infty). \end{cases}$$

Since we find

$$\begin{aligned} 0 &\leq \int_0^\infty g(t)dt = \int_0^1 h(t)t^{\left(a-\frac{\delta_0}{2}\right)-1}dt + \int_1^\infty h(t)t^{\left(a+\frac{\delta_0}{2}\right)-1}dt \\ &\leq \int_0^\infty h(t)t^{\left(a-\frac{\delta_0}{2}\right)-1}dt + \int_0^\infty h(t)t^{\left(a+\frac{\delta_0}{2}\right)-1}dt \\ &= k\left(a - \frac{\delta_0}{2}\right) + k\left(\alpha + \frac{\delta_0}{2}\right) \in \mathbf{R}, \end{aligned}$$

then by Lebesgue control convergence theorem (cf. [43]), it follows

$$\begin{aligned} k(a \pm \delta) &= \int_0^\infty h(t)t^{(a\pm\delta)-1}dt \\ &= \int_0^\infty h(t)t^{a-1}dt + o(1)(\delta \to 0^+), \end{aligned}$$

namely, (26) follows. The lemma is proved

Theorem 2 *If there exists a constant* $\delta_0 > 0$, *such that for any* $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1,2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, $k(\widetilde{\lambda}_1) = \int_0^\infty k_\lambda(u,1)u^{\widetilde{\lambda}_1-1}du \in \mathbf{R}_+, \theta_{\widetilde{\lambda}_1}(x) \in (0,1)$ *and*

$$k\left(\widetilde{\lambda}_1\right)(1 - \theta_{\widetilde{\lambda}_1}(x)) < \varpi_\lambda\left(\widetilde{\lambda}_1, x\right) < k\left(\widetilde{\lambda}_1\right)(x \in \mathbf{R}_+), \tag{27}$$

where, $\theta_{\widetilde{\lambda}_1}(x) = O\left(\frac{1}{x^{\delta(\widetilde{\lambda}_1)}}\right)(x \in [1,\infty); \delta\left(\widetilde{\lambda}_1\right) > 0)$, *then the constant factor* $k(\lambda_1)$ *in Theorem 1 is the best possible.*

Proof (i) For $p > 1$, by Hölder's inequality, we can obtain

$$I \leq J_1 \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{28}$$

$$I \leq \left\{ \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx \right\}^{\frac{1}{p}} J_2. \tag{29}$$

For $0 < \varepsilon < q\delta_0$, we set $\widetilde{f}(x), \widetilde{a}_n$ as follows:

$$\widetilde{f}(x) := \begin{cases} 0, 0 < x < 1, \\ x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, x \geq 1, \end{cases}$$

$$\widetilde{a}_n := n^{\left(\lambda_2 - \frac{\varepsilon}{q}\right) - 1}, n \in \mathbf{N}.$$

Then for $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}\left(\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}\right)$, by (27), we find

$$\begin{aligned}
&\left\{\int_0^\infty x^{p(1-\lambda_1)-1}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}\widetilde{a}_n^q\right\}^{\frac{1}{q}}\\
&=\left\{\int_1^\infty x^{-1-\varepsilon}dx\right\}^{\frac{1}{p}}\left\{1+\sum_{n=2}^\infty n^{-1-\varepsilon}\right\}^{\frac{1}{q}}\\
&<\left\{\frac{1}{\varepsilon}\right\}^{\frac{1}{p}}\left\{1+\int_1^\infty y^{-1-\varepsilon}dy\right\}^{\frac{1}{q}}=\frac{1}{\varepsilon}\{\varepsilon+1\}^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &:= \int_0^\infty \sum_{n=1}^\infty k_\lambda(x,n)\widetilde{a}_n\widetilde{f}(x)dx = \int_1^\infty x^{-1-\varepsilon}\varpi_\lambda\left(\widetilde{\lambda}_1, x\right)dx\\
&\geq k\left(\widetilde{\lambda}_1\right)\int_1^\infty x^{-1-\varepsilon}\left(1-O\left(\frac{1}{x^{\delta(\widetilde{\lambda}_1)}}\right)\right)dx\\
&=\frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left[1-\varepsilon O_{\widetilde{\lambda}_1}(1)\right].
\end{aligned}$$

If there exists a constant $k \leq k(\lambda_1)$, such that (20) is valid when replacing $k(\lambda_1)$ by k, then in particular, we have

$$\begin{aligned}
k\left(\widetilde{\lambda}_1\right)\left[1-\varepsilon O_{\widetilde{\lambda}_1}(1)\right] &\leq \varepsilon\widetilde{I} < \varepsilon k\left\{\int_0^\infty x^{p(1-\lambda_1)-1}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}}\\
&\times\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}\widetilde{a}_n^q\right\}^{\frac{1}{q}} < k\{\varepsilon+1\}^{\frac{1}{q}},
\end{aligned}$$

and then by (26), we find $k(\lambda_1) \leq k(\varepsilon \to 0^+)$. Hence $k = k(\lambda_1)$ is the best possible constant factor of (20).

By the equivalency, we can prove that the constant factor $k(\lambda_1)$ in (21) (22) is the best possible. Otherwise, we would reach a contradiction by (28) (29) that the constant factor $k(\lambda_1)$ in (20) is not the best possible.

(ii) For $p < 0$, by the reverse Hölder's inequality, we can obtain the reverses of (28) and (29). For $0 < \varepsilon < q\delta_0$, we set $\widetilde{f}(x), \widetilde{a}_n$ as (i). Then for $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}\left(\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}\right)$, by (27), we find

$$\left\{\int_0^\infty x^{p(1-\lambda_1)-1}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}\widetilde{a}_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{ \int_1^\infty x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{-1-\varepsilon} \right\}^{\frac{1}{q}}$$

$$> \left\{ \int_1^\infty x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty y^{-1-\varepsilon} dy \right\}^{\frac{1}{q}} = \frac{1}{\varepsilon},$$

$$\widetilde{I} = \int_0^\infty \sum_{n=1}^\infty k_\lambda(x,n) \widetilde{a}_n \widetilde{f}(x) dx = \int_1^\infty x^{-1-\varepsilon} \varpi_\lambda \left(\widetilde{\lambda}_1, x \right) dx$$

$$< k(\widetilde{\lambda}_1) \int_1^\infty x^{-1-\varepsilon} dx = \frac{1}{\varepsilon} k(\widetilde{\lambda}_1).$$

If there exists a constant $K \geq k(\lambda_1)$, such that the reverse of (20) is valid when replacing $k(\lambda_1)$ by K, then in particular, we have

$$k(\widetilde{\lambda}_1) > \varepsilon \widetilde{I}$$

$$> \varepsilon K \left\{ \int_0^\infty x^{p(1-\lambda_1)-1} \widetilde{f}^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} \widetilde{a}_n^q \right\}^{\frac{1}{q}} > K,$$

and then by (26), $k(\lambda_1) \geq K(\varepsilon \to 0^+)$. Hence $K = k(\lambda_1)$ is the best possible constant factor of the reverse of (20).

By the equivalency, we can prove that the constant factor $k(\lambda_1)$ in the reverses of (21) and (22) is the best possible. Otherwise, we would reach a contradiction by the reverses of (28) and (29) that the constant factor $k(\lambda_1)$ in the reverse of (20) is not the best possible.

(iii) For $0 < p < 1$, by the reverse Hölder's inequality, we can obtain

$$I \geq J_1 \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{30}$$

$$I \geq \left\{ \int_0^\infty (1 - \theta_{\lambda_1}(x)) x^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}} \widehat{J}_2. \tag{31}$$

For $0 < \varepsilon < |q|\delta_0$, we set $\widetilde{f}(x), \widetilde{a}_n$ as (i). Then for $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q} \left(\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q} \right)$, by (27), we find

$$\left\{ \int_0^\infty (1 - \theta_{\lambda_1}(x)) x^{p(1-\lambda_1)-1} \widetilde{f}^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} \widetilde{a}_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_1^\infty \left(1 - O\left(\frac{1}{x^{\delta(\lambda_1)}} \right) \right) x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ 1 + \sum_{n=2}^\infty n^{-1-\varepsilon} \right\}^{\frac{1}{q}}$$

$$> \left\{ \int_1^\infty \left(1 - O\left(\frac{1}{x^{\delta(\lambda_1)}} \right) \right) x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ 1 + \int_1^\infty y^{-1-\varepsilon} dy \right\}^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon} \{1 - \varepsilon O_{\lambda_1}(1)\}^{\frac{1}{p}} \{\varepsilon + 1\}^{\frac{1}{q}},$$

$$\widetilde{I} = \int_0^\infty \sum_{n=1}^\infty k_\lambda(x,n) \widetilde{a}_n \widetilde{f}(x) dx$$

$$= \int_1^\infty x^{-1-\varepsilon} \varpi_\lambda(\widetilde{\lambda}_1, x) dx < \frac{1}{\varepsilon} k(\widetilde{\lambda}_1).$$

If there exists a constant $K \geq k(\lambda_1)$, such that the (23) is valid when replacing $k(\lambda_1)$ by K, then in particular, we have

$$k(\widetilde{\lambda}_1) > \varepsilon \widetilde{I} > \varepsilon K \left\{ \int_0^\infty (1 - \theta_{\lambda_1}(x)) x^{p(1-\lambda_1)-1} \widetilde{f}^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} \widetilde{a}_n^q \right\}^{\frac{1}{q}} > K \{1 - \varepsilon O_{\lambda_1}(1)\}^{\frac{1}{p}} \{\varepsilon + 1\}^{\frac{1}{q}},$$

and then by (26), $k(\lambda_1) \geq K(\varepsilon \to 0^+)$. Hence $K = k(\lambda_1)$ is the best possible constant factor of (23).

By the equivalency, we can prove that the constant factor $k(\lambda_1)$ in (24) (25) is the best possible. Otherwise, we would reach a contradiction by (30) (31) that the constant factor $k(\lambda_1)$ in (23) is not the best possible. The theorem is proved.

Lemma 4 *Suppose that $h(t)(>0)$ is strictly decreasing with respect to $t \in \mathbf{R}_+$. If $\int_0^\infty h(t)dt < \infty$, then we have*

$$\int_1^\infty h(t)dt < \sum_{n=1}^\infty h(n) < \int_0^\infty h(t)dt. \tag{32}$$

Proof Since $h(t)$ is a strict decreasing function, we have

$$h(t) < h(n) < h(t-1)(t \in (n, n+1); n \in \mathbf{N}),$$

$$\int_n^{n+1} h(t)dt < \int_n^{n+1} h(n)dt = h(n) < \int_n^{n+1} h(t-1)dt,$$

and then

$$\int_1^\infty h(t)dt = \sum_{n=1}^\infty \int_n^{n+1} h(t)dt < \sum_{n=1}^\infty h(n)$$

$$< \sum_{n=1}^\infty \int_n^{n+1} h(t-1)dt = \int_1^\infty h(t-1)dt = \int_0^\infty h(u)du.$$

Hence (32) follows. The lemma is proved.

Corollary 1 *If there exists a constant* $\delta_0 > 0$, *such that for any* $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1, 2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, $k(\widetilde{\lambda}_1) \in \mathbf{R}_+$, $k_\lambda(x, y)y^{\widetilde{\lambda}_2 - 1}$ *is strictly decreasing with respect to* $y \in \mathbf{R}_+$, *and there exist constants* $L > 0$ *and* $\eta_1 > \widetilde{\lambda}_1$, *satisfying*

$$k_\lambda(u, 1) \le \frac{L}{u^{\eta_1}}(u \in [1, \infty)),$$

then the constant factor $k(\lambda_1)$ *in Theorem 1 is the best possible.*

Proof In view of (32), we find

$$\begin{aligned}\varpi_\lambda\left(\widetilde{\lambda}_1, x\right) &= x^{\widetilde{\lambda}_1} \sum_{n=1}^{\infty} k_\lambda(x, n) \frac{1}{n^{1-\widetilde{\lambda}_2}} \\ &< x^{\widetilde{\lambda}_1} \int_0^\infty k_\lambda(x, y) \frac{1}{y^{1-\widetilde{\lambda}_2}} dy \\ &= \int_0^\infty k_\lambda(u, 1) \frac{1}{u^{1-\widetilde{\lambda}_1}} du = k(\widetilde{\lambda}_1),\end{aligned}$$

$$\begin{aligned}\varpi_\lambda(\widetilde{\lambda}_1, x) &> x^{\widetilde{\lambda}_1} \int_1^\infty k_\lambda(x, y) \frac{1}{y^{1-\widetilde{\lambda}_2}} dy \\ &= \int_0^x k_\lambda(u, 1) \frac{1}{u^{1-\widetilde{\lambda}_1}} du \\ &= k(\widetilde{\lambda}_1)\left[(1 - \theta_{\widetilde{\lambda}_1}(x))\right](x \in \mathbf{R}_+),\end{aligned}$$

where,

$$\theta_{\widetilde{\lambda}_1}(x) := \frac{1}{k(\widetilde{\lambda}_1)} \int_x^\infty k_\lambda(u, 1) \frac{1}{u^{1-\widetilde{\lambda}_1}} du \in (0, 1).$$

For $x \in [1, \infty)$, we find

$$\begin{aligned}0 < \theta_{\widetilde{\lambda}_1}(x) &\le \frac{1}{k(\widetilde{\lambda}_1)} \int_x^\infty \frac{L}{u^{\eta_1}} \frac{1}{u^{1-\widetilde{\lambda}_1}} du \\ &= \frac{L}{(\eta_1 - \widetilde{\lambda}_1)k(\widetilde{\lambda}_1)} \frac{1}{x^{\delta(\widetilde{\lambda}_1)}} \left(\delta(\widetilde{\lambda}_1) = \eta_1 - \widetilde{\lambda}_1\right),\end{aligned}$$

namely, $\theta_{\widetilde{\lambda}_1}(x) = O\left(\frac{1}{x^{\delta(\widetilde{\lambda}_1)}}\right)(x \in [1, \infty); \delta(\widetilde{\lambda}_1) > 0)$. Then we have (27). Therefore, the constant factor $k(\lambda_1)$ in Theorem 1 is the best possible. The corollary is proved.

2.2 Operator Expressions and Some Particular Examples

For $p > 1$, we set $\varphi(x) = x^{p(1-\lambda_1)-1}(x \in \mathbf{R}_+)$ and $\psi(n) = n^{q(1-\lambda_2)-1}(n \in \mathbf{N})$, wherefrom

$$[\psi(n)]^{1-p} = n^{p\lambda_2-1}, [\varphi(x)]^{1-q} = x^{q\lambda_1-1}.$$

We define two real weight normal spaces $L_{p,\varphi}(\mathbf{R}_+)$ and $l_{q,\psi}$ as follows:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{ f; ||f||_{p,\varphi} = \left\{ \int_0^\infty \varphi(x)|f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ a = \{a_n\}; ||a||_{q,\psi} = \left\{ \sum_{n=1}^\infty \psi(n)|a_n|^q \right\}^{\frac{1}{q}} < \infty \right\}.$$

As the assumptions of Theorem 1, in view of

$$J_1 < k(\lambda_1)||f||_{p,\varphi}, J_2 < k(\lambda_1)||a||_{q,\psi},$$

we may give the following definition:

Definition 2 Define a first kind of half-discrete Hilbert-type operator $T_1 : L_{p,\varphi}(\mathbf{R}_+) \to l_{p,\psi^{1-p}}$ as follows: For $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1 f \in l_{p,\psi^{1-p}}$, satisfying

$$(T_1 f)(n) := \int_0^\infty k_\lambda(x,n) f(x) dx (n \in \mathbf{N}). \tag{33}$$

For $a \in l_{q,\psi}$, we define the following formal inner product of $T_1 f$ and a as follows:

$$(T_1 f, a) := \sum_{n=1}^\infty a_n \int_0^\infty k_\lambda(x,n) f(x) dx.$$

Define a second kind of half-discrete Hilbert-type operator $T_2 : l_{q,\psi} \to L_{q,\varphi^{1-q}}(\mathbf{R}_+)$ as follows: For $a \in l_{q,\psi}$, there exists a unique representation $T_2 a \in L_{q,\varphi^{1-q}}(\mathbf{R}_+)$, satisfying

$$(T_2 a)(x) := \sum_{n=1}^\infty k_\lambda(x,n) a_n (x \in \mathbf{R}_+). \tag{34}$$

For $f \in L_{p,\varphi}(\mathbf{R}_+)$, we define the following formal inner product of f and $T_2 a$ as follows:

$$(f, T_2 a) := \int_0^\infty k_\lambda(x,n) a_n f(x) dx.$$

Then by Theorem 1, for $0 < ||f||_{p,\varphi}, ||a||_{q,\psi} < \infty$, we have the following equivalent inequalities:

$$(T_1 f, a) = (T_2 a, f) < k(\lambda_1)||f||_{p,\varphi}||a||_{q,\psi}, \tag{35}$$

$$||T_1 f||_{p,\psi^{1-p}} < k(\lambda_1)||f||_{p,\varphi}, \tag{36}$$

$$||T_2 a||_{q,\varphi^{1-q}} < k(\lambda_1)||a||_{q,\psi}. \tag{37}$$

It follows that T_1 and T_2 are bounded with

$$||T_1|| := \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_1 f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq k(\lambda_1),$$

$$||T_2|| := \sup_{a(\neq\theta)\in l_{q,\psi}} \frac{||T_2 a||_{q,\varphi^{1-q}}}{||a||_{q,\psi}} \leq k(\lambda_1).$$

Since in Theorem 2 or Corollary 1, the constant factor $k(\lambda_1)$ in (36) and (37) is the best possible, we have

$$||T_1|| = ||T_2|| = k(\lambda_1) = \int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du. \tag{38}$$

Note. If we define

$$(T_1 f)(n) := n^{\lambda-1}\int_0^\infty k_\lambda(x,n)f(x)dx(n\in\mathbf{N}),$$

then we have $||T_1 f||_{p,\varphi} < k(\lambda_1)||f||_{p,\varphi}$, and then $T_1 f \in l_{p,\varphi}$; if we define

$$(T_2 a)(x) := x^{\lambda-1}\sum_{n=1}^\infty k_\lambda(x,n)a_n(x\in\mathbf{R}_+),$$

then we have $||T_2 a||_{q,\psi} < k(\lambda_1)||a||_{q,\psi}$ and $T_2 a \in L_{q,\psi}(\mathbf{R}_+)$.

Example 1 (i) We set

$$k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}(\lambda,\lambda_1 > 0, 0 < \lambda_2 < 1).$$

For $\delta_0 = \frac{1}{2}\min\{\lambda_1,\lambda_2,1-\lambda_2\} > 0$, and $\widetilde{\lambda}_i \in (\lambda_i-\delta_0,\lambda_i+\delta_0)(i=1,2), \widetilde{\lambda}_1+\widetilde{\lambda}_2=\lambda$, it follows

$$k(\widetilde{\lambda}_1) = \int_0^\infty \frac{1}{(t+1)^\lambda}t^{\widetilde{\lambda}_1-1}dt = B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\in\mathbf{R}_+,$$

and

$$\frac{\partial}{\partial y}\left(\frac{1}{(x+y)^\lambda}y^{\widetilde{\lambda}_2-1}\right) < 0.$$

Setting $\eta_1 \in (\lambda_1+\delta_0,\lambda)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since $\frac{u^{\eta_1}}{(u+1)^\lambda} \to 0 (u\to\infty)$, there exists a constant $L > 0$, such that

$$k_\lambda(u,1) = \frac{1}{(u+1)^\lambda} \leq \frac{L}{u^{\eta_1}}(u\in[1,\infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = B(\lambda_1, \lambda_2).$$

(ii) We set

$$k_\lambda(x, y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}(\lambda, \lambda_1 > 0, 0 < \lambda_2 < 1).$$

For $\delta_0 = \frac{1}{2}\min\{\lambda_1, \lambda_2, 1 - \lambda_2\} > 0$ and $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1, 2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, it follows

$$\begin{aligned} k(\widetilde{\lambda}_1) &= \int_0^\infty \frac{\ln t}{t^\lambda - 1} t^{\widetilde{\lambda}_1 - 1} dt \\ &= \frac{1}{\lambda^2} \int_0^\infty \frac{\ln v}{v - 1} v^{\frac{\widetilde{\lambda}_1}{\lambda} - 1} dv \\ &= \left[\frac{\pi}{\lambda \sin\pi(\widetilde{\lambda}_1/\lambda)}\right]^2 \in \mathbf{R}_+, \end{aligned}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\ln(x/y)}{x^\lambda - y^\lambda} y^{\widetilde{\lambda}_2 - 1}\right) < 0.$$

Setting $\eta_1 \in (\lambda_1 + \delta_0, \lambda)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since $\frac{(\ln u)u^{\eta_1}}{u^\lambda - 1} \to 0 (u \to \infty)$, there exists a constant $L > 0$, such that

$$k_\lambda(u, 1) = \frac{\ln u}{u^\lambda - 1} \le \frac{L}{u^{\eta_1}} (u \in [1, \infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = \left[\frac{\pi}{\lambda \sin\pi\left(\frac{\lambda_1}{\lambda}\right)}\right]^2. \tag{39}$$

Lemma 5 *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, $z_k \in \mathbf{C}\backslash\{z|Rez \ge 0, Imz = 0\}$ $(k = 1, 2, \cdots, n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* $z_i (i = 1, 2, \cdots, n)$, *and* $z = \infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, then for* $\alpha \in \mathbf{R}$, *we have*

$$\int_0^\infty f(x)x^{\alpha - 1} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{k=1}^n Res\,[f(z)z^{\alpha - 1}, z_k], \tag{40}$$

where, $0 < Im \ln z = \arg z < 2\pi$. *In particular, if* $z_k (k = 1, \cdots, n)$ *are all poles of order 1, setting* $\varphi_k(z) = (z - z_k) f(z) (\varphi_k(z_k) \ne 0)$, *then*

$$\int_0^\infty f(x)x^{\alpha - 1} dx = \frac{\pi}{\sin\pi\alpha} \sum_{k=1}^n (-z_k)^{\alpha - 1} \varphi_k(z_k). \tag{41}$$

Proof By [44] (P.118), we have (40). We find

$$\begin{aligned}1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i\ \sin 2\pi\alpha \\ &= -2i\ \sin\pi\alpha(\cos\pi\alpha + i\ \sin\pi\alpha) = -2ie^{i\pi\alpha}\ \sin\pi\alpha.\end{aligned}$$

In particular, since $f(z)z^{\alpha-1} = \frac{1}{z-z_k}\left(\varphi_k(z)z^{\alpha-1}\right)$, it is evident that

$$Res\left[f(z)z^{\alpha-1}, -a_k\right] = z_k{}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (40), we obtain (41). The lemma is proved.

Example 2 (i) For $s \in \mathbf{N}$, we set

$$\begin{aligned}k_\lambda(x, y) = \frac{1}{\prod_{k=1}^{s}(x^{\lambda/s} + a_k y^{\lambda/s})}&(0 < a_1 < \cdots < a_s, \\ &\lambda, \lambda_1 > 0, 0 < \lambda_2 < 1).\end{aligned}$$

For $\delta_0 = \frac{1}{2}\min\{\lambda_1, \lambda_2, 1 - \lambda_2\} > 0$ and $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1, 2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, by (41), it follows

$$\begin{aligned}k_s(\widetilde{\lambda}_1) &= \int_0^\infty \frac{1}{\prod_{k=1}^{s}(t^{\lambda/s} + a_k)} t^{\widetilde{\lambda}_1 - 1} dt \\ &= \frac{s}{\lambda}\int_0^\infty \frac{1}{\prod_{k=1}^{s}(u + a_k)} u^{\frac{s\widetilde{\lambda}_1}{\lambda} - 1} du \\ &= \frac{\pi s}{\lambda\ \sin\left(\frac{\pi s\widetilde{\lambda}_1}{\lambda}\right)} \sum_{k=1}^{s} a_k^{\frac{s\widetilde{\lambda}_1}{\lambda} - 1} \prod_{j=1(j\neq k)}^{s} \frac{1}{a_j - a_k} \in \mathbf{R}_+,\end{aligned}$$

and

$$\frac{\partial}{\partial y}\left(\frac{y^{\widetilde{\lambda}_2 - 1}}{\prod_{k=1}^{s}\left(x^{\lambda/s} + a_k y^{\lambda/s}\right)}\right) < 0.$$

Setting $\eta_1 \in (\lambda_1 + \delta_0, \lambda)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since

$$\frac{u^{\eta_1}}{\prod_{k=1}^{s}(u^{\lambda/s} + a_k)} \to 0(u \to \infty),$$

there exists a constant $L > 0$, such that

$$k_\lambda(u, 1) = \frac{1}{\prod_{k=1}^{s}(u^{\lambda/s} + a_k)} \leq \frac{L}{u^{\eta_1}}(u \in [1, \infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = \frac{\pi s}{\lambda\ \sin\left(\frac{\pi s\lambda_1}{\lambda}\right)} \sum_{k=1}^{s} a_k^{\frac{s\lambda_1}{\lambda} - 1} \prod_{j=1(j\neq k)}^{s} \frac{1}{a_j - a_k}. \tag{42}$$

In particular, for $s = a_1 = 1$, we have $k_\lambda(x,y) = \frac{1}{x^\lambda + y^\lambda}$ and $||T_1|| = ||T_2|| = \frac{\pi}{\lambda \sin\pi(\lambda_1/\lambda)}$.

(ii) We set

$$k_\lambda(x,y) = \frac{1}{x^\lambda + \sqrt{c}(xy)^{\lambda/2}\cos\gamma + \frac{c}{4}y^\lambda}$$
$$(0 < \gamma < \frac{\pi}{2}, \lambda, \lambda_1 > 0, 0 < \lambda_2 < 1).$$

For $\delta_0 = \frac{1}{2}\min\{\lambda_1, \lambda_2, 1-\lambda_2\} > 0$ and $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1,2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, by (41), it follows

$$k(\widetilde{\lambda}_1) = \int_0^\infty \frac{1}{t^\lambda + \sqrt{c}t^{\lambda/2}\cos\gamma + \frac{c}{4}} t^{\widetilde{\lambda}_1 - 1} dt$$
$$= \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\widetilde{\lambda}_1}{\lambda}} \frac{2\pi \sin\gamma\left(1 - \frac{2\widetilde{\lambda}_1}{\lambda}\right)}{\lambda \sin\gamma \sin\left(\frac{2\pi\widetilde{\lambda}_1}{\lambda}\right)} \in \mathbf{R}_+,$$

and

$$\frac{\partial}{\partial y}\left(\frac{y^{\widetilde{\lambda}_2 - 1}}{x^\lambda + \sqrt{c}(xy)^{\lambda/2}\cos\gamma + \frac{c}{4}y^\lambda}\right) < 0.$$

Setting $\eta_1 \in (\lambda_1 + \delta_0, \lambda)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since

$$\frac{u^{\eta_1}}{u^\lambda + \sqrt{c}u^{\lambda/2}\cos\gamma + \frac{c}{4}} \to 0(u \to \infty),$$

there exists a constant $L > 0$, such that

$$k_\lambda(u,1) = \frac{1}{u^\lambda + \sqrt{c}u^{\lambda/2}\cos\gamma + \frac{c}{4}} \le \frac{L}{u^{\eta_1}}(u \in [1,\infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\lambda_1}{\lambda}} \frac{2\pi \sin\gamma\left(1 - \frac{2\lambda_1}{\lambda}\right)}{\lambda \sin\gamma \sin\left(\frac{2\pi\lambda_1}{\lambda}\right)}. \tag{43}$$

Example 3 (i) We set

$$k_0(x,y) = \ln\left(\frac{bx^\gamma + y^\gamma}{ax^\gamma + y^\gamma}\right)(0 \le a < b, \gamma > 0,$$
$$0 < -\lambda_1 = \lambda_2 = \sigma < \gamma).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \gamma - \sigma\} > 0$, and $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1, 2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, it follows

$$\begin{aligned}
k(\widetilde{\lambda}_1) &= \int_0^\infty \ln\left(\frac{bt^\gamma + 1}{at^\gamma + 1}\right) t^{\widetilde{\lambda}_1 - 1} dt = \frac{1}{\widetilde{\lambda}_1} \int_0^\infty \ln\left(\frac{bt^\gamma + 1}{at^\gamma + 1}\right) dt^{\widetilde{\lambda}_1} \\
&= \frac{1}{\widetilde{\lambda}_1}\left[t^{\widetilde{\lambda}_1} \ln\left(\frac{bt^\gamma + 1}{at^\gamma + 1}\right)|_0^\infty\right. \\
&\quad \left. -\gamma \int_0^\infty \left(\frac{b}{bt^\gamma + 1} - \frac{a}{at^\gamma + 1}\right) t^{\widetilde{\lambda}_1 + \gamma - 1} dt\right] \\
&= \frac{1}{\widetilde{\lambda}_1}\left(b^{-\frac{\widetilde{\lambda}_1}{\gamma}} - a^{-\frac{\widetilde{\lambda}_1}{\gamma}}\right) \int_0^\infty \frac{1}{u+1} u^{\left(1 + \frac{\widetilde{\lambda}_1}{\gamma}\right) - 1} du \\
&= \frac{-1}{\widetilde{\lambda}_1}\left(b^{-\frac{\widetilde{\lambda}_1}{\gamma}} - a^{-\frac{\widetilde{\lambda}_1}{\gamma}}\right) \frac{\pi}{\sin\pi\left(\frac{-\widetilde{\lambda}_1}{\gamma}\right)} \in \mathbf{R}_+,
\end{aligned}$$

and

$$\frac{\partial}{\partial y}\left(\ln\left(\frac{bx^\gamma + y^\gamma}{ax^\gamma + y^\gamma}\right) y^{\widetilde{\lambda}_2 - 1}\right) < 0.$$

Setting $\eta_1 \in (-\sigma + \delta_0, 0)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since

$$u^{\eta_1} \ln\left(\frac{bu^\gamma + 1}{au^\gamma + 1}\right) \to 0 (u \to \infty),$$

there exists a constant $L > 0$, such that

$$k_0(u, 1) = \ln\left(\frac{bu^\gamma + 1}{au^\gamma + 1}\right) \le \frac{L}{u^{\eta_1}} (u \in [1, \infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin\pi(\frac{\sigma}{\gamma})}. \tag{44}$$

(ii) We set

$$k_0(x, y) = e^{-\rho(\frac{y}{x})^\gamma} (\rho > 0, 0 < -\lambda_1 = \lambda_2 = \sigma < \gamma).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \gamma - \sigma\} > 0$ and $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1, 2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, it follows

$$\begin{aligned}
k(\widetilde{\lambda}_1) &= \int_0^\infty e^{-\frac{\rho}{t^\gamma}} t^{\widetilde{\lambda}_1 - 1} dt = \frac{1}{\gamma} \rho^{\frac{\widetilde{\lambda}_1}{\gamma}} \int_0^\infty e^{-u} u^{-\frac{\widetilde{\lambda}_1}{\gamma} - 1} du \\
&= \frac{1}{\gamma} \rho^{\frac{\widetilde{\lambda}_1}{\gamma}} \Gamma\left(-\frac{\widetilde{\lambda}_1}{\gamma}\right) \in \mathbf{R}_+,
\end{aligned}$$

and $\frac{\partial}{\partial y}(e^{-\rho(\frac{y}{x})^{\gamma}} y^{\widetilde{\lambda}_2-1}) < 0$. Setting $\eta_1 \in (-\sigma + \delta_0, 0)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since $u^{\eta_1} e^{-\frac{\rho}{u^{\gamma}}} \to 0 (u \to \infty)$, there exists a constant $L > 0$, such that

$$k_0(u,1) = e^{-\frac{\rho}{u^{\gamma}}} \leq \frac{L}{u^{\eta_1}} (u \in [1,\infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = \frac{1}{\gamma \rho^{\frac{\sigma}{\gamma}}} \Gamma\left(\frac{\sigma}{\gamma}\right). \tag{45}$$

(iii) We set

$$k_0(x,y) = \arctan \rho \left(\frac{x}{y}\right)^{\gamma} (\rho > 0, 0 < -\lambda_1 = \lambda_2 = \sigma < \gamma).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \gamma - \sigma\} > 0$ and $\widetilde{\lambda}_i \in (\lambda_i - \delta_0, \lambda_i + \delta_0)(i = 1, 2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, it follows

$$\begin{aligned}
k(\widetilde{\lambda}_1) &= \int_0^{\infty} t^{\widetilde{\lambda}_1 - 1} (\arctan \rho t^{\gamma}) dt = \frac{1}{\widetilde{\lambda}_1} \int_0^{\infty} (\arctan \rho t^{\gamma}) dt^{\widetilde{\lambda}_1} \\
&= \frac{1}{\widetilde{\lambda}_1} \left[(\arctan \rho t^{\gamma}) t^{\widetilde{\lambda}_1} |_0^{\infty} - \int_0^{\infty} \frac{\gamma \rho t^{\widetilde{\lambda}_1 + \gamma - 1}}{1 + (\rho t^{\gamma})^2} dt \right] \\
&= \frac{-\rho^{-\frac{\widetilde{\lambda}_1}{\gamma}}}{2\widetilde{\lambda}_1} \int_0^{\infty} \frac{1}{1+u} u^{\left(\frac{\widetilde{\lambda}_1}{2\gamma} + \frac{1}{2}\right) - 1} du \\
&= \frac{-\rho^{-\frac{\widetilde{\lambda}_1}{\gamma}} \pi}{2\widetilde{\lambda}_1 \sin \pi \left(\frac{\widetilde{\lambda}_1}{2\gamma} + \frac{1}{2}\right)} = \frac{-\rho^{-\frac{\widetilde{\lambda}_1}{\gamma}} \pi}{2\widetilde{\lambda}_1 \cos \pi \left(\frac{\widetilde{\lambda}_1}{2\gamma}\right)} \in \mathbf{R}_+,
\end{aligned}$$

and $\frac{\partial}{\partial y}(y^{\widetilde{\lambda}_2 - 1} \arctan \rho (\frac{x}{y})^{\gamma}) < 0$. Setting $\eta_1 \in (-\sigma + \delta_0, 0)$, then it follows $\eta_1 > \widetilde{\lambda}_1$. Since $u^{\eta_1} \arctan \rho u^{\gamma} \to 0 (u \to \infty)$, there exists a constant $L > 0$, such that

$$k_0(u,1) = \arctan \rho u^{\gamma} \leq \frac{L}{u^{\eta_1}} (u \in [1,\infty)).$$

Then by Corollary 1 and (38), we have

$$||T_1|| = ||T_2|| = \frac{\rho^{\frac{\sigma}{\gamma}} \pi}{2\sigma \cos \pi \left(\frac{\sigma}{2\gamma}\right)}. \tag{46}$$

Example 4 (1) We set

$$k_\lambda(x,y) = \frac{(\min\{x,y\})^\gamma}{(\max\{x,y\})^{\lambda+\gamma}} (\lambda_1 > -\gamma, -\gamma < \lambda_2 < 1-\gamma).$$

For $\delta_0 = \frac{1}{2}\min\{\lambda_1+\gamma, \lambda_2+\gamma, 1-\gamma-\lambda_2\} > 0$ and $\widetilde{\lambda}_i \in (\lambda_i-\delta_0, \lambda_i+\delta_0)(i=1,2)$, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, it follows

$$\begin{aligned} k(\widetilde{\lambda}_1) &= \int_0^\infty \frac{(\min\{t,1\})^\gamma}{(\max\{t,1\})^{\lambda+\gamma}} t^{\widetilde{\lambda}_1-1} dt \\ &= \int_0^1 t^{\widetilde{\lambda}_1+\gamma-1} dt + \int_1^\infty \frac{1}{t^{\lambda+\gamma}} t^{\widetilde{\lambda}_1-1} dt \\ &= \frac{\lambda+2\gamma}{(\widetilde{\lambda}_1+\gamma)(\widetilde{\lambda}_2+\gamma)} \in \mathbf{R}_+. \end{aligned}$$

We find that

$$\begin{aligned} k_\lambda(x,y) y^{\widetilde{\lambda}_2-1} &= \frac{(\min\{x,y\})^\gamma}{(\max\{x,y\})^{\lambda+\gamma}} y^{\widetilde{\lambda}_2-1} \\ &= \begin{cases} \frac{y^{\gamma+\widetilde{\lambda}_2-1}}{x^{\lambda+\gamma}}, 0 < y < x, \\ \frac{x^\gamma}{y^{\widetilde{\lambda}_1+\gamma+1}}, y \geq x \end{cases} \end{aligned}$$

is a strict decreasing function with respect to $y \in \mathbf{R}_+$. There exists a constant $\eta_1 \in (\lambda_1+\delta_0, \lambda+\gamma)$, such that $\eta_1 > \lambda_1+\delta_0 > \widetilde{\lambda}_1$ and $\lambda+\gamma-\eta_1 > 0$. Hence, in view of

$$u^{\eta_1} k_\lambda(u,1) = \frac{u^{\eta_1}(\min\{u,1\})^\gamma}{(\max\{u,1\})^{\lambda+\gamma}} = \begin{cases} u^{\gamma+\eta_1}, 0 < u < 1, \\ \frac{1}{u^{\lambda+\gamma-\eta_1}}, u \geq 1, \end{cases}$$

we have $u^{\eta_1} k_\lambda(u,1) \to 0 (u \to \infty)$, and then there exists a constant $L > 0$, satisfying

$$k_\lambda(u,1) \leq \frac{L}{u^{\eta_1}} (u \in [1,\infty)).$$

Therefore, by Corollary 1 and (38), it follows

$$||T_1|| = ||T_2|| = \frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)}. \tag{47}$$

2.3 Some Strengthened Versions of Half-Discrete Hilbert's Inequality

Definition 3 For $r > 1, \frac{1}{r}+\frac{1}{s} = 1$, we define the following weight functions:

$$\omega(s,n) := n^{\frac{1}{s}} \int_1^\infty \frac{1}{(x+n)x^{1/s}} dx (n \in \mathbf{N}), \tag{48}$$

$$\varpi(r,x) := x^{\frac{1}{r}} \sum_{n=1}^{\infty} \frac{1}{(x+n)n^{1/r}} (x \in [1,\infty)). \tag{49}$$

Setting $u = x/n$, we find

$$\begin{aligned}\omega(s,n) &= \int_{1/n}^{\infty} \frac{1}{(u+1)u^{1/s}} du \\ &= \int_0^{\infty} \frac{du}{(u+1)u^{1/s}} - \int_0^{1/n} \frac{du}{(u+1)u^{1/s}} \\ &> \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \int_0^{1/n} \frac{du}{u^{1/s}} = \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{r}{n^{1/r}}.\end{aligned} \tag{50}$$

We set the following decomposition:

$$\omega(s,n) = \int_{\frac{1}{n}}^{\infty} \frac{du}{(u+1)u^{1/s}} = \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\theta_r(n)}{n^{1/r}}, \tag{51}$$

where,

$$\theta_r(x) := x^{\frac{1}{r}} \int_0^{\frac{1}{x}} \frac{du}{(u+1)u^{1/s}} (x \geq 1).$$

Then we obtain

$$\frac{\partial}{\partial x}\theta_r(x) = \frac{1}{r} x^{\frac{-1}{s}} \int_0^{\frac{1}{x}} \frac{du}{(u+1)u^{1/s}} - \frac{1}{x+1}.$$

Setting $f(y)$ as follows

$$f(y) := \int_0^{y} \frac{du}{(u+1)u^{1/s}} - \frac{ry^{1/r}}{1+y} (0 \leq y \leq 1),$$

we find $f(0) = 0$ and

$$f'(y) = \frac{1}{(y+1)y^{1/s}} - \frac{y^{-1/s}}{1+y} + \frac{ry^{1/r}}{(1+y)^2} > 0.$$

Then it follows $f(y) > 0 (0 < y \leq 1)$ and

$$\frac{\partial}{\partial x}\theta_r(x) = \frac{1}{r} x^{\frac{-1}{s}} f\left(\frac{1}{x}\right) > 0 (x \geq 1). \tag{52}$$

Therefore,

$$\theta_r(x) \geq \inf_{x \geq 1} \theta_r(x) = \theta_r(1) = \int_0^1 \frac{u^{\frac{1}{r}-1}}{u+1} du. \tag{53}$$

Since we obtain

$$\left(\int_0^1 \frac{u^{\frac{1}{r}-1}du}{u+1}\right)'_r = -\frac{1}{r^2}\int_0^1 \frac{u^{\frac{1}{r}-1}\ln u du}{u+1} > 0,$$

then it follows

$$\theta_r(1) > \inf_{r>1}\theta_r(1) = \lim_{r\to 1^+}\theta_r(1)$$
$$= \int_0^1 \frac{du}{u+1} = \ln 2 = 0.6931^+. \tag{54}$$

By (50), (51), (53) and (54), we have

Lemma 6 *For* $n \in \mathbf{N}$,

$$\frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{r}{n^{1/r}} < \omega(s,n) < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\ln 2}{n^{1/r}}, \tag{55}$$

where the constant $\ln 2 = 0.6931^+$ *is the best possible.*

Lemma 7 *If* $(-1)^i F^{(i)}(t) > 0(t \in (0,\infty); i = 0,1,2,3)$, *then we have* (cf. [4, 45])

$$-\frac{1}{12}F(1) < \int_1^\infty P_1(t)F(t)dt < -\frac{1}{12}F\left(\frac{3}{2}\right), \tag{56}$$

where, $P_1(t) = t - [t] - \frac{1}{2}$ *is Bernoulli function of one-order.*

For $x \geq 1$, setting $f(t) := \frac{1}{(x+t)t^{1/r}}(t > 0)$, we find

$$f'(t) = \frac{-1}{(x+t)^2 t^{1/r}} - \frac{1}{r(x+t)t^{1+(1/r)}}$$
$$= -\frac{(r+1)t + y}{r(x+t)^2 t^{1+(1/r)}}.$$

By Euler–Maclaurin summation formula (cf. [4]), it follows

$$\varpi_\lambda(r,x) = x^{\frac{1}{r}}\sum_{n=1}^\infty \frac{1}{(x+n)n^{1/r}}$$
$$= x^{\frac{1}{r}}\left[\int_1^\infty f(t)dt + \frac{1}{2}f(1) + \int_1^\infty P_1(t)f'(t)dt\right]$$
$$= x^{\frac{1}{r}}\int_0^\infty f(t)dt - x^{\frac{1}{r}}\int_0^1 f(t)dt$$
$$+ \frac{1}{2}x^{\frac{1}{r}}f(1) + x^{\frac{1}{r}}\int_1^\infty P_1(t)f'(t)dt$$
$$= \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1}{x^{1/s}}\left[x^{\frac{1}{s}}\int_0^{\frac{1}{x}} \frac{u^{-\frac{1}{r}}du}{1+u} - \frac{x}{2(x+1)}\right.$$

$$+ x \int_1^\infty P_1(t) \frac{(r+1)t + x}{r(x+t)^2 t^{1+(1/r)}} dt \Bigg]. \tag{57}$$

Setting $G(t,x) := \frac{(r+1)tx + x^2}{r(x+t)^2 t^{1+(1/r)}}$,

$$A(x) := x^{1-\frac{1}{r}} \int_0^{\frac{1}{x}} \frac{u^{-\frac{1}{r}}}{1+u} du,$$

$B(x) := \int_1^\infty P_1(t) G(t,x) dt$ and

$$\theta(r,x) := A(x) + B(x) - \frac{x}{2(x+1)} (x \in [1,\infty)),$$

then by (57), we have the following decomposition:

$$\varpi(r,x) = \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\theta(r,x)}{x^{1/s}} (x \geq 1). \tag{58}$$

Lemma 8 *For $r > 1$, we have*

$$\min_{x \geq 1} \theta(r,x) = \theta(r,1) = \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \varpi(r,1). \tag{59}$$

Proof By Lemma 1 of [46], we have

$$\int_0^{\frac{1}{x}} \frac{u^{-\frac{1}{r}} du}{1+u} \geq \frac{r(2r-1)x^{\frac{1}{r}}}{(r-1)[(2r-1)x + r - 1]} (x \geq 1).$$

Then we find

$$\begin{aligned} A'(x) &= \left(1 - \frac{1}{r}\right) x^{\frac{-1}{r}} \int_0^{\frac{1}{x}} \frac{u^{-\frac{1}{r}} du}{1+u} - \frac{1}{x+1} \\ &\geq \frac{\left(1 - \frac{1}{r}\right) r(2r-1)}{(r-1)[(2r-1)x + r - 1]} - \frac{1}{x+1} \\ &= \frac{(2r-1)}{(2r-1)x + r - 1} - \frac{1}{x+1} \\ &= \frac{r}{(x+1)[(2r-1)x + r - 1]}. \end{aligned}$$

Setting $F_1(t) = \frac{1}{(x+t)^2 t^{1/r}}$ and $F_2(t) = \frac{1}{(x+t)^3 t^{1/r}}$, then by Lemma 7, it follows

$$\begin{aligned} B'(x) &= \int_1^\infty P_1(t) G'_x(t,x) dt \\ &= \frac{r+1}{r} \int_1^\infty P_1(t) F_1(t) dt - 2x \int_1^\infty P_1(t) F_2(t) dt \end{aligned}$$

$$> \frac{r+1}{r}\left(-\frac{1}{12}F_1(1)\right) + \frac{2x}{12}F_2\left(\frac{3}{2}\right)$$
$$= -\frac{r+1}{12r(x+1)^2} + \frac{4x}{3(2x+3)^3}\left(\frac{2}{3}\right)^{\frac{1}{r}}.$$

Then we have

$$\theta_x'(r,x) = A'(x) + B'(x) - \frac{1}{2(x+1)^2}$$

$$= \frac{r}{(x+1)[(2r-1)x+r-1]} - \frac{r+1}{12r(x+1)^2}$$
$$+ \frac{4x}{3(2x+3)^3}\left(\frac{2}{3}\right)^{\frac{1}{r}} - \frac{1}{2(x+1)^2}$$
$$= \frac{(-2r^2+5r+1)x + (5r^2+6r+1)}{12r(x+1)^2[(2r-1)x+r-1]}$$
$$+ \frac{4x}{3(2x+3)^3}\left(\frac{2}{3}\right)^{\frac{1}{r}}.$$

(1) If $1 < r < \frac{5}{2}, -2r^2+5r+1 > 0$, then we have $\theta_y'(r,x) > 0$;
(2) If $r \geq \frac{5}{2}, \left(\frac{2}{3}\right)^{\frac{1}{r}} > \frac{4}{5}$, then we obtain

$$\theta_x'(r,x) > \frac{\left(-2r^2+5r+1\right)x + \left(5r^2+6r+1\right)}{12r(x+1)^2[(2r-1)x+r-1]} + \frac{16x}{15(2x+3)^3}$$
$$= \frac{5[(-2r^2+5r+1)x + (5r^2+6r+1)](2x+3)^3}{60r(x+1)^2(2x+3)^3[(2r-1)x+r-1]}$$
$$+ \frac{64rx(x+1)^2[(2r-1)x+r-1]}{60r(x+1)^2(2x+3)^3[(2r-1)x+r-1]}$$
$$> \frac{(48r^2-44r+40)x^4 + (160r^2+1076r+92)x^3}{60r(x+1)^2(2x+3)^3[(2r-1)x+r-1]}$$
$$> 0 (x \geq 1).$$

Hence, $\theta(r,x)$ is strictly increasing with respect to $x \in [1,\infty)$, and then we have (59). The lemma is proved.

Lemma 9 *If $k \in \mathbf{N}, k \geq 5$, then the function*

$$I(r,k) := \int_0^k \frac{u^{-\frac{1}{r}}du}{1+u} - \frac{k^{-\frac{1}{r}}}{2(1+k)} - \sum_{m=1}^{k-1} \frac{m^{-\frac{1}{r}}}{1+m}$$

is strictly decreasing with respect to $r \in (1,\infty)$.

Proof For $k \geq 5$, we find

$$I_r'(r,k) = \frac{1}{r^2}\left\{-\frac{k^{-\frac{1}{r}}\ln k}{2(1+k)}\right.$$
$$+\left[\int_0^4 \frac{u^{-\frac{1}{r}}\ln u}{1+u}du - \frac{\ln 2}{3\cdot 2^{\frac{1}{r}}} - \frac{\ln 3}{4\cdot 3^{\frac{1}{r}}}\right]$$
$$\left.-\left[\sum_{m=4}^{k-1}\frac{m^{-\frac{1}{r}}\ln m}{1+m} - \int_4^k \frac{u^{-\frac{1}{r}}\ln u}{1+u}du\right]\right\}.$$

It is obvious that for $u \geq 4$,

$$\frac{d}{du}\left(\frac{u^{-\frac{1}{r}}\ln u}{1+u}\right) = \frac{u^{-\frac{1}{r}}}{1+u}\left(-\frac{\ln u}{ru} - \frac{\ln u}{1+u} + \frac{1}{u}\right)$$
$$< \frac{u^{-\frac{1}{r}}}{1+u}\left(\frac{1}{u} - \frac{\ln u}{1+u}\right) < 0,$$

and then $\frac{u^{-\frac{1}{r}}\ln u}{1+u}$ is decreasing with respect to $u \geq 4$. It follows that

$$\sum_{m=4}^{k-1}\frac{m^{-1/r}\ln m}{1+m} - \int_4^k \frac{u^{-\frac{1}{r}}\ln u}{1+u}du \geq 0.$$

Setting $u = e^{-y}$, we obtain

$$J(r) := \int_0^4 \frac{u^{-\frac{1}{r}}\ln u}{1+u}du = -\int_{-\ln 4}^{\infty}\frac{ye^{(-1+\frac{1}{r})y}}{1+e^{-y}}dy$$
$$< -\frac{1}{5}\int_{-\ln 4}^{\infty} ye^{(-1+\frac{1}{r})y}dy$$
$$= \frac{r4^{1-\frac{1}{r}}}{5(r-1)}\left(\ln 4 - \frac{r}{r-1}\right) = \frac{s4^{\frac{1}{s}}}{5}(\ln 4 - s).$$

If $1 < s = \frac{r}{r-1} < \ln 4$, namely, $r > \frac{\ln 4}{\ln 4 - 1} = 3.5887^+$, then we find

$$\frac{d}{ds}\left[\frac{s4^{\frac{1}{s}}}{5}(\ln 4 - s)\right]$$
$$= \frac{4^{\frac{1}{s}}}{5}(\ln 4 - s)\left(1 - \frac{\ln 4}{s}\right) - \frac{s4^{\frac{1}{s}}}{5} < 0,$$

and $\frac{s4^{\frac{1}{s}}}{5}(\ln 4 - s) < \frac{4}{5}(\ln 4 - 1)$. In this case,

$$\int_0^4 \frac{u^{-\frac{1}{r}}\ln u}{1+u}du - \frac{\ln 2}{3\cdot 2^{\frac{1}{r}}} - \frac{\ln 3}{4\cdot 3^{\frac{1}{r}}}$$

$$< \frac{4}{5}(\ln 4 - 1) - \frac{\ln 2}{3 \cdot 2^{1/3.5887}} - \frac{\ln 3}{4 \cdot 3^{1/3.5887}}$$
$$< -0.083996 < 0.$$

If $\ln 4 \leq \frac{r}{r-1}$, then it is obvious that $J(r) < 0$.

Therefore, we have $I'_r(r,k) < 0$ and then $I(r,k)$ is strictly decreasing with respect to $r \in (1, \infty)$. The lemma is proved.

Lemma 10 *If* $r > 1, \frac{1}{r} + \frac{1}{s} = 1$, *then for* $x \geq 1$, *we have the following inequalities:*

$$\frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{s}{x^{1/s}} < \varpi(r,x) < \frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{1-\gamma}{x^{1/s}}, \tag{60}$$

where, $1 - \gamma = 0.4227^{+}$ *is the best value (*γ *is Euler constant).*

Proof Similarly to (50), it follows

$$\varpi(r,x) > x^{\frac{1}{r}} \int_1^\infty \frac{dy}{(x+y)y^{1/r}} = \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{s}{x^{1/s}}.$$

For $k \in \mathbf{N}, k \geq 5$, we have

$$\begin{aligned}
\theta(r,1) &= \frac{\pi}{\sin(\frac{\pi}{r})} - \varpi(r,1) \\
&= \int_0^\infty \frac{u^{-\frac{1}{r}}}{1+u} du - \sum_{m=1}^\infty \frac{m^{-\frac{1}{r}}}{1+m} \\
&= \int_0^k \frac{u^{-\frac{1}{r}}}{1+u} du + \int_k^\infty \frac{u^{-\frac{1}{r}}}{1+u} du \\
&\quad - \sum_{m=1}^{k-1} \frac{m^{-\frac{1}{r}}}{1+m} - \sum_{m=k}^\infty \frac{m^{-\frac{1}{r}}}{1+m}.
\end{aligned}$$

Setting $g(t) := \frac{1}{(1+t)t^{1/r}}$, then by Euler–Maclaurin summation formula (cf. [4]), we have

$$\int_k^\infty \frac{u^{-\frac{1}{r}} du}{1+u} + \frac{u^{-\frac{1}{r}}}{2(1+k)} < \sum_{m=k}^\infty \frac{m^{-\frac{1}{r}}}{1+m}$$
$$< \int_k^\infty \frac{u^{-\frac{1}{r}} du}{1+u} + \frac{u^{-\frac{1}{r}}}{2(1+k)} - \frac{g'(k)}{12}.$$

It follows

$$I(r,k) + \frac{g'(k)}{12} < \theta(r,1) < I(r,k),$$

$$\inf_{r>1} I(r,k) + \frac{1}{12} \inf_{r>1} g'(k)$$

$$\leq \inf_{r>1} \theta(r,1) \leq \inf_{r>1} I(r,k)(k \geq 5).$$

Since for any $k \geq 5$,

$$0 \geq \inf_{r>1} g'(k) = -\sup_{r>1} \left[\frac{1}{(1+k)^2 k^{1/r}} + \frac{1}{r(1+k)k^{1+(1/r)}} \right]$$
$$\geq -\left[\frac{1}{(1+k)^2} + \frac{1}{(1+k)k} \right] \to 0(k \to \infty),$$

then it follows $\lim_{k\to\infty} \inf_{r>1} g'(k) = 0$. Hence by Lemma 4, we obtain

$$\inf_{r>1} \theta(r,1) = \lim_{k\to\infty} \inf_{r>1} I(r,k) = \lim_{k\to\infty} \lim_{r\to\infty} I(r,k)$$
$$= \lim_{k\to\infty} \left[\int_0^k \frac{du}{1+u} - \frac{1}{2(1+k)} - \sum_{m=1}^{k-1} \frac{1}{1+m} \right]$$
$$= 1 - \lim_{k\to\infty} \left[\sum_{m=1}^{k+1} \frac{1}{m} - \ln(1+k) - \frac{1}{2(k+1)} \right]$$
$$= 1 - \gamma.$$

By (58), in view of $\inf_{x\geq 1} \theta(r,x) = \theta(r,1)$, we have

$$\varpi(r,x) \leq \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\theta(r,1)}{x^{1/s}}$$
$$< \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\inf_{r>1} \theta(r,1)}{x^{1/s}}$$
$$= \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1-\gamma}{x^{1/s}} (x \geq 1).$$

It is obvious that the constant $1-\gamma$ in (60) is the best possible. The lemma is proved.

Lemma 11 *If* $p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1, a_n \geq 0 (n \in \mathbf{N}), f(x)$ *is a non-negative measurable function in* $[1, \infty)$, *then we have the following equivalent inequalities:*

$$I := \int_1^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{n+x} dx$$
$$\leq \left\{ \sum_{n=1}^\infty \omega(s,n) n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \int_1^\infty \varpi(r,x) x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}}, \tag{61}$$

$$J_1 := \left\{ \int_1^\infty \frac{x^{\frac{p}{r}-1}}{[\varpi(r,x)]^{p-1}} \left[\sum_{n=1}^\infty \frac{a_n}{x+n} \right]^p dx \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}, \tag{62}$$

$$L_1 := \left\{ \sum_{n=1}^{\infty} \frac{n^{\frac{q}{s}-1}}{[\omega(s,n)]^{q-1}} \left[\int_1^{\infty} \frac{f(x)}{x+n} dx \right]^q \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_1^{\infty} \varpi(r,x) x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}}; \tag{63}$$

Proof By Hölder's inequality (cf. [47]), it follows

$$\sum_{n=1}^{\infty} \frac{a_n}{x+n} = \sum_{n=1}^{\infty} \frac{1}{x+n} \left[\frac{n^{\frac{1}{rq}}}{x^{\frac{1}{sp}}} a_n \right] \left[\frac{x^{\frac{1}{sp}}}{n^{\frac{1}{rq}}} \right]$$

$$\leq \left\{ \sum_{n=1}^{\infty} \frac{1}{x+n} \frac{n^{\frac{p}{rq}}}{x^{\frac{1}{s}}} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{x+n} \frac{x^{\frac{q}{sp}}}{n^{\frac{1}{r}}} \right\}^{\frac{1}{q}}$$

$$= x^{\frac{1}{s}-\frac{1}{q}} \{\varpi(r,x)\}^{\frac{1}{q}} \left\{ \frac{1}{x^{1/s}} \sum_{n=1}^{\infty} \frac{n^{\frac{1}{s}}}{x+n} n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}. \tag{64}$$

Then by Lebesgue term, by term integration theorem (cf. [43]), we have

$$J_1 \leq \left\{ \int_1^{\infty} \frac{1}{x^{1/s}} \sum_{n=1}^{\infty} \frac{n^{\frac{1}{s}}}{x+n} n^{\frac{p}{r}-1} a_n^p dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \sum_{n=1}^{\infty} \left[n^{\frac{1}{s}} \int_1^{\infty} \frac{1}{x+n} \frac{1}{x^{1/s}} dx \right] n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}$$

$$= \left\{ \sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}, \tag{65}$$

and then (62) follows.

By Hölder's inequality (cf. [47]), we have

$$I = \int_1^{\infty} \frac{x^{\frac{1}{q}-\frac{1}{s}}}{(\varpi(r,x))^{\frac{1}{q}}} \left[\sum_{n=1}^{\infty} \frac{a_n}{n+x} \right] \left[(\varpi(r,x))^{\frac{1}{q}} x^{\frac{1}{s}-\frac{1}{q}} f(x) \right] dx$$

$$\leq J_1 \left\{ \int_1^{\infty} \varpi(r,x) x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}}. \tag{66}$$

Then by (62), we have (61).

On the other hand, assuming that (61) is valid, we set

$$f(x) := \frac{x^{\frac{p}{r}-1}}{[\varpi(r,x)]^{p-1}}\left[\sum_{n=1}^{\infty}\frac{a_n}{x+n}\right]^{p-1}, x \geq 1.$$

Then we find

$$J_1^p = \int_1^{\infty}\varpi(r,x)x^{\frac{q}{s}-1}f^q(x)dx.$$

If $J_1 = 0$, then (62) is trivially valid; if $J_1 = \infty$, then by (65), (62) is the form of equality ($= \infty$). Suppose that $0 < J_1 < \infty$. By (61), it follows

$$\int_1^{\infty}\varpi(r,x)x^{\frac{q}{s}-1}f^q(x)dx = J_1^p = I$$

$$\leq \left\{\sum_{n=1}^{\infty}\omega(s,n)n^{\frac{p}{r}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\int_1^{\infty}\varpi(r,x)x^{\frac{q}{s}-1}f^q(x)dx\right\}^{\frac{1}{q}},$$

$$J_1 = \left\{\int_1^{\infty}\varpi(r,x)x^{\frac{q}{s}-1}f^q(x)dx\right\}^{\frac{1}{p}} \leq \left\{\sum_{n=1}^{\infty}\omega(s,n)n^{\frac{p}{r}-1}a_n^p\right\}^{\frac{1}{p}}.$$

Hence we have (65), and then (61) and (62) are equivalent.

Still by Hölder's inequality, it follows

$$\int_1^{\infty}\frac{f(x)}{x+n}dx = \int_1^{\infty}\frac{1}{x+n}\left[\frac{n^{\frac{1}{rq}}}{x^{\frac{1}{sp}}}\right]\left[\frac{x^{\frac{1}{sp}}}{n^{\frac{1}{rq}}}f(x)\right]dx$$

$$\leq \left\{\int_1^{\infty}\frac{1}{x+n}\frac{n^{\frac{p}{rq}}}{x^{1/s}}dx\right\}^{\frac{1}{p}}\left\{\int_1^{\infty}\frac{1}{x+n}\frac{x^{\frac{q}{sp}}}{n^{1/r}}f^q(x)dx\right\}^{\frac{1}{q}}$$

$$= n^{\frac{1}{r}-\frac{1}{p}}\{\omega(s,n)\}^{\frac{1}{p}}\left\{\int_1^{\infty}\frac{1}{x+n}\frac{x^{\frac{1}{r}}}{n^{1/r}}x^{\frac{q}{s}-1}f^q(x)dx\right\}^{\frac{1}{q}}. \quad (67)$$

Then by Lebesgue term, by term integration theorem (cf. [43]), we have

$$L_1 \leq \left\{\sum_{n=1}^{\infty}\int_1^{\infty}\frac{1}{x+n}\frac{x^{\frac{1}{r}}}{n^{1/r}}x^{\frac{q}{s}-1}f^q(x)dx\right\}^{\frac{1}{q}}$$

$$= \left\{\int_1^{\infty}\varpi(r,x)x^{\frac{q}{s}-1}f^q(x)dx\right\}^{\frac{1}{q}}, \quad (68)$$

and then (63) follows.

By Hölder's inequality, we have

$$I = \sum_{n=1}^{\infty}\left[(\omega(s,n))^{\frac{1}{p}}n^{\frac{1}{r}-\frac{1}{p}}a_n\right]\left[\frac{n^{\frac{1}{p}-\frac{1}{r}}}{(\omega(s,n))^{\frac{1}{p}}}\int_1^{\infty}\frac{f(x)}{n+x}dx\right]$$

$$\leq \left\{ \sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}} L_1. \tag{69}$$

Then by (63), we have (61).

On the other hand, assuming that (61) is valid, we set

$$a_n := \frac{n^{\frac{q}{s}-1}}{[\omega(s,n)]^{q-1}} \left[\int_1^{\infty} \frac{f(x)}{x+n} dx \right]^{q-1}, n \in \mathbf{N}.$$

Then we find

$$L_1^q = \sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p.$$

If $L_1 = 0$, then (63) is trivially valid; if $L_1 = \infty$, then by (68), (63) is the form of equality. Suppose that $0 < L_1 < \infty$. By (61), it follows

$$\sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p = L_1^q = I$$

$$\leq \left\{ \sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \int_1^{\infty} \varpi(r,x) x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}},$$

$$L_1 = \left\{ \sum_{n=1}^{\infty} \omega(s,n) n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{q}} \leq \left\{ \int_1^{\infty} \varpi(r,x) x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}}.$$

Hence we have (63), and then (61) and (63) are equivalent.

Therefore, (61), (62) and (63) are equivalent. The lemma is proved.

Theorem 3 *If* $p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$, $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} n^{\frac{p}{r}-1} a_n^p < \infty$, $f(x) \geq 0$, $0 < \int_0^{\infty} x^{\frac{q}{s}-1} f^q(x) dx < \infty$, *then we have the following equivalent inequalities* (cf. [19])*:*

$$I = \int_1^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{n+x} dx$$

$$< \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\ln 2}{n^{1/r}} \right] n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_1^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1-\gamma}{x^{1/s}} \right] x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}}, \tag{70}$$

$$J := \left\{ \int_1^{\infty} \frac{x^{\frac{p}{r}-1}}{[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1-\gamma}{x^{1/s}}]^{p-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{x+n} \right]^p dx \right\}^{\frac{1}{p}}$$

$$< \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\ln 2}{n^{1/r}} \right] n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}, \tag{71}$$

$$L := \left\{ \sum_{n=1}^{\infty} \frac{n^{\frac{q}{s}-1}}{[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\ln 2}{n^{1/r}}]^{q-1}} \left[\int_1^{\infty} \frac{f(x)}{x+n} dx \right]^q \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_1^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1-\gamma}{x^{1/s}} \right] x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}}, \tag{72}$$

with the same best possible constant factor $\frac{\pi}{\sin(\frac{\pi}{r})}$.

Proof In view of the assumptions, (65), (60), (61), (62) and (63), we have the equivalent inequalities (70), (71) and (72).

For any $0 < \varepsilon < \frac{p}{s}$, we set $\widetilde{a}_n$, $\widetilde{f}(x)$ as follows:

$$\widetilde{a}_n := n^{\frac{-1}{r} - \frac{\varepsilon}{p}} (n \in \mathbf{N}), \; \widetilde{f}(x) := x^{\frac{-1}{s} - \frac{\varepsilon}{q}} (x \geq 1).$$

Putting $R = (\frac{1}{r} + \frac{\varepsilon}{p})^{-1}$, $S = (\frac{1}{s} - \frac{\varepsilon}{p})^{-1}$, then we have $R > 1$, $\frac{1}{R} + \frac{1}{S} = 1$ and by (60), it follows

$$\widetilde{I} := \int_1^{\infty} \sum_{n=1}^{\infty} \frac{\widetilde{a}_n \widetilde{f}(x)}{n+x} dx = \int_1^{\infty} \sum_{n=1}^{\infty} \frac{n^{\frac{-1}{r} - \frac{\varepsilon}{p}}}{n+x} x^{\frac{-1}{s} - \frac{\varepsilon}{q}} dx$$

$$= \int_1^{\infty} x^{-1-\varepsilon} \left[\sum_{n=1}^{\infty} \frac{x^{1/R}}{n+x} \frac{1}{n^{1/R}} \right] dx$$

$$= \int_1^{\infty} x^{-1-\varepsilon} \varpi(R, x) dx > \int_1^{\infty} x^{-1-\varepsilon} \left[\frac{\pi}{\sin(\frac{\pi}{R})} - \frac{S}{x^{1/S}} \right] dx$$

$$= \frac{1}{\varepsilon} \frac{\pi}{\sin(\frac{\pi}{R})} - \frac{S^2}{S\varepsilon + 1}. \tag{73}$$

If there exists a constant $k \leq \frac{\pi}{\sin(\frac{\pi}{r})}$, such that (70) is valid when replacing $\frac{\pi}{\sin(\frac{\pi}{r})}$ by k, then in particular, by (73), we have

$$\frac{\pi}{\sin(\frac{\pi}{R})} - \frac{\varepsilon S^2}{S\varepsilon + 1}$$

$$< \varepsilon \widetilde{I} < \varepsilon k \left\{ \sum_{n=1}^{\infty} n^{\frac{p}{r}-1} \widetilde{a}_n^p \right\}^{\frac{1}{p}} \left\{ \int_1^{\infty} x^{\frac{q}{s}-1} \widetilde{f}^q(x) dx \right\}^{\frac{1}{q}}$$

$$= \varepsilon k \left\{ 1 + \sum_{n=2}^{\infty} n^{-\varepsilon-1} \right\}^{\frac{1}{p}} \left\{ \int_1^{\infty} x^{-\varepsilon-1} dx \right\}^{\frac{1}{q}}$$

$$< \varepsilon k \left\{1 + \int_1^\infty y^{-\varepsilon-1} dy\right\}^{\frac{1}{p}} \left\{\int_1^\infty x^{-\varepsilon-1} dx\right\}^{\frac{1}{q}}$$
$$= k(\varepsilon + 1)^{\frac{1}{p}},$$

and then $\frac{\pi}{\sin(\frac{\pi}{r})} \leq k(\varepsilon \to 0^+)$. Hence $k = \frac{\pi}{\sin(\frac{\pi}{r})}$ is the best possible constant factor of (70).

We confirm that the constant factor $\frac{\pi}{\sin(\frac{\pi}{r})}$ in (71) (72) is the best possible. Otherwise, by (66) (69), we would reach a contradiction that the constant factor $\frac{\pi}{\sin(\frac{\pi}{r})}$ in (70) is not the best possible. The theorem is proved.

Corollary 2 *If* $p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1, a_n \geq 0, 0 < \sum_{n=1}^\infty n^{\frac{p}{r}-1} a_n^p < \infty$, $f(x) \geq 0$, $0 < \int_1^\infty x^{\frac{q}{s}-1} f^q(x) dx < \infty$, *then we have the following equivalent inequalities with the best possible constant factor* $\frac{\pi}{\sin(\frac{\pi}{r})}$*:*

$$I < \left\{\sum_{n=1}^\infty \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{r}{n^{1/r} + n^{-1/s}}\right] n^{\frac{p}{r}-1} a_n^p\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_1^\infty \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1}{2x^{1/s} + x^{-1/r}}\right] x^{\frac{q}{s}-1} f^q(x) dx\right\}^{\frac{1}{q}}. \tag{74}$$

$$\left\{\int_1^\infty \frac{x^{\frac{p}{r}-1}}{\left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1}{2x^{1/s}+x^{-1/r}}\right]^{p-1}} \left[\sum_{n=1}^\infty \frac{a_n}{x+n}\right]^p dx\right\}^{\frac{1}{p}}$$

$$< \left\{\sum_{n=1}^\infty \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{r}{n^{1/r} + n^{-1/s}}\right] n^{\frac{p}{r}-1} a_n^p\right\}^{\frac{1}{p}}, \tag{75}$$

$$\left\{\sum_{n=1}^\infty \frac{n^{\frac{q}{s}-1}}{\left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{r}{n^{1/r}+n^{-1/s}}\right]^{q-1}} \left[\int_1^\infty \frac{f(x)}{x+n} dx\right]^q\right\}^{\frac{1}{q}}$$

$$\leq \left\{\int_1^\infty \left[\frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1}{2x^{1/s} + x^{-1/r}}\right] x^{\frac{q}{s}-1} f^q(x) dx\right\}^{\frac{1}{q}}. \tag{76}$$

Proof By (52), $\frac{\partial}{\partial x}\theta_r(x) > 0$ implies $\theta_r(n) > \frac{rn}{n+1}$. By (51), we have

$$\omega(s, n) < \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{rn}{n^{1/r}(n+1)}$$

$$= \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{r}{n^{1/r} + n^{-1/s}} (n \in \mathbf{N}). \tag{77}$$

In view of (61), for showing the corollary, we need to prove only the following inequality:

$$\varpi(r,x) < \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1}{2x^{1/s} + x^{-1/r}}(x \geq 1). \tag{78}$$

For $x \geq 1$, we find

$$\begin{aligned}
A(x) &= x^{1-\frac{1}{r}} \int_0^{\frac{1}{x}} \frac{u^{-\frac{1}{r}}}{1+u} du \\
&= x^{1-\frac{1}{r}} \int_0^{\frac{1}{x}} \sum_{k=0}^{\infty} (-1)^k u^{k-\frac{1}{r}} du \\
&= x^{1-\frac{1}{r}} \sum_{k=0}^{\infty} (-1)^k \int_0^{\frac{1}{x}} u^{k-\frac{1}{r}} du \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+\frac{1}{s}\right) x^k} > \sum_{k=0}^{3} \frac{(-1)^k}{\left(k+\frac{1}{s}\right) x^k},
\end{aligned}$$

$$\begin{aligned}
B(x) &= \int_1^{\infty} P_1(t) G(t,x) dt \\
&= \int_1^{\infty} P_1(t) \left[\frac{x}{(x+t)^2 t^{1/r}} + \frac{x}{r(x+t)t^{1+1/r}}\right] dt \\
&> -\frac{1}{12}\left[\frac{x}{(x+1)^2} + \frac{x}{r(x+1)}\right].
\end{aligned}$$

For $x \geq 1$, $\frac{1}{x^2} > 0$ is equivalent to

$$\frac{x}{x+1} < 1 - \frac{1}{x} + \frac{1}{x^2},$$

and $\frac{4}{x^2} + \frac{3}{x^3} > 0$ is equivalent to

$$\frac{x}{(x+1)^2} < \frac{1}{x}\left(1 - \frac{2}{x} + \frac{3}{x^2}\right).$$

Then we have

$$\begin{aligned}
\theta(r,x) &= A(x) + B(x) - \frac{x}{2(x+1)} \\
&> f(s,x) + g(s,x)(x \geq 1),
\end{aligned}$$

where,

$$f(s,x) := s + \frac{1}{12s} + \frac{1}{(1+s)x}$$

$$+\frac{1}{12sx^2}+\frac{1}{3(1+3s)x^3},$$

$$g(s,y):=-\frac{1}{12sx}-\frac{1}{2(1+2s)x^2}$$
$$-\frac{7}{12}-\frac{1}{2x}+\frac{1}{12x^2}-\frac{7}{12x^3}.$$

For $s>1, x\geq 1$, we find

$$f_s'(s,x)=1-\frac{1}{12s^2}-\frac{1}{(1+s)^2x}$$
$$-\frac{1}{12s^2x^2}-\frac{1}{(1+3s)^2x^3}$$
$$>1-\frac{1}{12}-\frac{1}{4}-\frac{1}{12}-\frac{1}{16}>0,$$
$$g_s'(s,x)=\frac{1}{12s^2x}+\frac{1}{(1+2s)^2x^2}>0.$$

Then we obtain

$$\theta(r,x)>f(s,x)+g(s,x)>\lim_{s\to 1^+}(f(s,x)+g(s,x))$$
$$=\frac{1}{2}-\frac{1}{12x}-\frac{1}{2x^3}.$$

For $x\geq 2.5$, since

$$\left(\frac{1}{2}-\frac{1}{12x}-\frac{1}{2x^3}\right)\left(1+\frac{1}{2x}\right)$$
$$=\frac{1}{2}+\frac{1}{x}\left(\frac{1}{6}-\frac{1}{24x}-\frac{1}{2x^2}-\frac{1}{4x^3}\right)>\frac{1}{2},$$

we have

$$\frac{1}{2}-\frac{1}{12x}-\frac{1}{2x^3}>\frac{1}{2\left(1+\frac{1}{2x}\right)}=\frac{1}{2+x^{-1}},$$

and then we find

$$\varpi(r,x)=\frac{\pi}{\sin(\frac{\pi}{r})}-\frac{\theta(r,x)}{x^{\frac{1}{s}}}$$
$$<\frac{\pi}{\sin(\frac{\pi}{r})}-\frac{1}{x^{\frac{1}{s}}}\left(\frac{1}{2}-\frac{1}{12x}-\frac{1}{2x^3}\right)$$
$$<\frac{\pi}{\sin(\frac{\pi}{r})}-\frac{1}{x^{\frac{1}{s}}(2+x^{-1})}$$
$$=\frac{\pi}{\sin(\frac{\pi}{r})}-\frac{1}{2x^{\frac{1}{s}}+x^{-\frac{1}{r}}}(x\geq 2.5).$$

For $1 \leq x < 2.5$, $x < \frac{1-\gamma}{2\gamma-1} = 2.73^{+}$, we find

$$\frac{1-\gamma}{x^{\frac{1}{s}}} > \frac{1}{2x^{\frac{1}{s}} + x^{-\frac{1}{r}}}$$

and

$$\begin{aligned}\varpi(r,x) &< \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1-\gamma}{x^{\frac{1}{s}}} \\ &< \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{1}{2x^{\frac{1}{s}} + x^{-\frac{1}{r}}}.\end{aligned}$$

Hence, (78) is valid for $x \geq 1$. Then by the same way of proving Theorem 3, we can prove the corollary.

By Theorem 3, we can reduce the following corollary:

Corollary 3 *If* $p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1, a_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{\frac{p}{r}-1} a_n^p < \infty$, $f(x) \geq 0$, $0 < \int_1^{\infty} x^{\frac{q}{s}-1} f^q(x)dx < \infty$, *then we have the following equivalent inequalities with the same best possible constant factor* $\frac{\pi}{\sin(\frac{\pi}{r})}$*:*

$$\begin{aligned}\int_1^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{n+x} dx &= \sum_{n=1}^{\infty} a_n \int_1^{\infty} \frac{f(x)}{n+x} dx \\ &< \frac{\pi}{\sin(\frac{\pi}{r})} \left\{ \sum_{n=1}^{\infty} n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \int_1^{\infty} x^{\frac{q}{s}-1} f^q(x)dx \right\}^{\frac{1}{q}},\end{aligned} \tag{79}$$

$$\left\{ \int_1^{\infty} x^{\frac{p}{r}-1} \left[\sum_{n=1}^{\infty} \frac{a_n}{x+n} \right]^p dx \right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{r})} \left\{ \sum_{n=1}^{\infty} n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}}, \tag{80}$$

$$\left\{ \sum_{n=1}^{\infty} n^{\frac{q}{s}-1} \left[\int_1^{\infty} \frac{f(x)dx}{x+n} \right]^q \right\}^{\frac{1}{q}} < \left\{ \int_1^{\infty} x^{\frac{q}{s}-1} f^q(x)dx \right\}^{\frac{1}{q}}. \tag{81}$$

In particular, for $r = p, s = q$, we have the following equivalent half-discrete Hilbert's inequalities:

$$\begin{aligned}\int_1^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{n+x} dx &= \sum_{n=1}^{\infty} a_n \int_1^{\infty} \frac{f(x)}{n+x} dx \\ &< \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \int_1^{\infty} f^q(x)dx \right\}^{\frac{1}{q}},\end{aligned} \tag{82}$$

$$\left\{ \int_1^{\infty} \left[\sum_{n=1}^{\infty} \frac{a_n}{x+n} \right]^p dx \right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}}, \tag{83}$$

$$\left\{\sum_{n=1}^{\infty}\left[\int_{1}^{\infty}\frac{f(x)}{x+n}dx\right]^{q}\right\}^{\frac{1}{q}}<\left\{\int_{1}^{\infty}f^{q}(x)dx\right\}^{\frac{1}{q}};\tag{84}$$

for $r=q, s=p$, we have the equivalent dual forms as follows:

$$\int_{1}^{\infty}f(x)\sum_{n=1}^{\infty}\frac{a_n}{n+x}dx=\sum_{n=1}^{\infty}a_n\int_{1}^{\infty}\frac{f(x)}{n+x}dx$$

$$<\frac{\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{n=1}^{\infty}n^{p-2}a_n^p\right\}^{\frac{1}{p}}\left\{\int_{1}^{\infty}x^{q-2}f^{q}(x)dx\right\}^{\frac{1}{q}},\tag{85}$$

$$\left\{\int_{1}^{\infty}x^{p-2}\left[\sum_{n=1}^{\infty}\frac{a_n}{x+n}\right]^{p}dx\right\}^{\frac{1}{p}}<\frac{\pi}{\sin(\frac{\pi}{r})}\left\{\sum_{n=1}^{\infty}n^{p-2}a_n^p\right\}^{\frac{1}{p}},\tag{86}$$

$$\left\{\sum_{n=1}^{\infty}n^{q-2}\left[\int_{1}^{\infty}\frac{f(x)}{x+n}dx\right]^{q}\right\}^{\frac{1}{q}}<\left\{\int_{1}^{\infty}x^{q-2}f^{q}(x)dx\right\}^{\frac{1}{q}}.\tag{87}$$

Remark 2 Inequalities (70) and (74) are different strengthened versions of (79) with the same best constant factor $\frac{\pi}{\sin(\frac{\pi}{r})}$.

3 Half-Discrete Hilbert-Type Inequalities with the General Non-Homogeneous Kernel and Operator Expressions

In this section, we agree that $p\in\mathbf{R}\backslash\{0,1\}, \frac{1}{p}+\frac{1}{q}=1$, $h(t)(>0)$ is a finite measurable function with respect to $t\in\mathbf{R}_{+}$.

3.1 Some Equivalent Inequalities

Definition 4 For $\sigma, x\in\mathbf{R}_{+}, n\in\mathbf{N}$, we define two weight functions $\omega(\sigma,n)$ and $\varpi(\sigma,x)$ as follows:

$$\omega(\sigma,n):=n^{\sigma}\int_{0}^{\infty}h(xn)\frac{dx}{x^{1-\alpha}},\tag{88}$$

$$\varpi(\sigma,x):=x^{\sigma}\sum_{n=1}^{\infty}h(xn)\frac{1}{n^{1-\sigma}}.\tag{89}$$

Setting $u=xn$, we find

$$\omega(\sigma,n)=n^{\sigma}\int_{0}^{\infty}h(u)\frac{n^{1-\sigma}du}{nu^{1-\sigma}}=\int_{0}^{\infty}h(u)u^{\sigma-1}du.\tag{90}$$

Lemma 12 *As the assumptions of Definition* 4*, if*

$$K(\sigma) := \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+, \tag{91}$$

$f(x), a_n \geq 0$, *then* (i) *for* $p > 1$, *we have the following inequality:*

$$H_1 := \left\{\sum_{n=1}^\infty n^{p\sigma-1}\left(\int_0^\infty h(xn)f(x)dx\right)^p\right\}^{\frac{1}{p}}$$

$$\leq [K(\sigma)]^{\frac{1}{q}}\left\{\int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}}, \tag{92}$$

$$\widetilde{H}_2 := \left\{\int_0^\infty \frac{x^{q\sigma-1}}{[\varpi(\sigma,x)]^{q-1}}\left(\sum_{n=1}^\infty h(xn)a_n\right)^q dx\right\}^{\frac{1}{q}}$$

$$\leq \left\{K(\sigma)\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right\}^{\frac{1}{q}}; \tag{93}$$

(ii) for $p < 0$, or $0 < p < 1$, we have the reverses of (92) and (93).

Proof (i) For $p > 1$, by Hölder's inequality with weight (cf. [47]), it follows

$$\int_0^\infty h(xn)f(x)dx = \int_0^\infty h(xn)\left[\frac{x^{(1-\sigma)/q}}{n^{(1-\sigma)/p}}f(x)\right]\left[\frac{n^{(1-\sigma)/p}}{x^{(1-\sigma)/q}}\right]dx$$

$$\leq \left\{\int_0^\infty h(xn)\frac{x^{(1-\sigma)(p-1)}}{n^{1-\sigma}}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty h(xn)\frac{n^{(1-\sigma)(q-1)}}{x^{1-\sigma}}\right\}^{\frac{1}{q}}$$

$$= [\omega(\sigma,n)]^{\frac{1}{q}}n^{\frac{1}{p}-\sigma}\left\{\int_0^\infty h(xn)\frac{x^{(1-\sigma)(p-1)}}{n^{1-\sigma}}f^p(x)dx\right\}^{\frac{1}{p}}. \tag{94}$$

Then by Lebesgue term, by term integration theorem (cf. [43]) and (91), we have

$$J_1 \leq [K(\sigma)]^{\frac{1}{q}}\left\{\sum_{n=1}^\infty\int_0^\infty h(xn)\frac{x^{(1-\sigma)(p-1)}}{n^{1-\sigma}}f^p(x)dx\right\}^{\frac{1}{p}}$$

$$= [K(\sigma)]^{\frac{1}{q}}\left\{\int_0^\infty\sum_{n=1}^\infty h(xn)\frac{x^{(1-\sigma)(p-1)}}{n^{1-\sigma}}f^p(x)dx\right\}^{\frac{1}{p}}$$

$$= [K(\sigma)]^{\frac{1}{q}}\left\{\int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}}. \tag{95}$$

Hence, (92) follows.

By the same way, we obtain

$$\sum_{n=1}^{\infty} h(xn)a_n \leq [\varpi(\sigma,x)]^{\frac{1}{p}} x^{\frac{1}{q}-\sigma} \times \left\{ \sum_{n=1}^{\infty} h(xn) \frac{n^{(1-\sigma)(q-1)}}{x^{1-\sigma}} a_n^q \right\}^{\frac{1}{q}}, \tag{96}$$

then by Lebesgue term, by term integration theorem and the same way as in obtaining (95), we have (93).

(ii) For $p < 0$ or $0 < p < 1$, by the reverse Hölder's inequality with weight (cf. [47]), we obtain the reverses of (94) and (96). Then by Lebesgue term, by term integration theorem, we still can obtain the reverses of (92) and (93).

Lemma 13 *As the assumptions of Lemma 12, (i) for $p > 1$, we have the following inequality equivalent to* (92) *and* (93)*:*

$$H := \sum_{n=1}^{\infty} \int_0^{\infty} h(xn)a_n f(x)dx \leq \left\{ \int_0^{\infty} \varpi(\sigma,x) x^{p(1-\sigma)-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ K(\sigma) \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}; \tag{97}$$

(ii) for $p < 0$ or $0 < p < 1$, we have the reverse of (97) equivalent to the reverses of (92) and (93).

Proof (i) For $p > 1$, by Hölder's inequality (cf. [47]), it follows

$$H = \sum_{n=1}^{\infty} n^{\frac{1}{q}-(1-\sigma)} \left[\int_0^{\infty} h(xn) f(x)dx \right] \left[n^{(1-\sigma)-\frac{1}{q}} a_n \right] \leq H_1 \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}. \tag{98}$$

Then by (92), we have (97). On the other hand, assuming that (97) is valid, we set

$$b_n := n^{p\sigma-1} \left(\int_0^{\infty} h(xn) f(x)dx \right)^{p-1}, n \in \mathbf{N}.$$

Then it follows $H_1^p = \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q$. If $H_1 = 0$, then (92) is trivially valid; if $H_1 = \infty$, then by (95), (92) keeps the form of equality ($= \infty$). Suppose that $0 < H_1 < \infty$. By (97), we have

$$0 < \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q = H_1^p = H$$

$$\leq \left\{\int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$\times \left\{K(\sigma)\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right\}^{\frac{1}{q}} < \infty.$$

It follows

$$H_1 = [K(\sigma)]^{\frac{1}{q}}\left\{\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right\}^{\frac{1}{p}}$$
$$\leq [K(\sigma)]^{\frac{1}{q}}\left\{\int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}},$$

and then (92) follows. Hence, (92) and (97) are equivalent.

By Hölder's inequality and the same way, we can obtain

$$H \leq \left\{\int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{p}}\widetilde{H}_2. \tag{99}$$

Then by (93), we have (97). On the other hand, assuming that (97) is valid, we set

$$f(x) = \frac{x^{q\sigma-1}}{[\varpi(\sigma,x)]^{q-1}}\left(\sum_{n=1}^\infty h(xn)a_n\right)^{q-1} \quad (x \in \mathbf{R}_+).$$

Then it follows

$$\widetilde{H}_2^q = \int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx.$$

By (97) and the same way, we can obtain

$$\widetilde{H}_2 = \left\{\int_0^\infty \varpi(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx\right\}^{\frac{1}{q}}$$
$$\leq \left\{K(\sigma)\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right\}^{\frac{1}{q}},$$

and then (93) is equivalent to (97).

Hence (92), (93) and (97) are equivalent.

(2) for $p < 0$ or $0 < p < 1$, by the same way, we have the reverse of (97) equivalent to the reverses of (92) and (93). The lemma is proved.

Theorem 4 *As the assumptions of Lemma 12, if* $\theta_\sigma(x) \in (0,1)$,

$$K(\sigma)(1-\theta_\sigma(x)) < \varpi(\sigma,x) < K(\sigma)(x \in \mathbf{R}_+), \tag{100}$$

$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty, 0 < \sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q < \infty$, *then (i) for* $p > 1$, *we have the following equivalent inequalities:*

$$H = \sum_{n=1}^{\infty} \int_0^{\infty} h(xn) a_n f(x) dx$$
$$< K(\sigma) \left\{ \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{101}$$

$$H_1 = \left\{ \sum_{n=1}^{\infty} n^{p\sigma-1} \left(\int_0^{\infty} h(xn) f(x) dx \right)^p \right\}^{\frac{1}{p}}$$
$$< K(\sigma) \left\{ \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{102}$$

$$H_2 := \left\{ \int_0^{\infty} x^{q\sigma-1} \left(\sum_{n=1}^{\infty} h(xn) a_n \right)^q dx \right\}^{\frac{1}{q}}$$
$$< K(\sigma) \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}; \tag{103}$$

(ii) for $p < 0 (0 < q < 1)$, we have the equivalent reverses of (101), (102) and (103);

(iii) for $0 < p < 1 (q < 0)$, we have the following equivalent inequalities:

$$H = \sum_{n=1}^{\infty} \int_0^{\infty} h(xn) a_n f(x) dx$$
$$> K(\sigma) \left\{ \int_0^{\infty} (1 - \theta_\sigma(x)) x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{104}$$

$$H_1 = \left\{ \sum_{n=1}^{\infty} n^{p\sigma-1} \left(\int_0^{\infty} h(xn) f(x) dx \right)^p \right\}^{\frac{1}{p}}$$
$$> K(\sigma) \left\{ \int_0^{\infty} (1 - \theta_\sigma(x)) x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{105}$$

$$\widehat{H}_2 := \left\{ \int_0^{\infty} \frac{x^{q\sigma-1}}{(1 - \theta_\sigma(x))^{q-1}} \left(\sum_{n=1}^{\infty} h(xn) a_n \right)^q dx \right\}^{\frac{1}{q}}$$

$$> K(\sigma)\left\{\sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q\right\}^{\frac{1}{p}}. \tag{106}$$

Theorem 5 *If there exists a constant* $\delta_0 > 0$, *such that for any* $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $K(\widetilde{\sigma}) = \int_0^{\infty} h(u)u^{\widetilde{\sigma}-1}du \in \mathbf{R}_+, \theta_{\widetilde{\sigma}}(x) \in (0,1)$ *and*

$$K(\widetilde{\sigma})(1 - \theta_{\widetilde{\sigma}}(x)) < \varpi(\widetilde{\sigma}, x) < K(\widetilde{\sigma})(x \in \mathbf{R}_+), \tag{107}$$

where, $\theta_{\widetilde{\sigma}}(x) = O(x^{\delta(\widetilde{\sigma})})(x \in (0,1]; \delta(\widetilde{\sigma}) > 0)$, *then the constant factor* $K(\sigma)$ *in Theorem 4 is the best possible.*

Proof (i) For $p > 1$, by Hölder's inequality, we find

$$H \leq H_1 \left\{\sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q\right\}^{\frac{1}{q}}, \tag{108}$$

$$H \leq \left\{\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x)dx\right\}^{\frac{1}{p}} H_2. \tag{109}$$

For $0 < \varepsilon < q\delta_0$, we set $\widetilde{f}(x), \widetilde{a}_n$ as follows:

$$\widetilde{f}(x) := \begin{cases} x^{\sigma + \frac{\varepsilon}{p} - 1}, 0 < x \leq 1, \\ 0, x > 1, \end{cases}$$

$$\widetilde{a}_n := n^{(\sigma - \frac{\varepsilon}{q}) - 1}, n \in \mathbf{N}.$$

Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$, by (107), we find

$$\left\{\int_0^{\infty} x^{p(1-\sigma)-1} \widetilde{f}^p(x)dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\sigma)-1} \widetilde{a}_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{\int_0^1 x^{-1+\varepsilon} dx\right\}^{\frac{1}{p}} \left\{1 + \sum_{n=2}^{\infty} n^{-1-\varepsilon}\right\}^{\frac{1}{q}}$$

$$< \left\{\frac{1}{\varepsilon}\right\}^{\frac{1}{p}} \left\{1 + \int_1^{\infty} y^{-1-\varepsilon} dy\right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}\{\varepsilon + 1\}^{\frac{1}{q}},$$

$$\widetilde{H} := \int_0^{\infty} \sum_{n=1}^{\infty} h(xn)\widetilde{a}_n \widetilde{f}(x)dx = \int_0^1 x^{-1+\varepsilon} \varpi_{\lambda}(\widetilde{\sigma}, x)dx$$

$$\geq K(\widetilde{\sigma}) \int_0^1 x^{-1+\varepsilon} \left(1 - O(x^{\delta(\widetilde{\sigma})})\right) dx$$

$$= \frac{K(\widetilde{\sigma})}{\varepsilon}\left[1 - \varepsilon O_{\widetilde{\sigma}}(1)\right].$$

If there exists a constant $k \leq K(\sigma)$, such that (104) is valid when replacing $K(\sigma)$ by k, then in particular, we have

$$K(\widetilde{\sigma})\left[1 - \varepsilon O_{\widetilde{\sigma}}(1)\right] \leq \varepsilon \widetilde{H} < \varepsilon k \left\{\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}^p(x)dx\right\}^{\frac{1}{p}}$$

$$\times \left\{\sum_{n=1}^\infty n^{q(1-\sigma)-1} \widetilde{a}_n^q\right\}^{\frac{1}{q}} < k\left\{\varepsilon + 1\right\}^{\frac{1}{q}},$$

and then by (26), $K(\sigma) \leq k(\varepsilon \to 0^+)$. Hence $k = K(\sigma)$ is the best possible constant factor of (101).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (102) and (103) is the best possible. Otherwise, we would reach a contradiction by (108) and (109) that the constant factor $K(\sigma)$ in (101) is not the best possible.

(ii) For $p < 0$, by the reverse Hölder's inequality, we have the reverses of (108) and (109). For $0 < \varepsilon < q\delta_0$, we set $\widetilde{f}(x), \widetilde{a}_n$ as (i). Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$, by (107), we find

$$\left\{\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}^p(x)dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q(1-\sigma)-1} \widetilde{a}_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{\int_0^1 x^{-1+\varepsilon}dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{-1-\varepsilon}\right\}^{\frac{1}{q}}$$

$$> \left\{\frac{1}{\varepsilon}\right\}^{\frac{1}{p}} \left\{\int_1^\infty y^{-1-\varepsilon}dy\right\}^{\frac{1}{q}} = \frac{1}{\varepsilon},$$

$$\widetilde{H} = \int_0^\infty \sum_{n=1}^\infty h(xn)\widetilde{a}_n \widetilde{f}(x)dx = \int_0^1 x^{-1+\varepsilon} \varpi_\lambda(\widetilde{\sigma}, x)dx < \frac{1}{\varepsilon} K(\widetilde{\sigma}).$$

If there exists a constant $K \geq K(\sigma)$, such that the reverse of (101) is valid when replacing $K(\sigma)$ by K, then in particular, we have

$$K(\widetilde{\sigma}) > \varepsilon \widetilde{H}$$

$$> \varepsilon K \left\{\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}^p(x)dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q(1-\sigma)-1} \widetilde{a}_n^q\right\}^{\frac{1}{q}} > K,$$

and then by (26), $K(\sigma) \geq K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of the reverse of (101).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in the reverses of (102) and (103) is the best possible. Otherwise, we would reach a contradiction by the reverses of (108) and (109) that the constant factor $K(\sigma)$ in the reverse of (101) is not the best possible.

(iii) For $0 < p < 1$, by the reverse Hölder's inequality, we find

$$H \geq H_1 \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right\}^{\frac{1}{q}}, \tag{110}$$

$$H \geq \left\{ \int_0^{\infty} (1-\theta_\sigma(x)) x^{p(1-\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \widehat{H}_2. \tag{111}$$

For $0 < \varepsilon < |q|\delta_0$, we set $\widetilde{f}(x), \widetilde{a}_n$ as (i). Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$, by (107), we find

$$\begin{aligned}
&\left\{ \int_0^{\infty} (1-\theta_\sigma(x)) x^{p(1-\sigma)-1} \widetilde{f}^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} \widetilde{a}_n^q \right\}^{\frac{1}{q}} \\
&= \left\{ \int_0^1 (1 - O(x^{\delta(\sigma)})) x^{-1+\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ 1 + \sum_{n=2}^{\infty} n^{-1-\varepsilon} \right\}^{\frac{1}{q}} \\
&> \left\{ \int_0^1 (1 - O(x^{\delta(\sigma)})) x^{-1+\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ 1 + \int_1^{\infty} y^{-1-\varepsilon} dy \right\}^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon} \{1 - \varepsilon O_\sigma(1)\}^{\frac{1}{p}} \{\varepsilon + 1\}^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{H} &= \int_0^{\infty} \sum_{n=1}^{\infty} h(xn) \widetilde{a}_n \widetilde{f}(x) dx = \int_0^1 x^{-1+\varepsilon} \varpi(\widetilde{\sigma}, x) dx \\
&< K(\widetilde{\sigma}) \int_0^1 x^{-1+\varepsilon} dx = \frac{1}{\varepsilon} K(\widetilde{\sigma}).
\end{aligned}$$

If there exists a constant $K \geq K(\sigma)$, such that (104) is valid when replacing $K(\sigma)$ by K, then in particular, we have

$$\begin{aligned}
K(\widetilde{\sigma}) > \varepsilon \widetilde{H} > \varepsilon K &\left\{ \int_0^{\infty} (1-\theta_\sigma(x)) x^{p(1-\sigma)-1} \widetilde{f}^p(x) dx \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} \widetilde{a}_n^q \right\}^{\frac{1}{q}} > K\{1 - \varepsilon O_\sigma(1)\}^{\frac{1}{p}} \{\varepsilon + 1\}^{\frac{1}{q}},
\end{aligned}$$

and then by (26), $K(\sigma) \geq K(\varepsilon \to 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of (104).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (105) (106) is the best possible. Otherwise, we would reach a contradiction by (110) (111) that the constant factor $K(\sigma)$ in (104) is not the best possible. The theorem is proved.

Corollary 4 *If there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, $K(\widetilde{\sigma}) \in \mathbf{R}_+$, $h(xy)y^{\widetilde{\sigma}-1}$ is strictly decreasing with respect to $y \in \mathbf{R}_+$, and there exist constants $L > 0$ and $\eta_0 > -\widetilde{\sigma}$, satisfying*

$$h(u) \leq Lu^{\eta_0}(u \in (0,1]), \tag{112}$$

then the constant factor $K(\sigma)$ in Theorem 5 is the best possible.

Proof In view of (32), we find

$$\varpi(\widetilde{\sigma}, x) = x^{\widetilde{\sigma}} \sum_{n=1}^{\infty} h(xn) \frac{1}{n^{1-\widetilde{\sigma}}} < x^{\widetilde{\sigma}} \int_0^{\infty} h(xy) \frac{1}{y^{1-\widetilde{\sigma}}} dy$$

$$= \int_0^{\infty} h(u) u^{\widetilde{\sigma}-1} du = K(\widetilde{\sigma}),$$

$$\varpi(\widetilde{\sigma}, x) > x^{\widetilde{\sigma}} \int_1^{\infty} h(xy) \frac{1}{y^{1-\widetilde{\sigma}}} dy$$

$$= \int_x^{\infty} h(u) u^{\widetilde{\sigma}-1} du = K(\widetilde{\sigma})[(1 - \theta_{\widetilde{\sigma}}(x))](x \in \mathbf{R}_+),$$

where,

$$\theta_{\widetilde{\sigma}}(x) := \frac{1}{K(\widetilde{\sigma})} \int_0^x h(u) u^{\widetilde{\sigma}-1} du \in (0,1).$$

For $x \in (0,1]$,

$$0 < \theta_{\widetilde{\sigma}}(x) \leq \frac{L}{K(\widetilde{\sigma})} \int_0^x u^{\eta_0} u^{\widetilde{\sigma}-1} du$$

$$= \frac{L}{(\eta_0 + \widetilde{\sigma})} x^{\delta(\widetilde{\sigma})} (\delta(\widetilde{\sigma}) = \eta_0 + \widetilde{\sigma}),$$

namely, $\theta_{\widetilde{\sigma}}(x) = O(x^{\delta(\widetilde{\sigma})})(x \in (0,1]; \delta(\widetilde{\sigma}) > 0)$. Then we have (107). Therefore, the constant factor $K(\sigma)$ in Theorem 5 is the best possible. The corollary is proved.

3.2 Operator Expressions and Examples

For $p > 1$, we set $\Phi(x) = x^{p(1-\sigma)-1}(x \in \mathbf{R}_+)$ and $\Psi(n) = n^{q(1-\sigma)-1}(n \in \mathbf{N})$, wherefrom

$$[\Psi(n)]^{1-p} = n^{p\sigma-1}, [\Phi(x)]^{1-q} = x^{q\sigma-1}.$$

We define two real weight normal spaces $L_{p,\Phi}(\mathbf{R}_+)$ and $l_{q,\Psi}$ as follows:

$$L_{p,\Phi}(\mathbf{R}_+) : = \left\{ f; ||f||_{p,\Phi} = \left\{ \int_0^{\infty} \Phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\Psi} := \left\{ a = \{a_n\}; ||a||_{q,\Psi} = \left\{ \sum_{n=1}^{\infty} \Psi(n)|a_n|^q \right\}^{\frac{1}{q}} < \infty \right\}.$$

As the assumptions of Theorem 4, in view of

$$H_1 < K(\sigma)||f||_{p,\Phi}, H_2 < K(\sigma)||a||_{q,\Psi},$$

we can give the following definition:

Definition 5 Define a first kind of half-discrete Hilbert-type operator $\widetilde{T}_1 : L_{p,\Phi}(\mathbf{R}_+) \to l_{p,\Psi^{1-p}}$ as follows: For $f \in L_{p,\Phi}(\mathbf{R}_+)$, there exists a unique representation $\widetilde{T}_1 f \in l_{p,\Psi^{1-p}}$, satisfying

$$(\widetilde{T}_1 f)(n) := \int_0^{\infty} h(xn) f(x) dx (n \in \mathbf{N}). \tag{113}$$

For $a \in l_{q,\Psi}$, we define the following formal inner product of $\widetilde{T}_1 f$ and a as follows:

$$(\widetilde{T}_1 f, a) := \sum_{n=1}^{\infty} a_n \int_0^{\infty} h(xn) f(x) dx. \tag{114}$$

Define a second kind of half-discrete Hilbert-type operator $\widetilde{T}_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(\mathbf{R}_+)$ as follows: For $a \in l_{q,\Psi}$, there exists a unique representation $\widetilde{T}_2 a \in L_{q,\Phi^{1-q}}(\mathbf{R}_+)$, satisfying

$$\left(\widetilde{T}_2 a\right)(x) := \sum_{n=1}^{\infty} k_\lambda(x, n) a_n (x \in \mathbf{R}_+). \tag{115}$$

For $f \in L_{p,\Phi}(\mathbf{R}_+)$, we define the following formal inner product of f and $\widetilde{T}_2 a$ as follows:

$$\left(f, \widetilde{T}_2 a\right) := \int_0^{\infty} k_\lambda(x, n) a_n f(x) dx. \tag{116}$$

Then by Theorem 4, for $0 < ||f||_{p,\Phi}, ||a||_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$(\widetilde{T}_1 f, a) = (\widetilde{T}_2 a, f) < K(\sigma)||f||_{p,\Phi}||a||_{q,\Psi}, \tag{117}$$

$$||\widetilde{T}_1 f||_{p,\Psi^{1-p}} < K(\sigma)||f||_{p,\Phi}, \tag{118}$$

$$||\widetilde{T}_2 a||_{q,\Phi^{1-q}} < K(\sigma)||a||_{q,\Psi}. \tag{119}$$

It follows that $\widetilde{T}_1$ and $\widetilde{T}_2$ are bounded with

$$||\widetilde{T}_1|| := \sup_{f(\neq\theta) \in L_{p,\Phi}(\mathbf{R}_+)} \frac{||\widetilde{T}_1 f||_{p,\Psi^{1-p}}}{||f||_{p,\Phi}} \leq K(\sigma),$$

$$||\widetilde{T}_2|| := \sup_{a(\neq\theta)\in l_{q,\Psi}} \frac{||\widetilde{T}_2 a||_{q,\Phi^{1-q}}}{||a||_{q,\Psi}} \leq K(\sigma).$$

Since by Theorem 5 or Corollary 4, the constant factor $K(\sigma)$ in (118) and (119) is the best possible, we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = K(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du. \tag{120}$$

Note. If we define

$$(\widetilde{T}_1 f)(n) := n^{2\sigma-1}\int_0^\infty h(xn)f(x)dx (n \in \mathbf{N}),$$

then we have $||\widetilde{T}_1 f||_{p,\Phi} < K(\sigma)||f||_{p,\Phi}$ and $\widetilde{T}_1 f \in l_{p,\Phi}$; if we define

$$(\widetilde{T}_2 a)(x) := x^{2\sigma-1}\sum_{n=1}^\infty k_\lambda(x,n)a_n (x \in \mathbf{R}_+),$$

then we have $||\widetilde{T}_2 a||_{q,\Psi} < K(\sigma)||a||_{q,\Psi}$ and $\widetilde{T}_2 a \in L_{q,\Psi}(\mathbf{R}_+)$.

Example 5 (i) We set

$$h(t) = \frac{1}{(t+1)^\lambda} (0 < \sigma < \min\{1,\lambda\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \lambda-\sigma, 1-\sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma-\delta_0, \sigma+\delta_0)$, it follows

$$K(\widetilde{\sigma}) = \int_0^\infty \frac{1}{(t+1)^\lambda} t^{\widetilde{\sigma}-1}dt = B(\widetilde{\sigma}, \lambda-\widetilde{\sigma}) \in \mathbf{R}_+,$$

and

$$\frac{\partial}{\partial y}\left(\frac{1}{(xy+1)^\lambda} y^{\widetilde{\sigma}-1}\right) < 0.$$

Setting $\eta_0 = 0 > -\widetilde{\sigma}$, there exists a constant $L > 0$, such that

$$h(u) = \frac{1}{(u+1)^\lambda} \leq L u^{\eta_0} (u \in (0,1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = B(\sigma, \lambda-\sigma). \tag{121}$$

(ii) We set

$$h(t) = \frac{\ln t}{t^\lambda - 1} (0 < \sigma < \min\{1,\lambda\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \lambda-\sigma, 1-\sigma\} > 0$ and $\widetilde{\sigma} \in (\sigma-\delta_0, \sigma+\delta_0)$, it follows

$$K(\widetilde{\sigma}) = \int_0^\infty \frac{t^{\widetilde{\sigma}-1}\ln t}{t^\lambda - 1}dt = \frac{1}{\lambda^2}\int_0^\infty \frac{v^{\frac{\widetilde{\sigma}}{\lambda}-1}\ln v}{v-1}dv$$

$$= \left[\frac{\pi}{\lambda \sin \pi(\widetilde{\sigma}/\lambda)}\right]^2 \in \mathbf{R}_+,$$

and $\frac{\partial}{\partial y}\left(\frac{\ln(xy)}{(xy)^\lambda - 1} y^{\widetilde{\sigma}-1}\right) < 0$. We set $\eta_0 = -\frac{\sigma}{2} > -\widetilde{\sigma}$. Since $\frac{u^{-\eta_0} \ln u}{u^\lambda - 1} \to 0 (u \to 0^+)$, there exists a constant $L > 0$, such that

$$h(u) = \frac{\ln u}{u^\lambda - 1} \leq L u^{\eta_0} (u \in (0, 1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \left[\frac{\pi}{\lambda \sin \pi(\frac{\sigma}{\lambda})}\right]^2.$$

Example 6 For $s \in \mathbf{N}$, we set

$$h(t) = \frac{1}{\prod_{k=1}^{s}(t^{\lambda/s} + a_k)} (0 < a_1 < \cdots < a_s,$$
$$0 < \sigma < \min\left\{1, \frac{\lambda}{s}\right\}\right).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \frac{\lambda}{s} - \sigma, 1 - \sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, by (41), it follows

$$K_s(\widetilde{\sigma}) = \int_0^\infty \frac{t^{\widetilde{\sigma}-1} dt}{\prod_{k=1}^{s}(t^{\lambda/s} + a_k)} = \frac{s}{\lambda} \int_0^\infty \frac{u^{\frac{s\widetilde{\sigma}}{\lambda}-1} du}{\prod_{k=1}^{s}(u + a_k)}$$
$$= \frac{\pi s}{\lambda \sin(\frac{\pi s \widetilde{\sigma}}{\lambda})} \sum_{k=1}^{s} a_k^{\frac{s\widetilde{\sigma}}{\lambda}-1} \prod_{j=1 (j \neq k)}^{s} \frac{1}{a_j - a_k} \in \mathbf{R}_+,$$

and

$$\frac{\partial}{\partial y}\left(\frac{y^{\widetilde{\sigma}-1}}{\prod_{k=1}^{s}[(xy)^{\lambda/s} + a_k]}\right) < 0.$$

Setting $\eta_0 = 0 > -\widetilde{\sigma}$, there exists a constant $L > 0$, such that

$$h(u) = \frac{1}{\prod_{k=1}^{s}(u^{\lambda/s} + a_k)} \leq L u^{\eta_0} (u \in (0, 1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \frac{\pi s}{\lambda \sin(\frac{\pi s \sigma}{\lambda})} \sum_{k=1}^{s} a_k^{\frac{s\sigma}{\lambda}-1} \prod_{j=1 (j \neq k)}^{s} \frac{1}{a_j - a_k}. \tag{122}$$

(ii) We set

$$h(t) = \frac{1}{t^\lambda + \sqrt{c} t^{\lambda/2} \cos \gamma + \frac{c}{4}}$$

$$(0 < \gamma < \frac{\pi}{2}, 0 < \sigma < \min\{1, \lambda\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \lambda - \sigma, 1 - \sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, by (41), it follows

$$K(\widetilde{\sigma}) = \int_0^\infty \frac{1}{t^\lambda + \sqrt{c}t^{\lambda/2}\cos\gamma + \frac{c}{4}} t^{\widetilde{\sigma}-1}dt$$
$$= \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\widetilde{\sigma}}{\lambda}} \frac{2\pi \sin\gamma\left(1 - \frac{2\widetilde{\sigma}}{\lambda}\right)}{\lambda \sin\gamma \sin\left(\frac{2\pi\widetilde{\sigma}}{\lambda}\right)} \in \mathbf{R}_+,$$

and

$$\frac{\partial}{\partial y}\left(\frac{y^{\widetilde{\sigma}-1}}{(xy)^\lambda + \sqrt{c}(xy)^{\lambda/2}\cos\gamma + \frac{c}{4}}\right) < 0.$$

Setting $\eta_0 = 0 > -\widetilde{\sigma}$, there exists a constant $L > 0$, such that

$$h(u) = \frac{1}{u^\lambda + \sqrt{c}u^{\lambda/2}\cos\gamma + \frac{c}{4}} \le Lu^{\eta_0}(u \in (0, 1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \left(\frac{\sqrt{c}}{2}\right)^{\frac{2\sigma}{\lambda}} \frac{2\pi \sin\gamma\left(1 - \frac{2\sigma}{\lambda}\right)}{\lambda \sin\gamma \sin\left(\frac{2\pi\sigma}{\lambda}\right)}. \tag{123}$$

Example 7 (i) We set

$$h(t) = \ln\left(\frac{b + t^\gamma}{a + t^\gamma}\right)(0 \le a < b, 0 < \sigma < \min\{1, \gamma\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \gamma - \sigma, 1 - \sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, by Example 3(i), it follows

$$K(\widetilde{\sigma}) = \int_0^\infty \ln\left(\frac{b + t^\gamma}{a + t^\gamma}\right) t^{\widetilde{\sigma}-1}dt$$
$$= \int_0^\infty \ln\left(\frac{by^\gamma + 1}{ay^\gamma + 1}\right) y^{-\widetilde{\sigma}-1}dy$$
$$= \frac{1}{\widetilde{\sigma}}\left(b^{\frac{\widetilde{\sigma}}{\gamma}} - a^{\frac{\widetilde{\sigma}}{\gamma}}\right)\frac{\pi}{\sin\pi(\frac{\widetilde{\sigma}}{\gamma})} \in \mathbf{R}_+,$$

and

$$\frac{\partial}{\partial y}\left[y^{\widetilde{\sigma}-1}\ln\left(\frac{b + (xy)^\gamma}{a + (xy)^\gamma}\right)\right] < 0.$$

Setting $\eta_0 = 0 > -\widetilde{\sigma}$, there exists a constant $L > 0$, such that

$$h(t) = \ln\left(\frac{b+t^\gamma}{a+t^\gamma}\right) \le Lt^{\eta_0}(t \in (0,1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \frac{\left(b^{\frac{\sigma}{\gamma}} - a^{\frac{\sigma}{\gamma}}\right)\pi}{\sigma \sin\pi(\frac{\sigma}{\gamma})}. \tag{124}$$

(ii) We set $h(t) = e^{-\rho t^\gamma}(\rho, \gamma > 0, 0 < \sigma < 1)$. For $\delta_0 = \frac{1}{2}\min\{\sigma, 1-\sigma\} > 0$, and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, it follows

$$\begin{aligned} K(\widetilde{\sigma}) &= \int_0^\infty e^{-\rho t^\gamma} t^{\widetilde{\sigma}-1} dt = \frac{1}{\gamma}\rho^{-\frac{\widetilde{\sigma}}{\gamma}} \int_0^\infty e^{-u} u^{\frac{\widetilde{\sigma}}{\gamma}-1} du \\ &= \frac{1}{\gamma\rho^{\frac{\widetilde{\sigma}}{\gamma}}}\Gamma\left(\frac{\widetilde{\sigma}}{\gamma}\right) \in \mathbf{R}_+, \end{aligned}$$

and $\frac{\partial}{\partial y}\left(e^{-\rho(xy)^\gamma} y^{\widetilde{\sigma}-1}\right) < 0$. Setting $\eta_0 = 0 > -\widetilde{\sigma}$, there exists a constant $L > 0$, such that

$$h(t) = e^{-\rho t^\gamma} \le Lt^{\eta_0}(t \in (0,1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \frac{1}{\gamma\rho^{\frac{\sigma}{\gamma}}}\Gamma\left(\frac{\sigma}{\gamma}\right). \tag{125}$$

(iii) We set

$$h(t) = \arctan \rho t^{-\gamma}(\rho, \gamma > 0, 0 < \sigma < \min\{1, \gamma\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma, \gamma - \sigma, 1 - \sigma\} > 0$ and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, it follows

$$\begin{aligned} K(\widetilde{\sigma}) &= \int_0^\infty t^{\widetilde{\sigma}-1}(\arctan \rho t^{-\gamma})dt = \frac{1}{\widetilde{\sigma}}\int_0^\infty (\arctan \rho t^{-\gamma})dt^{\widetilde{\sigma}} \\ &= \frac{1}{\widetilde{\sigma}}\left[(\arctan \rho t^{-\gamma})t^{\widetilde{\sigma}}|_0^\infty + \int_0^\infty \frac{\gamma\rho t^{\widetilde{\sigma}-\gamma-1}}{1+(\rho t^{-\gamma})^2}dt\right] \\ &= \frac{\rho^{\frac{\widetilde{\sigma}}{\gamma}}}{2\widetilde{\sigma}}\int_0^\infty \frac{1}{1+u}u^{\left(-\frac{\widetilde{\sigma}}{2\gamma}+\frac{1}{2}\right)-1}du \\ &= \frac{\rho^{\frac{\widetilde{\sigma}}{\gamma}}\pi}{2\widetilde{\sigma}\sin\pi\left(-\frac{\widetilde{\sigma}}{2\gamma}+\frac{1}{2}\right)} = \frac{\rho^{\frac{\widetilde{\sigma}}{\gamma}}\pi}{2\widetilde{\sigma}\cos\pi\left(\frac{\widetilde{\sigma}}{2\gamma}\right)} \in \mathbf{R}_+, \end{aligned}$$

and $\frac{\partial}{\partial y}(y^{\widetilde{\sigma}-1} \arctan \rho(xy)^{-\gamma}) < 0$. We set $\eta_0 = 0 > -\widetilde{\sigma}$. Since

$$t^{-\eta_0} \arctan \rho t^{-\gamma} \to \frac{\pi}{2}(t \to 0^+),$$

there exists a constant $L > 0$, such that

$$h(t) = \arctan \rho t^{-\gamma} \leq L t^{\eta_0}(t \in (0, 1]).$$

Then by Corollary 4 and (120), we have

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \frac{\rho^{\frac{\sigma}{\gamma}} \pi}{2\sigma \cos \pi\left(\frac{\sigma}{2\gamma}\right)}. \tag{126}$$

Example 8 We set

$$h(t) = \frac{(\min\{t, 1\})^{\gamma}}{(\max\{t, 1\})^{\lambda+\gamma}}(-\gamma < \sigma < \min\{\lambda + \gamma, 1 - \gamma\}).$$

For $\delta_0 = \frac{1}{2}\min\{\sigma + \gamma, \lambda + \gamma - \sigma, 1 - \sigma - \gamma\} > 0$ and $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, it follows

$$K(\widetilde{\sigma}) = \int_0^{\infty} \frac{(\min\{t, 1\})^{\gamma} t^{\widetilde{\sigma}-1}}{(\max\{t, 1\})^{\lambda+\gamma}} dt = \frac{\lambda + 2\gamma}{(\widetilde{\sigma} + \gamma)(\lambda - \widetilde{\sigma} + \gamma)} \in \mathbf{R}_+.$$

We find

$$\begin{aligned} h(xy)y^{\widetilde{\sigma}-1} &= \frac{(\min\{xy, 1\})^{\gamma}}{(\max\{xy, 1\})^{\lambda+\gamma}} y^{\widetilde{\sigma}-1} \\ &= \begin{cases} x^{\gamma} y^{\gamma+\widetilde{\sigma}-1}, 0 < y < x, \\ \frac{1}{x^{\lambda+\gamma} y^{\lambda+\gamma-\widetilde{\sigma}+1}}, y \geq x, \end{cases} \end{aligned}$$

is strictly decreasing with respect to $y \in \mathbf{R}_+$.

There exists a constant η_0, such that $\eta_0 \in (-\widetilde{\sigma}, \gamma)$. In view of

$$t^{-\eta_0} h(t) = \frac{t^{-\eta_0}(\min\{t, 1\})^{\gamma}}{(\max\{t, 1\})^{\lambda+\gamma}} = \begin{cases} t^{\gamma-\eta_0}, 0 < t < 1, \\ \frac{1}{t^{\lambda+\gamma+\eta_0}}, t \geq 1, \end{cases}$$

we have $t^{-\eta_0}h(t) \to 0(t \to 0^+)$, and then there exists a constant $L > 0$, satisfying $h(t) \leq L t^{\eta_0}(t \in (0, 1])$.

Therefore, by Corollary 4 and (120), it follows

$$||\widetilde{T}_1|| = ||\widetilde{T}_2|| = \frac{\lambda + 2\gamma}{(\sigma + \gamma)(\lambda - \sigma + \gamma)}. \tag{127}$$

4 Two Kinds of Compositions of Two Half-Discrete Hilbert-Type Operators

4.1 The Case That the First Kernel Is Homogeneous

For $p > 1$, we set $\varphi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1}(x, y \in \mathbf{R}_+)$, and define three normal spaces as follows:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{m=1}^{\infty}; ||a||_{p,\varphi} = \left\{ \sum_{m=1}^{\infty} \varphi(m)|a_m|^p \right\}^{\frac{1}{p}} < \infty \right\},$$

$$L_{p,\varphi} := \left\{ f; ||f||_{p,\varphi} = \left\{ \int_0^{\infty} \varphi(x)|f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}_{n=1}^{\infty}; ||b||_{q,\psi} = \left\{ \sum_{n=1}^{\infty} \psi(n)|b_n|^q \right\}^{\frac{1}{q}} < \infty \right\}.$$

In the following, we agree that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda, \lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda^{(i)}(x, y)$ $(i = 1, 2, 3)$ are non-negative finite homogeneous functions of degree $-\lambda$ in $\mathbf{R}_+^2$, with

$$k^{(i)}(\lambda_1) := \int_0^{\infty} k_\lambda^{(i)}(u, 1)u^{\lambda_1 - 1} du \in \mathbf{R}_+,$$

and $k_\lambda^{(1)}(x, y)$ is symmetric.

Definition 6 If $k \in \mathbf{N}$, we define two functions $\widetilde{F}_k(y)$ and $\widetilde{G}_k(x)$ as follows:

$$\widetilde{F}_k(y) : = y^{\lambda-1} \int_1^{\infty} k_\lambda^{(2)}(x_1, y) x_1^{\lambda_1 - \frac{1}{pk} - 1} dx_1, y \in [1, \infty), \tag{128}$$

$$\widetilde{G}_k(x) : = x^{\lambda-1} \int_1^{\infty} k_\lambda^{(3)}(x, y_1) y_1^{\lambda_2 - \frac{1}{qk} - 1} dy_1, x \in [1, \infty). \tag{129}$$

Lemma 14 *If there exists a constant* $\delta_0 > 0$, *such that* $k^{(i)}(\lambda_1 \pm \delta_0) \in \mathbf{R}_+$ $(i = 1, 2, 3)$, *and there exist constants* $\delta_1 \in (0, \delta_0)$ *and* $L > 0$, *satisfying for any* $u \in [1, \infty)$,

$$k_\lambda^{(2)}(1, u)u^{\lambda_2 + \delta_1} \le L, \ k_\lambda^{(3)}(u, 1)u^{\lambda_1 + \delta_1} \le L, \tag{130}$$

then for $k \in \mathbf{N}, k > \frac{1}{\delta_1}\max\{\frac{1}{p}, \frac{1}{q}\}$, *setting functions* $F_k(y)$ *and* $G_k(x)$ *as follows:*

$$F_k(y) : = y^{\lambda_1 - \frac{1}{pk} - 1} k^{(2)} \left(\lambda_1 - \frac{1}{pk} \right) - \widetilde{F}_k(y), y \in [1, \infty),$$

$$G_k(x) : = x^{\lambda_2 - \frac{1}{qk} - 1} k^{(3)} \left(\lambda_1 + \frac{1}{qk} \right) - \widetilde{G}_k(x), x \in [1, \infty),$$

we have

$$0 \le F_k(y) = O(y^{\lambda_1-\delta_1-1})(y \in [1,\infty)), \tag{131}$$

$$0 \le G_k(x) = O(x^{\lambda_2-\delta_1-1})(x \in [1,\infty)). \tag{132}$$

Proof Setting $u = x_1/y$ in (128), we obtain

$$\begin{aligned}\widetilde{F}_k(y) &= y^{\lambda_1-\frac{1}{pk}-1}\int_{1/y}^{\infty} k_\lambda^{(2)}(u,1)u^{\lambda_1-\frac{1}{pk}-1}du\\ &= y^{\lambda_1-\frac{1}{pk}-1}\int_{0}^{\infty} k_\lambda^{(2)}(u,1)u^{\lambda_1-\frac{1}{pk}-1}du\\ &- y^{\lambda_1-\frac{1}{pk}-1}\int_{0}^{1/y} k_\lambda^{(2)}(u,1)u^{\lambda_1-\frac{1}{pk}-1}du\\ &= y^{\lambda_1-\frac{1}{pk}-1}k^{(2)}\left(\lambda_1-\frac{1}{pk}\right)\\ &- y^{\lambda_1-\frac{1}{pk}-1}\int_{0}^{1/y} k_\lambda^{(2)}(u,1)u^{\lambda_1-\frac{1}{pk}-1}du.\end{aligned}$$

Hence, it follows

$$\begin{aligned}F_k(y) &= y^{\lambda_1-\frac{1}{pk}-1}k^{(2)}\left(\lambda_1-\frac{1}{pk}\right) - \widetilde{F}_k(y)\\ &= y^{\lambda_1-\frac{1}{pk}-1}\int_{0}^{1/y} k_\lambda^{(2)}(u,1)u^{\lambda_1-\frac{1}{pk}-1}du\\ &= y^{\lambda_1-\frac{1}{pk}-1}\int_{y}^{\infty} k_\lambda^{(2)}(1,v)v^{\lambda_2+\frac{1}{pk}-1}dv \ge 0(y \in [1,\infty)).\end{aligned}$$

In view of (130), we have

$$\begin{aligned}0 \le F_k(y) &\le y^{\lambda_1-\frac{1}{pk}-1}L\int_{y}^{\infty} v^{-\lambda_2-\delta_1}v^{\lambda_2+\frac{1}{pk}-1}dv\\ &= y^{\lambda_1-\frac{1}{pk}-1}L\int_{y}^{\infty} v^{-\delta_1+\frac{1}{pk}-1}dv = \frac{Ly^{\lambda_1-\delta_1-1}}{\delta_1-\frac{1}{pk}},\end{aligned}$$

and then $F_k(y) = O(y^{\lambda_1-\delta_1-1})(y \in [1,\infty))$.

Still setting $u = x/y_1$, we find

$$\begin{aligned}\widetilde{G}_k(x) &= x^{\lambda_2-\frac{1}{qk}-1}\int_{0}^{x} k_\lambda^{(3)}(u,1)u^{\lambda_1+\frac{1}{qk}-1}du\\ &= x^{\lambda_2-\frac{1}{qk}-1}k^{(3)}\left(\lambda_1+\frac{1}{qk}\right) - x^{\lambda_2-\frac{1}{qk}-1}\int_{x}^{\infty} k_\lambda^{(3)}(u,1)u^{\lambda_1+\frac{1}{qk}-1}du.\end{aligned}$$

Hence, it follows

$$G_k(x) = x^{\lambda_2-\frac{1}{qk}-1}k^{(3)}\left(\lambda_1+\frac{1}{qk}\right) - \widetilde{G}_k(x)$$

$$= x^{\lambda_2 - \frac{1}{qk} - 1} \int_x^\infty k_\lambda^{(3)}(u,1) u^{\lambda_1 + \frac{1}{qk} - 1} du \geq 0.$$

By (90), we have

$$0 \leq G_k(x) \leq x^{\lambda_2 - \frac{1}{qk} - 1} L \int_x^\infty u^{-\delta_1 + \frac{1}{qk} - 1} du = \frac{L x^{\lambda_2 - \delta_1 - 1}}{\delta_1 - \frac{1}{qk}},$$

and then $G_k(x) = O(x^{\lambda_2 - \delta_1 - 1})(x \in [1, \infty))$. The lemma is proved.

Lemma 15 *As the assumptions of Lemma 14, we have*

$$\begin{aligned} L_k &:= \frac{1}{k} \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y) x^{\lambda_2 - \frac{1}{qk} - 1} y^{\lambda_1 - \frac{1}{pk} - 1} dx \right) dy \\ &= k^{(1)}(\lambda_1) + o(1)(k \to \infty). \end{aligned} \tag{133}$$

Proof Setting $u = y/x$, since $k_\lambda^{(1)}(x,y)$ is symmetric, by (26), it follows

$$\begin{aligned} L_k &= \frac{1}{k} \int_1^\infty y^{-\frac{1}{k} - 1} \left(\int_0^y k_\lambda^{(1)}(1,u) u^{\lambda_1 + \frac{1}{qk} - 1} du \right) dy \\ &= \frac{1}{k} \left[\int_1^\infty y^{-\frac{1}{k} - 1} \left(\int_0^1 k_\lambda^{(1)}(u,1) u^{\lambda_1 + \frac{1}{qk} - 1} du \right) dy \right. \\ &\quad \left. + \int_1^\infty y^{-\frac{1}{k} - 1} \left(\int_1^y k_\lambda^{(1)}(u,1) u^{\lambda_1 + \frac{1}{qk} - 1} du \right) dy \right] \\ &= \int_0^1 k_\lambda^{(1)}(u,1) u^{\lambda_1 + \frac{1}{qk} - 1} du \\ &\quad + \frac{1}{k} \int_1^\infty \left(\int_u^\infty y^{-\frac{1}{k} - 1} dy \right) k_\lambda^{(1)}(u,1) u^{\lambda_1 + \frac{1}{qk} - 1} du \\ &= \int_0^1 k_\lambda^{(1)}(u,1) u^{\lambda_1 + \frac{1}{qk} - 1} du + \int_1^\infty k_\lambda^{(1)}(u,1) u^{\lambda_1 - \frac{1}{pk} - 1} du \\ &= \int_0^\infty k_\lambda^{(1)}(u,1) u^{\lambda_1 - 1} du + o(1). \end{aligned}$$

Hence, (133) is valid. The lemma is proved.

Lemma 16 *As the assumptions of Lemma* 14, *if* $\lambda, \lambda_1, \lambda_2 \leq 1, k_\lambda^{(i)}(x,y)$ $(i = 1,2,3)$ *are decreasing with respect to* x $(y) \in \mathbf{R}_+$, *setting*

$$\widetilde{A}_\lambda(n) := n^{\lambda - 1} \int_1^\infty k_\lambda^{(2)}(x_1, n) x_1^{\lambda_1 - \frac{1}{pk} - 1} dx_1,$$

$$\widetilde{B}_{\lambda}(x) := x^{\lambda-1} \sum_{n_1=1}^{\infty} k_{\lambda}^{(3)}(x, n_1) n_1^{\lambda_2 - \frac{1}{qk} - 1},$$

then we have

$$\widetilde{I}_k := \frac{1}{k} \int_0^{\infty} \sum_{n=1}^{\infty} k_{\lambda}^{(1)}(x, n) \widetilde{A}_{\lambda}(n) \widetilde{B}_{\lambda}(x) dx$$
$$\geq \prod_{i=1}^{3} k^{(i)}(\lambda_1) + o(1)(k \to \infty). \tag{134}$$

Proof By (32), Definition 6 and Lemma 14, it follows

$$\begin{aligned}
\widetilde{I}_k &\geq \frac{1}{k} \int_1^{\infty} \int_1^{\infty} k_{\lambda}^{(1)}(x, y) \widetilde{F}_k(y) \widetilde{G}_k(x) dx dy \\
&= \frac{1}{k} \int_1^{\infty} \int_1^{\infty} k_{\lambda}^{(1)}(x, y) \left[y^{\lambda_1 - \frac{1}{pk} - 1} k^{(2)} \left(\lambda_1 - \frac{1}{pk} \right) - F_k(y) \right] \\
&\quad \times \left[x^{\lambda_2 - \frac{1}{qk} - 1} k^{(3)} \left(\lambda_1 + \frac{1}{qk} \right) - G_k(x) \right] dx dy \\
&\geq I_1 - I_2 - I_3,
\end{aligned}$$

where, $I_i (i = 1, 2, 3)$ are defined by

$$\begin{aligned}
I_1 &:= k^{(2)} \left(\lambda_1 - \frac{1}{pk} \right) k^{(3)} \left(\lambda_1 + \frac{1}{qk} \right) \\
&\quad \times \frac{1}{k} \int_1^{\infty} \int_1^{\infty} k_{\lambda}^{(1)}(x, y) x^{\lambda_2 - \frac{1}{qk} - 1} y^{\lambda_1 - \frac{1}{pk} - 1} dx dy, \\
I_2 &:= k^{(3)} \left(\lambda_1 + \frac{1}{qk} \right) \\
&\quad \times \frac{1}{k} \int_1^{\infty} \left(\int_1^{\infty} k_{\lambda}^{(1)}(x, y) x^{\lambda_2 - \frac{1}{qk} - 1} dx \right) F_k(y) dy, \\
I_3 &:= k^{(2)} \left(\lambda_1 - \frac{1}{pk} \right) \\
&\quad \times \frac{1}{k} \int_1^{\infty} \left(\int_1^{\infty} k_{\lambda}^{(1)}(x, y) y^{\lambda_1 - \frac{1}{pk} - 1} dy \right) G_k(x) dx.
\end{aligned}$$

By Lemma 15, we have

$$I_1 = (k^{(1)}(\lambda_1) + o(1)) k^{(2)} \left(\lambda_1 - \frac{1}{pk} \right) k^{(3)} \left(\lambda_1 + \frac{1}{qk} \right).$$

Since $0 \leq F_k(y) = O(y^{\lambda_1 - \delta_1 - 1})$, there exists a constant $L_2 > 0$ such that $F_k(y) \leq L_2 y^{\lambda_1 - \delta_1 - 1} (y \in [1, \infty))$, and then

$$0 \leq I_2 \leq k^{(3)} \left(\lambda_1 + \frac{1}{qk} \right) \frac{L_2}{k} \int_1^{\infty} \left(\int_0^{\infty} k_{\lambda}^{(1)}(x, y) x^{\lambda_2 - \frac{1}{qk} - 1} dx \right) y^{\lambda_1 - \delta_1 - 1} dy$$

$$= k^{(3)}\left(\lambda_1 + \frac{1}{qk}\right)\frac{L_2}{k}\int_1^{\infty}\left(\int_0^{\infty} k_\lambda^{(1)}(u,1)u^{\lambda_1+\frac{1}{qk}-1}du\right)y^{-\delta_1-\frac{1}{qk}-1}dy$$
$$= \frac{1}{k}k^{(3)}\left(\lambda_1 + \frac{1}{qk}\right)k^{(1)}\left(\lambda_1 + \frac{1}{qk}\right)\frac{L_2}{\delta_1 + \frac{1}{qk}}.$$

Hence, $I_2 \to 0 (k \to \infty)$.

Since $0 \le G_k(x) = O(x^{\lambda_2-\delta_1-1})$, there exists a constant $L_3 > 0$ such that $G_k(x) \le L_3 x^{\lambda_2-\delta_1-1} (x \in [1,\infty))$, and then

$$0 \le I_3 \le k^{(2)}\left(\lambda_1 - \frac{1}{pk}\right)\frac{L_3}{k}$$
$$\times \int_1^{\infty}\left(\int_0^{\infty} k_\lambda^{(1)}(x,y)y^{\lambda_1-\frac{1}{pk}-1}dy\right)x^{\lambda_2-\delta_1-1}dx$$
$$= k^{(2)}\left(\lambda_1 - \frac{1}{pk}\right)\frac{L_3}{k}$$
$$\times \int_1^{\infty}\left(\int_0^{\infty} k_\lambda^{(1)}(u,1)u^{\lambda_1-\frac{1}{pk}-1}du\right)x^{-\delta_1-\frac{1}{pk}-1}dx$$
$$= \frac{1}{k}k^{(2)}\left(\lambda_1 - \frac{1}{pk}\right)k^{(1)}\left(\lambda_1 - \frac{1}{pk}\right)\frac{L_3}{\delta_1 + \frac{1}{pk}}.$$

Hence, $I_3 \to 0 (k \to \infty)$. Therefore,

$$\widetilde{I}_k \ge I_1 - I_2 - I_3 \to \prod_{i=1}^{3} k^{(i)}(\lambda_1) (k \to \infty),$$

and then (134) follows. The lemma is proved.

Theorem 6 *Suppose that for* $\lambda_1, \lambda_2 < 1, \lambda \le 1, k_\lambda^{(i)}(x,y)$ $(i = 1,2,3)$ *are decreasing with respect to* x $(y) \in \mathbf{R}_+$, *there exists a constant* $\delta_0 > 0$ *such that*

$$k^{(i)}(\lambda_1 \pm \delta_0) \in \mathbf{R}_+ (i = 1,2,3),$$

and there exist constants $\delta_1 \in (0, \delta_0)$ *and* $L > 0$ *satisfying for any* $u \in [1,\infty)$,

$$k_\lambda^{(2)}(1,u)u^{\lambda_2+\delta_1} \le L, k_\lambda^{(3)}(u,1)u^{\lambda_1+\delta_1} \le L.$$

If $f(x_1), B(x) \ge 0, f \in L_{p,\varphi}, B \in L_{q,\psi}, ||f||_{p,\varphi}, ||B||_{q,\psi} > 0$, *setting*

$$A_\lambda(n) := n^{\lambda-1}\int_0^{\infty} k_\lambda^{(2)}(x_1,n)f(x_1)dx_1 (n \in \mathbf{N}),$$

then we have the following equivalent inequalities:

$$I := \int_0^{\infty}\sum_{n=1}^{\infty} k_\lambda^{(1)}(x,n)A_\lambda(n)B(x)dx$$
$$< k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)||f||_{p,\varphi}||B||_{q,\psi}, \tag{135}$$

$$J_1 := \left[\int_0^\infty x^{p\lambda_2-1}\left(\sum_{n=1}^\infty k_\lambda^{(1)}(x,n)A_\lambda(n)\right)^p dx\right]^{\frac{1}{p}}$$

$$< k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)||f||_{p,\varphi}, \tag{136}$$

where the constant factor $k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)$ is the best possible.

In particular, if $b_{n_1} \geq 0, b = \{b_{n_1}\}_{n_1=1}^\infty \in l_{q,\psi}, ||b||_{q,\psi} > 0$, setting

$$B(x) = B_\lambda(x) := x^{\lambda-1}\sum_{n_1=1}^\infty k_\lambda^{(3)}(x,n_1)b_{n_1}(x \in \mathbf{R}_+),$$

then we still have

$$\int_0^\infty \sum_{n=1}^\infty k_\lambda^{(1)}(x,n)A_\lambda(n)B_\lambda(x)dx < \prod_{i=1}^3 k^{(i)}(\lambda_1)||f||_{p,\varphi}||b||_{q,\psi}, \tag{137}$$

where the constant factor $\prod_{i=1}^3 k^{(i)}(\lambda_1)$ is still the best possible.

Proof By (22) and (21), we have $J_1 \leq k^{(1)}(\lambda_1)||A_\lambda||_{p,\varphi}$, and

$$||A_\lambda||_{p,\varphi} = \left\{\sum_{n=1}^\infty n^{p(1-\lambda_1)-1}A_\lambda^p(n)\right\}^{\frac{1}{p}}$$

$$= \left\{\sum_{n=1}^\infty n^{p\lambda_2-1}\left(\int_0^\infty k_\lambda^{(2)}(x_1,n)f(x_1)dx_1\right)^p\right\}^{\frac{1}{p}} < k^{(2)}(\lambda_1)||f||_{p,\varphi},$$

then we have (136). By Hölder's inequality, we find

$$I = \int_0^\infty \left(x^{\lambda_2-\frac{1}{p}}\sum_{n=1}^\infty k_\lambda^{(1)}(x,n)A_\lambda(n)\right)\left(x^{\frac{1}{p}-\lambda_2}B(x)\right)dx \leq J||B||_{q,\psi}. \tag{138}$$

Then by (136), we have (135). On the other hand, assuming that (135) is valid, we set

$$B(x) := x^{p\lambda_2-1}\left(\sum_{n=1}^\infty k_\lambda^{(1)}(x,n)A_\lambda(n)\right)^{p-1} \quad (x \in \mathbf{R}_+).$$

Then we find $||B||_{q,\psi}^q = J_1^p$. If $J_1 = 0$, then (136) is trivially valid; if $J_1 = \infty$, then it is impossible to (136). For $0 < J_1 < \infty$, by (137), it follows

$$||B||_{q,\psi}^q = J_1^p = I < k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)||f||_{p,\varphi}||B||_{q,\psi},$$
$$J_1 = ||B||_{q,\psi}^{q-1} < k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)||f||_{p,\varphi},$$

and then we have (136). Hence, inequalities (135) and (136) are equivalent.

Since $||B_\lambda||_{q,\psi} \le k^{(3)}(\lambda_1)||b||_{q,\psi}$, for $B(x) = B_\lambda(x)$, by (135), we have (137).

In the following, we first prove that the constant factor in (137) is the best possible. For $k \in \mathbf{N}$, $k > \frac{1}{\delta_1}\max\{\frac{1}{p}, \frac{1}{q}\}$, we set

$$\widetilde{f}(x_1) := \begin{cases} 0, 0 < x_1 < 1, \\ x_1^{\lambda_1 - \frac{1}{pk} - 1}, x_1 \ge 1, \end{cases}$$

$$\widetilde{b}_{n_1} := n_1^{\lambda_2 - \frac{1}{qk} - 1} (n_1 \in \mathbf{N}).$$

Then it follows

$$\widetilde{A}_\lambda(n) = n^{\lambda - 1} \int_0^\infty k_\lambda^{(2)}(x_1, n) \widetilde{f}(x_1) dx_1,$$

$$\widetilde{B}_\lambda(x) = x^{\lambda - 1} \sum_{n_1 = 1}^\infty k_\lambda^{(3)}(x, n_1) \widetilde{b}_{n_1}.$$

If there exists a positive constant $K \le \prod_{i=1}^3 k^{(i)}(\lambda_1)$ such that (137) is valid when replacing $\prod_{i=1}^3 k^{(i)}(\lambda_1)$ by K, then in particular, it follows

$$\begin{aligned} \widetilde{I}_k &= \frac{1}{k} \int_0^\infty \sum_{n=1}^\infty k_\lambda^{(1)}(x, n) \widetilde{A}_\lambda(n) \widetilde{B}_\lambda(x) dx \\ &< \frac{1}{k} K ||\widetilde{f}||_{p,\varphi} ||\widetilde{b}||_{q,\psi} = \frac{K}{k} k^{\frac{1}{p}} \left(1 + \sum_{n_1 = 2}^\infty n_1^{-\frac{1}{k} - 1}\right)^{\frac{1}{q}} \\ &< \frac{K}{k} k^{\frac{1}{p}} \left(1 + \int_1^\infty y^{-\frac{1}{k} - 1} dy\right)^{\frac{1}{q}} = K \left(1 + \frac{1}{k}\right)^{\frac{1}{q}}. \end{aligned}$$

In view of (94), we find

$$\prod_{i=1}^3 k^{(i)}(\lambda_1) + o(1) \le \widetilde{I}_k = K \left(1 + \frac{1}{k}\right)^{\frac{1}{q}},$$

and then $\prod_{i=1}^3 k^{(i)}(\lambda_1) \le K (k \to \infty)$. Hence $K = \prod_{i=1}^3 k^{(i)}(\lambda_1)$ is the best possible constant factor of (137).

We can prove that the constant factor in (135) is the best possible. Otherwise, for $B(x) = B_\lambda(x)$, we would reach a contradiction that the constant factor in (137) is not the best possible. In the same way, we can prove that the constant factor in (136) is the best possible. Otherwise, we would reach a contradiction by (138) that the constant factor in (135) is not the best possible. The theorem is proved.

By the same way, we still have

Theorem 7 *Suppose that for* $\lambda_1, \lambda_2 < 1, \lambda \leq 1, k_\lambda^{(i)}(x, y)$ $(i = 1, 2, 3)$ *are decreasing with respect to* x $(y) \in \mathbf{R}_+$, *there exists a constant* $\delta_0 > 0$ *such that*

$$k^{(i)}(\lambda_1 \pm \delta_0) \in \mathbf{R}_+ (i = 1, 2, 3),$$

and there exist constants $\delta_1 \in (0, \delta_0)$ *and* $L > 0$ *satisfying for any* $u \in [1, \infty)$,

$$k_\lambda^{(2)}(1, u) u^{\lambda_2 + \delta_1} \leq L, k_\lambda^{(3)}(u, 1) u^{\lambda_1 + \delta_1} \leq L.$$

If $A(n), b_{n_1} \geq 0, b = \{b_{n_1}\}_{n_1=1}^\infty \in l_{q,\psi}, A = \{A(n)\}_{n=1}^\infty \in l_{p,\varphi}, ||b||_{q,\psi}, ||A||_{p,\varphi} > 0$, *setting*

$$B_\lambda(x) = x^{\lambda - 1} \sum_{n_1=1}^\infty k_\lambda^{(3)}(x, n_1) b_{n_1} (x \in \mathbf{R}_+),$$

then we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^\infty k_\lambda^{(1)}(x, n) A(n) B_\lambda(x) dx < k^{(1)}(\lambda_1) k^{(3)}(\lambda_1) ||A||_{p,\varphi} ||b||_{q,\psi}, \tag{139}$$

$$J_2 = \left[\sum_{n=1}^\infty n^{q\lambda_1 - 1} \left(\int_0^\infty k_\lambda^{(1)}(x, n) B_\lambda(x) dx \right)^q \right]^{\frac{1}{q}}$$

$$< k^{(1)}(\lambda_1) k^{(3)}(\lambda_1) ||b||_{q,\psi}, \tag{140}$$

where the constant factor $k^{(1)}(\lambda_1) k^{(3)}(\lambda_1)$ *is the best possible.*

In particular, if $f(x_1) \geq 0, f \in L_{p,\varphi}, ||f||_{p,\varphi} > 0$, setting

$$A(n) = A_\lambda(n) = n^{\lambda - 1} \int_0^\infty k_\lambda^{(2)}(x_1, n) f(x_1) dx_1 (n \in \mathbf{N}),$$

then we still have

$$\int_0^\infty \sum_{n=1}^\infty k_\lambda^{(1)}(x, n) A_\lambda(n) B_\lambda(x) dx < \prod_{i=1}^3 k^{(i)}(\lambda_1) ||f||_{p,\varphi} ||b||_{q,\psi}, \tag{141}$$

where the constant factor $\prod_{i=1}^3 k^{(i)}(\lambda_1)$ is still the best possible.

Definition 7 As the assumptions of Theorem 6, we define a Hilbert-type operator $T^{(1)} : l_{p,\varphi} \to L_{p,\varphi}$ as follows: For $A_\lambda = \{A_\lambda(n)\}_{n=1}^\infty \in l_{p,\varphi}$, there exists a unique representation $T^{(1)} A_\lambda \in L_{p,\varphi}$, satisfying

$$(T^{(1)} A_\lambda)(x) = x^{\lambda - 1} \sum_{n=1}^\infty k_\lambda^{(1)}(x, n) A_\lambda(n) (x \in \mathbf{R}_+). \tag{142}$$

Similarly to (22), we can find $||T^{(1)}A_\lambda||_{p,\varphi} \leq k^{(1)}(\lambda_1)||A_\lambda||_{p,\varphi}$, where the constant factor $k^{(1)}(\lambda_1)$ is the best possible. Hence, it follows

$$||T^{(1)}|| = k^{(1)}(\lambda_1) = \int_0^\infty k_\lambda^{(1)}(t,1)t^{\lambda_1-1}dt \in \mathbf{R}_+. \tag{143}$$

Definition 8 As the assumptions of Theorem 6, we define a Hilbert-type operator $T^{(2)} : L_{p,\varphi} \to l_{p,\varphi}$ as follows: For $f \in L_{p,\varphi}$, there exists a unique representation $T^{(2)}f \in l_{p,\varphi}$, satisfying

$$(T^{(2)}f)(n) = A_\lambda(n) = n^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x_1,n)f(x_1)dx_1 (n \in \mathbf{N}). \tag{144}$$

We can find $||T^{(2)}f||_{p,\varphi} \leq k^{(2)}(\lambda_1)||f||_{p,\varphi}$, where, the constant factor $k^{(2)}(\lambda_1)$ is the best possible. Hence, it follows

$$||T^{(2)}|| = k^{(2)}(\lambda_1) = \int_0^\infty k_\lambda^{(2)}(t,1)t^{\lambda_1-1}dt \in \mathbf{R}_+. \tag{145}$$

Definition 9 As the assumptions of Theorem 6, we define a Hilbert-type operator $T^{(0)} : L_{p,\varphi} \to L_{p,\varphi}$ as follows: For $f \in L_{p,\varphi}$, there exists a unique representation $T^{(0)}f \in l_{p,\varphi}$, satisfying

$$(T^{(0)}f)(x) = (T^{(1)}A_\lambda)(x) = x^{\lambda-1}\sum_{n=1}^\infty k_\lambda^{(1)}(x,n)A_\lambda(n)$$

$$= x^{\lambda-1}\sum_{n=1}^\infty k_\lambda^{(1)}(x,n)n^{\lambda-1}\left[\int_0^\infty k_\lambda^{(2)}(x_1,n)f(x_1)dx_1\right](x \in \mathbf{R}_+). \tag{146}$$

Since for any $f \in L_{p,\varphi}$, we have

$$T^{(0)}f = T^{(1)}A_\lambda = T^{(1)}(T^{(2)}f) = (T^{(1)}T^{(2)})f,$$

then it follows that $T^{(0)} = T^{(1)}T^{(2)}$, i.e. $T^{(0)}$ is a composition of $T^{(1)}$ and $T^{(2)}$. It is evident that

$$||T^{(0)}|| = ||T^{(1)}T^{(2)}|| \leq ||T^{(1)}|| \cdot ||T^{(2)}|| = k^{(1)}(\lambda_1)k^{(2)}(\lambda_1).$$

By (136), we have

$$||T^{(0)}f||_{p,\varphi} = ||T^{(1)}A_\lambda||_{p,\varphi} = J_1 < k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)||f||_{p,\varphi},$$

where, the constant factor $k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)$ is the best possible. It follows that $||T^{(0)}|| = k^{(1)}(\lambda_1)k^{(2)}(\lambda_1)$, and then we have the following theorem:

Theorem 8 *As the assumptions of Theorem* 6, *the operators* $T^{(1)}$ *and* $T^{(2)}$ *are respectively defined by Definitions* 7 *and* 8, *then we have*

$$||T^{(1)}T^{(2)}|| = ||T^{(1)}|| \cdot ||T^{(2)}|| = k^{(1)}(\lambda_1)k^{(2)}(\lambda_1). \tag{147}$$

Definition 10 As the assumptions of Theorem 7, we define a Hilbert-type operator $T_1 : L_{q,\psi} \to l_{q,\psi}$ as follows: For $B_\lambda \in L_{q,\psi}$, there exists a unique representation $T_1 B_\lambda \in l_{q,\psi}$, satisfying

$$(T_1 B_\lambda)(n) = n^{\lambda-1} \int_0^\infty k_\lambda^{(1)}(x,n) B_\lambda(x) dx (x \in \mathbf{N}). \tag{148}$$

We can find $||T_1 B_\lambda||_{q,\psi} \leq k^{(1)}(\lambda_1)||B_\lambda||_{q,\psi}$, where, the constant factor $k^{(1)}(\lambda_1)$ is the best possible. Hence, it follows

$$||T_1|| = k^{(1)}(\lambda_1) = \int_0^\infty k_\lambda^{(1)}(t,1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+. \tag{149}$$

Definition 11 As the assumptions of Theorem 7, we define a Hilbert-type operator $T_2 : l_{q,\psi} \to L_{q,\psi}$ as follows: For $b = \{b_{n_1}\} \in l_{q,\psi}$, there exists a unique representation $T_2 b \in L_{q,\psi}$, satisfying

$$(T_2 b)(x) = B_\lambda(x) = x^{\lambda-1} \sum_{n_1=1}^\infty k_\lambda^{(3)}(x, n_1) b_{n_1} (x \in \mathbf{R}_+). \tag{150}$$

We can find $||T_2 b||_{q,\psi} \leq k^{(3)}(\lambda_1)||b||_{q,\psi}$, where, the constant factor $k^{(3)}(\lambda_1)$ is the best possible. Hence, it follows

$$||T_2|| = k^{(3)}(\lambda_1) = \int_0^\infty k_\lambda^{(3)}(t,1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+. \tag{151}$$

Definition 12 As the assumptions of Theorem 7, we define a Hilbert-type operator $T_0 : l_{q,\psi} \to l_{q,\psi}$ as follows: For $b \in l_{q,\psi}$, there exists a unique representation $T_0 b \in l_{q,\psi}$, satisfying

$$\begin{aligned}(T_0 b)(n) &= (T_1 B_\lambda)(n) = n^{\lambda-1} \int_0^\infty k_\lambda^{(1)}(x,n) B_\lambda(x) dx \\ &= n^{\lambda-1} \int_0^\infty k_\lambda^{(1)}(x,n) x^{\lambda-1} \left[\sum_{n_1=1}^\infty k_\lambda^{(3)}(x,n_1) b_{n_1} \right] dx (n \in \mathbf{N}). \end{aligned} \tag{152}$$

Since for any $b \in l_{q,\psi}$, we have

$$T_0 b = T_1 B_\lambda = T_1(T_2 b) = (T_1 T_2) b,$$

then it follows that $T_0 = T_1 T_2$, i.e. T_0 is a composition of T_1 and T_2. It is evident that

$$||T_0|| = ||T_1 T_2|| \leq ||T_1|| \cdot ||T_2|| = k^{(1)}(\lambda_1) k^{(3)}(\lambda_1).$$

By (140), we have

$$||T_0 b||_{q,\psi} = ||T_1 B_\lambda||_{q,\psi} = J_2 < k^{(1)}(\lambda_1) k^{(3)}(\lambda_1) ||b||_{q,\psi},$$

where, the constant factor $k^{(1)}(\lambda_1)k^{(3)}(\lambda_1)$ is the best possible. It follows that $||T_0|| = k^{(1)}(\lambda_1)k^{(3)}(\lambda_1)$, and then we have the following theorem:

Theorem 9 *As the assumptions of Theorem* 7*, the operators* T_1 *and* T_2 *are respectively defined by Definitions* 10 *and* 11*, then we have*

$$||T_1T_2|| = ||T_1|| \cdot ||T_2|| = k^{(1)}(\lambda_1)k^{(3)}(\lambda_1). \tag{153}$$

Example 9 (i) For $0 < \lambda \leq 1, 0 < \lambda_1, \lambda_2 < 1$,

$$k_\lambda^{(i)}(x,y) = \frac{1}{x^\lambda + y^\lambda}, \frac{1}{(x+y)^\lambda},$$

$$\frac{\ln(x/y)}{x^\lambda - y^\lambda}, \frac{1}{(\max\{x,y\})^\lambda}(i = 1,2,3)$$

are satisfied using Theorems 8 and 9. If fact, since $0 < \lambda_i + \delta_1 < \lambda (i = 1,2)$,we find

$$k_\lambda^{(2)}(1,u)u^{\lambda_2+\delta_1} \to 0, k_\lambda^{(3)}(u,1)u^{\lambda_1+\delta_1} \to 0 (u \to \infty).$$

(ii) For

$$k_\lambda^{(1)}(x,y) = \frac{1}{x^\lambda + y^\lambda}, k_\lambda^{(2)}(x,y) = \frac{1}{(\max\{x,y\})^\lambda}$$

in Definitions 7, 8 and 9, it follows

$$(T^{(1)}A_\lambda)(x) = x^{\lambda-1}\sum_{n=1}^\infty \frac{1}{x^\lambda + n^\lambda}A_\lambda(n)(x \in \mathbf{R}_+),$$

$$(T^{(2)}f)(n) = n^{\lambda-1}\int_0^\infty \frac{1}{(\max\{x_1,n\})^\lambda}f(x_1)dx_1(n \in \mathbf{N}),$$

$$(T^{(0)}f)(x) = x^{\lambda-1}\sum_{n=1}^\infty \frac{n^{\lambda-1}}{x^\lambda + n^\lambda}\left[\int_0^\infty \frac{f(x_1)dx_1}{(\max\{x_1,n\})^\lambda}\right](x \in \mathbf{R}_+).$$

Then by Theorem 8, we have

$$||T^{(0)}|| = ||T^{(1)}T^{(2)}|| = ||T^{(1)}|| \cdot ||T^{(2)}|| = \frac{\pi}{\lambda \sin\pi(\frac{\lambda_1}{\lambda})}\frac{\lambda}{\lambda_1\lambda_2}$$

$$= \frac{\pi}{\lambda_1\lambda_2 \sin\pi\left(\frac{\lambda_1}{\lambda}\right)}, \tag{154}$$

(iii) For

$$k_\lambda^{(1)}(x,y) = \frac{1}{x^\lambda + y^\lambda}, k_\lambda^{(3)}(x,y) = \frac{1}{(\max\{x,y\})^\lambda}$$

in Definitions 10, 11 and 12, it follows

$$(T_1B_\lambda)(n) = n^{\lambda-1}\int_0^\infty \frac{1}{x^\lambda + n^\lambda}B_\lambda(x)dx(n \in \mathbf{N}),$$

$$(T_2 b)(x) = x^{\lambda-1} \sum_{n_1=1}^{\infty} \frac{1}{(\max\{x, n_1\})^{\lambda}} b_{n_1} (x \in \mathbf{R}_+),$$

$$(T_0 b)(n) = n^{\lambda-1} \int_0^{\infty} \frac{x^{\lambda-1}}{x^{\lambda} + n^{\lambda}} \left[\sum_{n_1=1}^{\infty} \frac{b_{n_1}}{(\max\{x, n_1\})^{\lambda}} \right] dx (n \in \mathbf{N}).$$

Then by Theorem 9, we have

$$||T_0|| = ||T_1 T_2|| = ||T_1|| \cdot ||T_2|| = \frac{\pi}{\lambda_1 \lambda_2 \sin \pi(\frac{\lambda_1}{\lambda})}. \tag{155}$$

4.2 The Case That the First Kernel Is Non-Homogeneous

For $p > 1$, set $\Phi(x) = x^{p(1-\frac{\lambda}{2})-1}, \Psi(y) = y^{q(1-\frac{\lambda}{2})-1} (x, y \in \mathbf{R}_+)$, and we define three normal spaces as follows:

$$l_{p,\Phi} := \left\{ a = \{a_m\}_{m=1}^{\infty}; ||a||_{p,\Phi} = \left\{ \sum_{m=1}^{\infty} \Phi(m) |a_m|^p \right\}^{\frac{1}{p}} < \infty \right\},$$

$$L_{p,\Phi} := \left\{ f; ||f||_{p,\Phi} = \left\{ \int_0^{\infty} \Phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\Psi} := \left\{ b = \{b_n\}_{n=1}^{\infty}; ||b||_{q,\Psi} = \left\{ \sum_{n=1}^{\infty} \Psi(n) |b_n|^q \right\}^{\frac{1}{q}} < \infty \right\}.$$

In the following, we agree that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda \in \mathbf{R}, k_{\lambda}^{(i)}(x, y)$ $(i = 2, 3)$ are non-negative finite homogeneous functions of degree $-\lambda$ in $\mathbf{R}_+^2$, with

$$K^{(i)}\left(\frac{\lambda}{2}\right) := \int_0^{\infty} k_{\lambda}^{(i)}(u, 1) u^{\frac{\lambda}{2}-1} du \in \mathbf{R}_+, \tag{156}$$

and $h(t)$ is a non-negative finite measurable function with

$$K^{(1)}\left(\frac{\lambda}{2}\right) := \int_0^{\infty} h(u) u^{\frac{\lambda}{2}-1} du \in \mathbf{R}_+. \tag{157}$$

Definition 13 If $k \in \mathbf{N}$, define two functions $\widehat{F}_k(y)$ and $\widehat{G}_k(x)$ as follows:

$$\widehat{F}_k(y) : = y^{\lambda-1} \int_0^1 k_{\lambda}^{(2)}(x_1, y) x_1^{\frac{\lambda}{2}+\frac{1}{pk}-1} dx_1, y \in (0, 1], \tag{158}$$

$$\widehat{G}_k(x) : = x^{\lambda-1} \int_1^{\infty} k_{\lambda}^{(3)}(x, y_1) y_1^{\frac{\lambda}{2}-\frac{1}{qk}-1} dy_1, x \in [1, \infty). \tag{159}$$

Lemma 17 *If there exists a constant $\delta_0 > 0$, such that $K^{(i)}\left(\frac{\lambda}{2} \pm \delta_0\right) \in \mathbf{R}_+$ ($i = 1, 2, 3$), and there exist constants $\delta_1 \in (0, \delta_0)$ and $L > 0$, satisfying for any $u \in [1, \infty)$,*

$$k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\delta_1} \leq L,\ k_\lambda^{(3)}(u,1)u^{\frac{\lambda}{2}+\delta_1} \leq L, \tag{160}$$

then for $k \in \mathbf{N}$, $k > \frac{1}{\delta_1}\max\{\frac{1}{p}, \frac{1}{q}\}$, setting functions $F_\lambda(y)$ and $G_\lambda(x)$ as follows:

$$F_\lambda(y) := y^{\frac{\lambda}{2}+\frac{1}{pk}-1} K^{(2)}\left(\frac{\lambda}{2} + \frac{1}{pk}\right) - \widehat{F}_k(y), y \in (0, 1],$$

$$G_\lambda(x) := x^{\frac{\lambda}{2}-\frac{1}{qk}-1} K^{(3)}\left(\frac{\lambda}{2} + \frac{1}{qk}\right) - \widehat{G}_k(x), x \in [1, \infty),$$

we have

$$0 \leq F_\lambda(y) = O\left(y^{\frac{\lambda}{2}+\delta_1-1}\right)(y \in (0, 1]), \tag{161}$$

$$0 \leq G_\lambda(x) = O\left(x^{\frac{\lambda}{2}-\delta_1-1}\right)(x \in [1, \infty)). \tag{162}$$

Proof Setting $u = x_1/y$, we obtain

$$\begin{aligned}
\widehat{F}_k(y) &= y^{\frac{\lambda}{2}+\frac{1}{pk}-1} \int_0^{1/y} k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du \\
&= y^{\frac{\lambda}{2}+\frac{1}{pk}-1} \int_0^{\infty} k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du \\
&\quad - y^{\frac{\lambda}{2}+\frac{1}{pk}-1} \int_{1/y}^{\infty} k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du \\
&= y^{\frac{\lambda}{2}+\frac{1}{pk}-1} K^{(2)}\left(\frac{\lambda}{2} + \frac{1}{pk}\right) \\
&\quad - y^{\frac{\lambda}{2}+\frac{1}{pk}-1} \int_{1/y}^{\infty} k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du.
\end{aligned}$$

Hence, it follows

$$\begin{aligned}
F_\lambda(y) &= y^{\frac{\lambda}{2}+\frac{1}{pk}-1} K^{(2)}\left(\frac{\lambda}{2} + \frac{1}{pk}\right) - \widehat{F}_k(y) \\
&= y^{\frac{\lambda}{2}+\frac{1}{pk}-1} \int_{1/y}^{\infty} k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du \\
&\geq 0 (y \in (0, 1]).
\end{aligned}$$

In view of (160), we have

$$0 \leq F_\lambda(y) \leq y^{\frac{\lambda}{2}+\frac{1}{pk}-1} L \int_{1/y}^{\infty} u^{-\frac{\lambda}{2}-\delta_1} u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du$$

$$= y^{\frac{\lambda}{2}+\frac{1}{pk}-1} L \int_{1/y}^{\infty} u^{-\delta_1+\frac{1}{pk}-1} dv = \frac{L y^{\frac{\lambda}{2}+\delta_1-1}}{\delta_1 - \frac{1}{pk}},$$

and then $F_\lambda(y) = O(y^{\frac{\lambda}{2}+\delta_1-1})(y \in (0, 1])$.

Still setting $u = x/y_1$, we find

$$\begin{aligned}\widehat{G}_k(x) &= x^{\frac{\lambda}{2}-\frac{1}{qk}-1} \int_0^x k_\lambda^{(3)}(u, 1) u^{\frac{\lambda}{2}+\frac{1}{qk}-1} du \\ &= x^{\frac{\lambda}{2}-\frac{1}{qk}-1} K^{(3)} \left(\frac{\lambda}{2} + \frac{1}{qk}\right) \\ &\quad - x^{\frac{\lambda}{2}-\frac{1}{qk}-1} \int_x^{\infty} k_\lambda^{(3)}(u, 1) u^{\frac{\lambda}{2}+\frac{1}{qk}-1} du.\end{aligned}$$

Hence it follows

$$\begin{aligned}G_\lambda(x) &= x^{\frac{\lambda}{2}-\frac{1}{qk}-1} K^{(3)} \left(\frac{\lambda}{2} + \frac{1}{qk}\right) - \widehat{G}_k(x) \\ &= x^{\frac{\lambda}{2}-\frac{1}{qk}-1} \int_x^{\infty} k_\lambda^{(3)}(u, 1) u^{\frac{\lambda}{2}+\frac{1}{qk}-1} du \geq 0.\end{aligned}$$

By (160), we have

$$0 \leq G_\lambda(x) \leq x^{\frac{\lambda}{2}-\frac{1}{qk}-1} L \int_x^{\infty} u^{-\delta_1+\frac{1}{qk}-1} du = \frac{L x^{\frac{\lambda}{2}-\delta_1-1}}{\delta_1 - \frac{1}{qk}},$$

and then $G_\lambda(x) = O(x^{\frac{\lambda}{2}-\delta_1-1})(x \in [1, \infty))$. The lemma is proved.

Lemma 18 *As the assumptions of Lemma 17, we have*

$$\begin{aligned}L_k &:= \frac{1}{k} \int_0^1 \left(\int_1^{\infty} h(xy) x^{\frac{\lambda}{2}-\frac{1}{qk}-1} y^{\frac{\lambda}{2}+\frac{1}{pk}-1} dx\right) dy \\ &= K^{(1)} \left(\frac{\lambda}{2}\right) + o(1)(k \to \infty).\end{aligned} \tag{163}$$

Proof Setting $u = xy$, by (26), it follows

$$\begin{aligned}L_k &= \frac{1}{k} \int_0^1 y^{\frac{1}{k}-1} \left(\int_y^{\infty} h(u) u^{\frac{\lambda}{2}-\frac{1}{qk}-1} du\right) dy \\ &= \frac{1}{k} \left[\int_0^1 y^{\frac{1}{k}-1} \left(\int_y^1 h(u) u^{\frac{\lambda}{2}-\frac{1}{qk}-1} du\right) dy \right. \\ &\quad \left. + \int_0^1 y^{\frac{1}{k}-1} \left(\int_1^{\infty} h(u) u^{\frac{\lambda}{2}-\frac{1}{qk}-1} du\right) dy\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k}\int_0^1 \left(\int_0^u y^{\frac{1}{k}-1}dy\right) h(u)u^{\frac{\lambda}{2}-\frac{1}{qk}-1}du \\
&\quad + \int_1^\infty h(u)u^{\frac{\lambda}{2}-\frac{1}{qk}-1}du \\
&= \int_0^1 h(u)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du + \int_1^\infty h(u)u^{\frac{\lambda}{2}-\frac{1}{qk}-1}du \\
&= \int_0^\infty h(u)u^{\frac{\lambda}{2}-1}du + o(1)(k\to\infty).
\end{aligned}
$$

Hence, (123) is valid. The lemma is proved.

Lemma 19 *As the assumptions of Lemma* 17*, if* $\lambda \le 1, h(xy)$ *is decreasing with respect to* $y \in \mathbf{R}_+$*, and* $k_\lambda^{(i)}(x,y)\,(i=2,3)$ *are decreasing with respect to* $x\,(y) \in \mathbf{R}_+$*, setting*

$$
\widehat{A}_\lambda(n) := n^{\lambda-1}\int_0^1 k_\lambda^{(2)}(x_1,n)x_1^{\frac{\lambda}{2}+\frac{1}{pk}-1}dx_1,
$$

$$
\widehat{B}_\lambda(x) := x^{\lambda-1}\sum_{n_1=1}^\infty k_\lambda^{(3)}(x,n_1)n_1^{\frac{\lambda}{2}-\frac{1}{qk}-1},
$$

then we have

$$
\begin{aligned}
\widehat{I}_k &:= \frac{1}{k}\int_0^\infty \sum_{n=1}^\infty h(xn)\widehat{A}_\lambda(n)\widehat{B}_\lambda(x)dx \\
&\ge \prod_{i=1}^3 K^{(i)}\left(\frac{\lambda}{2}\right) + o(1)(k\to\infty).
\end{aligned}
\tag{164}
$$

Proof By (32), Definition 13 and Lemma 14, it follows

$$
\begin{aligned}
\widehat{I}_k &\ge \frac{1}{k}\int_0^1\int_1^\infty h(xy)\widehat{F}_k(y)\widehat{G}_k(x)dxdy \\
&= \frac{1}{k}\int_0^1\int_1^\infty h(xy)\left[y^{\frac{\lambda}{2}+\frac{1}{pk}-1}K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right) - F_\lambda(y)\right] \\
&\quad \times \left[x^{\frac{\lambda}{2}-\frac{1}{qk}-1}K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right) - G_\lambda(x)\right]dxdy \\
&\ge \widehat{I}_1 - \widehat{I}_2 - \widehat{I}_3,
\end{aligned}
\tag{165}
$$

where, $\widehat{I}_i (i = 1, 2, 3)$ are defined by

$$\begin{aligned}
\widehat{I}_1 :&= K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right)\\
&\quad\times\frac{1}{k}\int_0^1\left(\int_1^\infty h(xy)x^{\frac{\lambda}{2}-\frac{1}{qk}-1}y^{\frac{\lambda}{2}+\frac{1}{pk}-1}dx\right)dy,\\
\widehat{I}_2 :&= K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right)\\
&\quad\times\frac{1}{k}\int_0^1\left(\int_1^\infty h(xy)x^{\frac{\lambda}{2}-\frac{1}{qk}-1}dx\right)F_\lambda(y)dy,\\
\widehat{I}_3 :&= K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)\\
&\quad\times\frac{1}{k}\int_1^\infty\left(\int_0^1 h(xy)y^{\frac{\lambda}{2}+\frac{1}{pk}-1}dy\right)G_\lambda(x)dx.
\end{aligned}$$

By Lemma 18, we have

$$\widehat{I}_1 = (K^{(1)}\left(\frac{\lambda}{2}\right)+o(1))K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right).$$

Since $0 \le F_\lambda(y) = O(y^{\frac{\lambda}{2}+\delta_1-1})$, there exists a constant $L_2 > 0$ such that $F_\lambda(y) \le L_2 y^{\frac{\lambda}{2}+\delta_1-1}(y \in (0,1])$, and then

$$\begin{aligned}
0 \le \widehat{I}_2 &\le K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right)\frac{L_2}{k}\int_0^1\left(\int_0^\infty h(xy)x^{\frac{\lambda}{2}-\frac{1}{qk}-1}dx\right)y^{\frac{\lambda}{2}+\delta_1-1}dy\\
&= K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right)\frac{L_2}{k}\int_0^1\left(\int_0^\infty h(u)u^{\frac{\lambda}{2}-\frac{1}{qk}-1}du\right)y^{\delta_1+\frac{1}{qk}-1}dy\\
&= \frac{1}{k}K^{(3)}\left(\frac{\lambda}{2}+\frac{1}{qk}\right)K^{(1)}\left(\frac{\lambda}{2}-\frac{1}{qk}\right)\frac{L_2}{\delta_1+\frac{1}{qk}}.
\end{aligned}$$

Hence, $\widehat{I}_2 \to 0 (k \to \infty)$.

Since $0 \le G_\lambda(x) = O\left(x^{\frac{\lambda}{2}-\delta_1-1}\right)$, there exists a constant $L_3 > 0$ such that $G_k(x) \le L_3 x^{\frac{\lambda}{2}-\delta_1-1}(x \in [1,\infty))$, and then

$$\begin{aligned}
0 \le \widehat{I}_3 &\le K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)\frac{L_3}{k}\int_1^\infty\left(\int_0^\infty h(xy)y^{\frac{\lambda}{2}+\frac{1}{pk}-1}dy\right)x^{\frac{\lambda}{2}-\delta_1-1}dx\\
&= K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)\frac{L_3}{k}\int_1^\infty\left(\int_0^\infty h(u)u^{\frac{\lambda}{2}+\frac{1}{pk}-1}du\right)x^{-\delta_1-\frac{1}{pk}-1}dx\\
&= \frac{1}{k}K^{(2)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)K^{(1)}\left(\frac{\lambda}{2}+\frac{1}{pk}\right)\frac{L_3}{\delta_1+\frac{1}{pk}}.
\end{aligned}$$

Hence, $\widehat{I}_3 \to 0(k \to \infty)$. Therefore,

$$\widehat{I}_k \geq \widehat{I}_1 - \widehat{I}_2 - \widehat{I}_3 \to \prod_{i=1}^{3} K^{(i)}\left(\frac{\lambda}{2}\right)(k \to \infty),$$

namely, (165) follows. The lemma is proved.

Theorem 10 *Suppose that for* $\lambda \leq 1, h(xy)$ *is decreasing with respect to* $y \in \mathbf{R}_+$, *and* $k_\lambda^{(i)}(x, y)$ $(i = 2, 3)$ *are decreasing with respect to* x $(y) \in \mathbf{R}_+$, *there exists a constant* $\delta_0 > 0$ *such that*

$$K^{(i)}\left(\frac{\lambda}{2} \pm \delta_0\right) \in \mathbf{R}_+(i = 1, 2, 3),$$

and there exist constants $\delta_1 \in (0, \delta_0)$ *and* $L > 0$ *satisfying for any* $u \in [1, \infty)$,

$$k_\lambda^{(2)}(u, 1)u^{\frac{\lambda}{2}+\delta_1} \leq L, k_\lambda^{(3)}(u, 1)u^{\frac{\lambda}{2}+\delta_1} \leq L.$$

If $f(x_1), B(x) \geq 0$, $f \in L_{p,\Phi}, B \in L_{q,\Psi}, ||f||_{p,\Phi}, ||B||_{q,\Psi} > 0$, *setting*

$$A_\lambda(n) = n^{\lambda-1} \int_0^\infty k^{(2)}(x_1, n) f(x_1) dx_1 (n \in \mathbf{N}),$$

then we have the following equivalent inequalities:

$$\widehat{I} := \int_0^\infty \sum_{n=1}^{\infty} h(xn) A_\lambda(n) B(x) dx$$
$$< K^{(1)}\left(\frac{\lambda}{2}\right) K^{(2)}\left(\frac{\lambda}{2}\right) ||f||_{p,\Phi} ||B||_{q,\Psi}, \tag{166}$$

$$\widehat{J}_1 := \left[\int_0^\infty x^{\frac{p\lambda}{2}-1} \left(\sum_{n=1}^{\infty} h(xn) A_\lambda(n)\right)^p dx\right]^{\frac{1}{p}}$$
$$< K^{(1)}\left(\frac{\lambda}{2}\right) K^{(2)}\left(\frac{\lambda}{2}\right) ||f||_{p,\Psi}, \tag{167}$$

where the constant factor $K^{(1)}\left(\frac{\lambda}{2}\right) K^{(2)}\left(\frac{\lambda}{2}\right)$ *is the best possible.*

In particular, if $b_{n_1} \geq 0, b = \{b_{n_1}\}_{n_1=1}^\infty \in l_{q,\Psi}, ||b||_{q,\Psi} > 0$, setting

$$B(x) = B_\lambda(x) = x^{\lambda-1} \sum_{n_1=1}^{\infty} k_\lambda^{(3)}(x, n_1) b_{n_1} (x \in \mathbf{R}_+),$$

then we still have

$$\int_0^\infty \sum_{n=1}^{\infty} h(xn) A_\lambda(n) B_\lambda(x) dx < \prod_{i=1}^{3} K^{(i)}\left(\frac{\lambda}{2}\right) ||f||_{p,\Phi} ||b||_{q,\Psi}, \tag{168}$$

where the constant factor $\prod_{i=1}^{3} K^{(i)}\left(\frac{\lambda}{2}\right)$ is still the best possible.

Proof Since we have $\widehat{J}_1 \leq K^{(1)}\left(\frac{\lambda}{2}\right)||A_\lambda||_{p,\Phi}$, and the following inequality:

$$||A_\lambda||_{p,\Phi} = \left\{\sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} A_\lambda^p(n)\right\}^{\frac{1}{p}}$$

$$= \left\{\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1}\left(\int_0^\infty k_\lambda^{(2)}(x_1,n)f(x_1)dx_1\right)^p\right\}^{\frac{1}{p}} < K^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi},$$

then we have (167). By Hölder's inequality, we find

$$\widehat{I} = \int_0^\infty \left(x^{\frac{\lambda}{2}-\frac{1}{p}}\sum_{n=1}^{\infty} h(xn)A_\lambda(n)\right)\left(x^{\frac{1}{p}-\frac{\lambda}{2}}B(x)\right)dx \leq \widehat{J}_1||B||_{q,\Psi}. \qquad (169)$$

Then by (167), we have (166). On the other hand, assuming that (166) is valid, we set

$$B(x) := x^{\frac{p\lambda}{2}-1}\left(\sum_{n=1}^{\infty} h(xn)A_\lambda(n)\right)^{p-1} \quad (x \in \mathbf{R}_+).$$

Then we find $||B||_{q,\Psi}^q = \widehat{J}_1^p$. If $\widehat{J}_1 = 0$, then (167) is trivially valid; if $\widehat{J}_1 = \infty$, then it is impossible to (167).

For $0 < \widehat{J}_1 < \infty$, by (166), it follows

$$||B||_{q,\Psi}^q = \widehat{J}_1^p = \widehat{I} < K^{(1)}\left(\frac{\lambda}{2}\right)K^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Phi}||B||_{q,\Psi},$$

$$\widehat{J}_1 = ||B||_{q,\Psi}^{q-1} < K^{(1)}\left(\frac{\lambda}{2}\right)K^{(2)}\left(\frac{\lambda}{2}\right)||f||_{p,\Psi},$$

and then we have (167). Hence, inequalities (166) and (167) are equivalent.

Since $||B||_{q,\Psi} \leq K^{(3)}(\frac{\lambda}{2})||b||_{q,\Psi}$, by (166), we have (168). In the following, we first prove that the constant factor in (168) is the best possible. For $k \in \mathbf{N}$, $k > \frac{1}{\delta_1}\max\{\frac{1}{p},\frac{1}{q}\}$, we set

$$\widehat{f}(x_1) := \begin{cases} x_1^{\frac{\lambda}{2}+\frac{1}{pk}-1}, 0 < x_1 \leq 1, \\ 0, x_1 > 1, \end{cases}$$

$$\widehat{b}_{n_1} := n_1^{\frac{\lambda}{2}-\frac{1}{qk}-1}(n_1 \in \mathbf{N}).$$

Then it follows

$$\widehat{A}_\lambda(n) = n^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x_1,n)\widehat{f}(x_1)dx_1,$$

$$\widehat{B}_\lambda(x) = x^{\lambda-1}\sum_{n_1=1}^{\infty} k_\lambda^{(3)}(x,n_1)\widehat{b}_{n_1}.$$

If there exists a positive constant $K \leq \prod_{i=1}^{3} K^{(i)}(\frac{\lambda}{2})$ such that (168) is valid when replacing $\prod_{i=1}^{3} K^{(i)}(\frac{\lambda}{2})$ by K, then in particular, it follows that

$$\begin{aligned}\widehat{I}_k &= \frac{1}{k}\int_0^\infty \sum_{n=1}^{\infty} h(xn)\widehat{A}_\lambda(n)\widehat{B}_\lambda(x)dx \\ &< \frac{1}{k}K||\widehat{f}||_{p,\Phi}||\widehat{b}||_{q,\Psi} = \frac{K}{k}k^{\frac{1}{p}}\left(1+\sum_{n_1=2}^{\infty} n_1^{-\frac{1}{k}-1}\right)^{\frac{1}{q}} \\ &< \frac{K}{k}k^{\frac{1}{p}}\left(1+\int_1^\infty y^{-\frac{1}{k}-1}dy\right)^{\frac{1}{q}} = K\left(1+\frac{1}{k}\right)^{\frac{1}{q}}.\end{aligned}$$

By (165), we find

$$\prod_{i=1}^{3} K^{(i)}\left(\frac{\lambda}{2}\right) + o(1) \leq \widehat{I}_k = K\left(1+\frac{1}{k}\right)^{\frac{1}{q}},$$

and then $\prod_{i=1}^{3} K^{(i)}(\frac{\lambda}{2}) \leq K (k \to \infty)$. Hence $K = \prod_{i=1}^{3} K^{(i)}\left(\frac{\lambda}{2}\right)$ is the best possible constant factor of (168).

We can prove that the constant factor in (166) is the best possible. Otherwise, for $B(x) = B_\lambda(x)$, we would reach a contradiction that the constant factor in (168) is not the best possible. In the same way, we can prove that the constant factor in (167) is the best possible. Otherwise, we would reach a contradiction by (169) that the constant factor in (166) is not the best possible. The theorem is proved.

By the same way, we still have

Theorem 11 *Suppose that for* $\lambda \leq 1, h(xy)$ *is decreasing with respect to* $y \in \mathbf{R}_+, k_\lambda^{(i)}(x,y)$ $(i = 2,3)$ *are decreasing with respect to* x $(y) \in \mathbf{R}_+$, *there exists a constant* $\delta_0 > 0$ *such that*

$$K^{(i)}\left(\frac{\lambda}{2} \pm \delta_0\right) \in \mathbf{R}_+ (i = 1,2,3),$$

and there exist constants $\delta_1 \in (0,\delta_0)$ *and* $L > 0$ *satisfying for any* $u \in [1,\infty)$,

$$k_\lambda^{(2)}(u,1)u^{\frac{\lambda}{2}+\delta_1} \leq L, k_\lambda^{(3)}(u,1)u^{\frac{\lambda}{2}+\delta_1} \leq L.$$

If $A(n), b_{n_1} \geq 0, b = \{b_{n_1}\}_{n_1=1}^\infty \in l_{q,\Psi}, A = \{A(n)\}_{n=1}^\infty \in l_{p,\Phi}, ||b||_{q,\Psi}, ||A||_{p,\Phi} > 0$, *setting*

$$B_\lambda(x) = x^{\lambda-1}\sum_{n_1=1}^{\infty} k_\lambda^{(3)}(x,n_1)b_{n_1} (x \in \mathbf{R}_+),$$

then we have the following equivalent inequalities:

$$\int_0^\infty \sum_{n=1}^{\infty} h(xn)A(n)B_\lambda(x)dx < K^{(1)}\left(\frac{\lambda}{2}\right)K^{(3)}\left(\frac{\lambda}{2}\right)||A||_{p,\Phi}||b||_{q,\Psi}, \qquad (170)$$

$$\widehat{J}_2 = \left[\sum_{n=1}^{\infty} n^{\frac{q\lambda}{2}-1} \left(\int_0^{\infty} h(xn) B_\lambda(x) dx \right)^q \right]^{\frac{1}{q}}$$

$$< K^{(1)} \left(\frac{\lambda}{2} \right) K^{(3)} \left(\frac{\lambda}{2} \right) ||b||_{q,\Psi}, \tag{171}$$

where the constant factor $K^{(1)}\left(\frac{\lambda}{2}\right) K^{(3)}\left(\frac{\lambda}{2}\right)$ *is the best possible.*

In particular, if $f(x_1) \geq 0$, $f \in L_{p,\Phi}$, $||f||_{p,\Phi} > 0$, setting

$$A(n) = A_\lambda(n) = n^{\lambda-1} \int_0^{\infty} k_\lambda^{(2)}(x_1, n) f(x_1) dx_1 (n \in \mathbf{N}),$$

then we still have

$$\int_0^{\infty} \sum_{n=1}^{\infty} h(xn) A_\lambda(n) B_\lambda(x) dx < \prod_{i=1}^{3} K^{(i)} \left(\frac{\lambda}{2} \right) ||f||_{p,\Phi} ||b||_{q,\Psi}, \tag{172}$$

where the constant factor $\prod_{i=1}^{3} K^{(i)}\left(\frac{\lambda}{2}\right)$ is the best possible.

Definition 14 As the assumptions of Theorem 10, we define a Hilbert-type operator $\widehat{T}^{(1)} : l_{p,\Phi} \to L_{p,\Phi}$ as follows: For $A_\lambda = \{A_\lambda(n)\}_{n=1}^{\infty} \in l_{p,\Phi}$, there exists a unique representation $\widehat{T}^{(1)} A_\lambda \in L_{p,\Phi}$, satisfying

$$(\widehat{T}^{(1)} A_\lambda)(x) = x^{\lambda-1} \sum_{n=1}^{\infty} h(xn) A_\lambda(n) (x \in \mathbf{R}_+). \tag{173}$$

We can find

$$||\widehat{T}^{(1)} A_\lambda||_{p,\Phi} \leq K^{(1)} \left(\frac{\lambda}{2} \right) ||A_\lambda||_{p,\Phi},$$

where, the constant factor $K^{(1)}\left(\frac{\lambda}{2}\right)$ is the best possible. Hence, it follows

$$||\widehat{T}^{(1)}|| = K^{(1)} \left(\frac{\lambda}{2} \right) = \int_0^{\infty} h(t) t^{\frac{\lambda}{2}-1} dt \in \mathbf{R}_+. \tag{174}$$

Definition 15 As the assumptions of Theorem 10, we define a Hilbert-type operator $\widehat{T}^{(2)} : L_{p,\Phi} \to l_{p,\Phi}$ as follows: For $f \in L_{p,\Phi}$, there exists a unique representation $\widehat{T}^{(2)} f \in l_{p,\Phi}$, satisfying

$$(\widehat{T}^{(2)} f)(n) = A_\lambda(n) = n^{\lambda-1} \int_0^{\infty} k_\lambda^{(2)}(x_1, n) f(x_1) dx_1 (n \in \mathbf{N}). \tag{175}$$

We can find

$$||\widehat{T}^{(2)} f||_{p,\Phi} \leq K^{(2)} \left(\frac{\lambda}{2} \right) ||f||_{p,\Phi},$$

where, the constant factor $K^{(2)}\left(\frac{\lambda}{2}\right)$ is the best possible. Hence, it follows

$$||\widehat{T}^{(2)}|| = K^{(2)} \left(\frac{\lambda}{2} \right) = \int_0^{\infty} k_\lambda^{(2)}(t, 1) t^{\frac{\lambda}{2}-1} dt \in \mathbf{R}_+. \tag{176}$$

Definition 16 As the assumptions of Theorem 10, we define a Hilbert-type operator $\widehat{T}^{(0)} : L_{p,\Phi} \to L_{p,\Phi}$ as follows: For $f \in L_{p,\Phi}$, there exists a unique representation $\widehat{T}^{(0)} f \in l_{p,\Phi}$, satisfying

$$(\widehat{T}^{(0)} f)(x) = (\widehat{T}^{(1)} A_\lambda)(x) = x^{\lambda-1} \sum_{n=1}^{\infty} h(xn) A_\lambda(n)$$

$$= x^{\lambda-1} \sum_{n=1}^{\infty} h(xn) n^{\lambda-1} \left[\int_0^\infty k_\lambda^{(2)}(x_1, n) f(x_1) dx_1 \right] (x \in \mathbf{R}_+). \quad (177)$$

Since for any $f \in L_{p,\Phi}$, we have

$$\widehat{T}^{(0)} f = \widehat{T}^{(1)} A_\lambda = \widehat{T}^{(1)}(\widehat{T}^{(2)} f) = (\widehat{T}^{(1)} \widehat{T}^{(2)}) f,$$

then it follows that $\widehat{T}^{(0)} = \widehat{T}^{(1)} \widehat{T}^{(2)}$, i.e. $\widehat{T}^{(0)}$ is a composition of $\widehat{T}^{(1)}$ and $\widehat{T}^{(2)}$. It is evident that

$$||\widehat{T}^{(0)}|| = ||\widehat{T}^{(1)} \widehat{T}^{(2)}|| \le ||\widehat{T}^{(1)}|| \cdot ||\widehat{T}^{(2)}|| = K^{(1)} \left(\frac{\lambda}{2} \right) k^{(2)} \left(\frac{\lambda}{2} \right).$$

By (167), we have

$$||\widehat{T}^{(0)} f||_{p,\Phi} = ||\widehat{T}^{(1)} A_\lambda||_{p,\Phi} = \widehat{J}_1 < K^{(1)} \left(\frac{\lambda}{2} \right) k^{(2)} \left(\frac{\lambda}{2} \right) ||f||_{p,\Phi},$$

where, the constant factor $K^{(1)} \left(\frac{\lambda}{2} \right) k^{(2)} \left(\frac{\lambda}{2} \right)$ is the best possible. It follows that $||\widehat{T}^{(0)}|| = K^{(1)} \left(\frac{\lambda}{2} \right) k^{(2)} \left(\frac{\lambda}{2} \right)$, and then we have the following theorem:

Theorem 12 *As the assumptions of Theorem 10, the operators $\widehat{T}^{(1)}$ and $\widehat{T}^{(2)}$ are respectively defined by Definitions 14 and 15, then we have*

$$||\widehat{T}^{(1)} \widehat{T}^{(2)}|| = ||\widehat{T}^{(1)}|| \cdot ||\widehat{T}^{(2)}|| = K^{(1)} \left(\frac{\lambda}{2} \right) k^{(2)} \left(\frac{\lambda}{2} \right). \quad (178)$$

Definition 17 As the assumptions of Theorem 11, we define a Hilbert-type operator $\widehat{T}_1 : L_{q,\Psi} \to l_{q,\Psi}$ as follows: For $B_\lambda \in L_{q,\Psi}$, there exists a unique representation $\widehat{T}_1 B_\lambda \in l_{q,\Psi}$, satisfying

$$(\widehat{T}_1 B_\lambda)(n) = n^{\lambda-1} \int_0^\infty h(xn) B_\lambda(x) dx (x \in \mathbf{R}_+). \quad (179)$$

We can find $||\widehat{T}_1 B_\lambda||_{q,\Psi} \le K^{(1)} \left(\frac{\lambda}{2} \right) ||B_\lambda||_{q,\Psi}$, where the constant factor $K^{(1)} \left(\frac{\lambda}{2} \right)$ is the best possible. Hence, it follows

$$||\widehat{T}_1|| = K^{(1)} \left(\frac{\lambda}{2} \right) = \int_0^\infty h(t) t^{\frac{\lambda}{2}-1} dt \in \mathbf{R}_+. \quad (180)$$

Definition 18 As the assumptions of Theorem 11, we define a Hilbert-type operator $\widehat{T}_2 : l_{q,\Psi} \to L_{q,\Psi}$ as follows: For $b = \{b_{n_1}\}_{n_1=1}^{\infty} \in l_{q,\Psi}$, there exists a unique representation $\widehat{T}_2 b \in L_{q,\Psi}$, satisfying

$$(\widehat{T}_2 b)(x) = B_\lambda(x) = x^{\lambda-1} \sum_{n_1=1}^{\infty} k_\lambda^{(3)}(x, n_1) b_{n_1} (x \in \mathbf{R}_+). \tag{181}$$

We can find $||\widehat{T}_2 b||_{q,\Psi} \le K^{(3)}(\frac{\lambda}{2})||b||_{q,\Psi}$, where, the constant factor $K^{(3)}\left(\frac{\lambda}{2}\right)$ is the best possible. Hence, it follows

$$||\widehat{T}_2|| = K^{(3)}\left(\frac{\lambda}{2}\right) = \int_0^\infty k_\lambda^{(3)}(t,1) t^{\frac{\lambda}{2}-1} dt \in \mathbf{R}_+. \tag{182}$$

Definition 19 As the assumptions of Theorem 11, we define a Hilbert-type operator $\widehat{T}_0 : l_{q,\Psi} \to l_{q,\Psi}$ as follows: For $b \in l_{q,\Psi}$, there exists a unique representation $\widehat{T}_0 b \in l_{q,\Psi}$, satisfying

$$\begin{aligned}(\widehat{T}_0 b)(n) &= (\widehat{T}_1 B_\lambda)(n) = n^{\lambda-1} \int_0^\infty h(xn) B_\lambda(x) dx \\ &= n^{\lambda-1} \int_0^\infty h(xn) x^{\lambda-1} \left[\sum_{n_1=1}^{\infty} k_\lambda^{(3)}(x, n_1) b_{n_1} \right] dx (x \in \mathbf{R}_+). \end{aligned}\tag{183}$$

Since for any $b \in l_{q,\Psi}$, we have

$$\widehat{T}_0 b = \widehat{T}_1 B_\lambda = \widehat{T}_1(\widehat{T}_2 b) = (\widehat{T}_1 \widehat{T}_2) b,$$

then it follows that $\widehat{T}_0 = \widehat{T}_1 \widehat{T}_2$, i.e. $\widehat{T}_0$ is a composition of $\widehat{T}_1$ and $\widehat{T}_2$. It is obvious that

$$||\widehat{T}_0|| = ||\widehat{T}_1 \widehat{T}_2|| \le ||\widehat{T}_1|| \cdot ||\widehat{T}_2|| = K^{(1)}\left(\frac{\lambda}{2}\right) K^{(3)}\left(\frac{\lambda}{2}\right).$$

By (171), we have

$$||\widehat{T}_0 b||_{q,\Psi} = ||\widehat{T}_1 B_\lambda||_{q,\Psi} = \widehat{J}_2 < K^{(1)}\left(\frac{\lambda}{2}\right) K^{(3)}\left(\frac{\lambda}{2}\right) ||b||_{q,\Psi},$$

where, the constant factor $K^{(1)}\left(\frac{\lambda}{2}\right) K^{(3)}\left(\frac{\lambda}{2}\right)$ is the best possible. It follows that $||\widehat{T}_0|| = K^{(1)}\left(\frac{\lambda}{2}\right) K^{(3)}\left(\frac{\lambda}{2}\right)$, and then we have the following theorem:

Theorem 13 *As the assumptions of Theorem* 11*, the operators* $\widehat{T}_1$ *and* $\widehat{T}_2$ *are respectively defined by Definitions* 17 *and* 18*, then we have*

$$||\widehat{T}_1 \widehat{T}_2|| = ||\widehat{T}_1|| \cdot ||\widehat{T}_2|| = K^{(1)}\left(\frac{\lambda}{2}\right) K^{(3)}\left(\frac{\lambda}{2}\right). \tag{184}$$

Example 10 (i) For $0 < \lambda \leq 1, 0 < \lambda_1, \lambda_2 < 1$,

$$h(xy) = \frac{1}{(xy)^{\lambda}+1}, \frac{1}{(xy+1)^{\lambda}}, \frac{\ln(xy)}{(xy)^{\lambda}-1}, \frac{1}{(\max\{xy,1\})^{\lambda}},$$

$$k_{\lambda}^{(i)}(x,y) = \frac{1}{x^{\lambda}+y^{\lambda}}, \frac{1}{(x+y)^{\lambda}},$$

$$\frac{\ln(x/y)}{x^{\lambda}-y^{\lambda}}, \frac{1}{(\max\{x,y\})^{\lambda}}(i=2,3)$$

are satisfied using Theorems 12 and 13. In fact, since $0 < \frac{\lambda}{2} + \delta_1 < \lambda$, we find

$$k_{\lambda}^{(2)}(u,1)u^{\frac{\lambda}{2}+\delta_1} \to 0, k_{\lambda}^{(3)}(u,1)u^{\frac{\lambda}{2}+\delta_1} \to 0(u \to \infty).$$

(ii) For

$$h(xy) = \frac{1}{(xy)^{\lambda}+1}, k_{\lambda}^{(2)}(x,y) = \frac{1}{(\max\{x,y\})^{\lambda}}$$

in Definitions 14, 15 and 16, it follows

$$(\widehat{T}^{(1)}A_{\lambda})(x) = x^{\lambda-1}\sum_{n=1}^{\infty}\frac{1}{(xn)^{\lambda}+1}A_{\lambda}(n)(x \in \mathbf{R}_+),$$

$$(\widehat{T}^{(2)}f)(n) = n^{\lambda-1}\int_0^{\infty}\frac{1}{(\max\{x_1,n\})^{\lambda}}f(x_1)dx_1(n \in \mathbf{N}),$$

$$(\widehat{T}^{(0)}f)(x) = x^{\lambda-1}\sum_{n=1}^{\infty}\frac{n^{\lambda-1}}{(xn)^{\lambda}+1}\left[\int_0^{\infty}\frac{f(x_1)dx_1}{(\max\{x_1,n\})^{\lambda}}\right](x \in \mathbf{R}_+).$$

Then by Theorem 12, we have

$$||\widehat{T}^{(0)}|| = ||\widehat{T}^{(1)}\widehat{T}^{(2)}|| = ||\widehat{T}^{(1)}|| \cdot ||\widehat{T}^{(2)}|| = \frac{\pi}{\lambda}\frac{4}{\lambda} = \frac{4\pi}{\lambda^2}. \tag{185}$$

(iii) For

$$h(xy) = \frac{1}{(xy)^{\lambda}+1}, k_{\lambda}^{(3)}(x,y) = \frac{1}{(\max\{x,y\})^{\lambda}}$$

in Definitions 17, 18 and 19, it follows

$$(\widehat{T}_1 B_{\lambda})(n) = n^{\lambda-1}\int_0^{\infty}\frac{1}{(xn)^{\lambda}+1}B_{\lambda}(x)dx(x \in \mathbf{R}_+),$$

$$(\widehat{T}_2 b)(x) = x^{\lambda-1}\sum_{n_1=1}^{\infty}\frac{1}{(\max\{x,n_1\})^{\lambda}}b_{n_1}(x \in \mathbf{R}_+),$$

$$(\widehat{T}_0 b)(n) = n^{\lambda-1}\int_0^{\infty}\frac{x^{\lambda-1}}{(xn)^{\lambda}+1}\left[\sum_{n_1=1}^{\infty}\frac{b_{n_1}}{(\max\{x,n_1\})^{\lambda}}\right]dx(x \in \mathbf{R}_+).$$

Then by Theorem 13, we have

$$||\widehat{T}_0|| = |||\widehat{T}_1\widehat{T}_2|| = ||\widehat{T}_1|| \cdot ||\widehat{T}_2|| = \frac{\pi}{\lambda}\frac{4}{\lambda} = \frac{4\pi}{\lambda^2}. \tag{186}$$

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Some Results Concerning Hardy and Hardy Type Inequalities

Nikolaos B. Zographopoulos

Abstract We review some recent results concerning functional aspects of the Hardy and Hardy type inequalities. Our main focus is the formulation of such inequalities, for functions having bad behavior at the singularity points. It turns out that Hardy's singularity terms appear in certain cases as a loss to the Hardy's functional, while in other cases are additive to it. Surprisingly, in the latter case, Hardy's functional may be negative. Thus, the validity of the Hardy's inequality is actually based on these singularity terms.

We also discuss the two topics: nonexistence of H_0^1 minimizers and improved Hardy–Sobolev inequalities. These topics may be seen as a consequence of the connection of the Hardy and Hardy type inequalities with the Sobolev inequality defined in the whole space.

Keywords Hardy inequality · Sobolev inequality · Optimal inequalities

1 Introduction

In this work we review some recent results concerning functional properties of the Hardy's inequality

$$\int_\Omega |\nabla u|^2 \, dx > \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \, dx, \tag{1}$$

which is well known to hold for any $u \in C_0^\infty(\Omega)$. The constant

$$\frac{(N-2)^2}{4}$$

in (1) is sharp and not achieved. The literature concerning Hardy and Hardy type inequalities and their applications is extensive; it is not in the purpose of this work to cover this. For some relevant works, cf. [15, 26, 38, 44, 45, 49].

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We introduce the Hardy functional

$$I_\Omega[\phi] := \int_\Omega |\nabla\phi|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\phi^2}{|x|^2}\,dx\,, \tag{2}$$

$\phi \in C_0^\infty(\Omega)$, which is positive and different lower bounds have been obtained (see discussion below). Note that the expression is finite for $u \in H_0^1(\Omega)$, but it can also be finite as an improper integral for other functions having a strong singularity at $x = 0$, due to cancelations between the two terms. Our goal will be the generalization for functions, for which the Hardy functional is well defined in the sense of principal value or is not well defined or is infinite.

The motivation for this is explained in [53]; In the study of the corresponding parabolic problem, we have to work with functions u which do not belong to $H_0^1(\Omega)$. More precisely, it came from a functional difficulty we found in interpreting the work [55], where the following singular evolution problem was studied:

$$\begin{cases} u_t &= \Delta u + c_*\,|x|^{-2}\,u,\ x \in \Omega,\ t > 0,\\ u(x,0) &= u_0(x), \quad \text{for } x \in \Omega,\\ u(x,t) &= 0 \ \text{ in } \partial\Omega,\ t > 0\,. \end{cases} \tag{3}$$

with critical coefficient $c_* = (N-2)^2/4$. The space dimension is $N \geq 3$ and Ω is a bounded domain in $\mathbb{R}^N$ containing 0, or $\Omega = \mathbb{R}^N$.

The separation of variables analysis produces some singular solutions. In particular, the maximal singularity (corresponding to the first mode of separation of variables) behaves like $|x|^{-(N-2)/2}$ near $x = 0$, and this function does not belong to $H_0^1(\Omega)$. Now, this solution must belong to the space H associated to the quadratic form, hence the conclusion $H \neq H_0^1(\Omega)$. We recall that this is a peculiar phenomenon for the equation with critical exponent $c_* = (N-2)^2/4$. For values of $c < c_*$, the maximal singularity is still in $H_0^1(\Omega)$. To consider this possibility into account, the Hilbert space H was introduced in [55] as the completion of the $C_0^\infty(\Omega)$ functions under the norm

$$||\phi||^2_{H(\Omega)} = I_\Omega[\phi], \quad \phi \in C_0^\infty(\Omega). \tag{4}$$

However, we have realized that with the proposed definition of H, there exists a problem with the solutions of the evolution problem having the maximal singularity. The verification is quite simple in the case where $\Omega = B_1$, the unit ball in $\mathbb{R}^N$ centered at the origin. In that case, the minimization problem

$$\min_{u\in H} \frac{||u||_H^2}{||u||_{L^2}^2} \tag{5}$$

admits as a solution to the function $e_1(r) = r^{-(N-2)/2}\,J_0(z_{0,1}\,r)$, where $r = |x|$, J_0 is the Bessel function with $J_0(0) = 1$, up to normalization and $z_{0,1}$ denotes the first zero of J_0. This function plays a big role in the asymptotic behavior of general solutions of

Problem (3). The minimum value of (5) is $\mu_1 = z_{0,1}^2$. Moreover, the quantity $I_{B_1}(e_1)$ is well defined as a principal value. Assuming that

$$||e_1||_H^2 = I_{B_1}(e_1), \tag{6}$$

from the definition of H, for any $\varepsilon > 0$, we should find a C_0^∞-function ϕ, such that $||e_1 - \phi||_H^2 < \varepsilon$. However, we may prove that $||e_1 - \phi||_H^2 \geq c > 0$, for any C_0^∞-function ϕ, which is a contradiction. It seems that e_1 fails to be in H, since it cannot be approximated by C_0^∞-functions and this will happen for every function with the maximal singularity. What is really happening in this case is that for functions with certain bad behavior, the norm of H is not given by (4).

Next we present the results of Vázquez and Zographopoulos [53, 54], which have their own interest, as they may be seen as generalizations of the Hardy and Hardy type inequalities.

1. We start with the Hardy inequality (1) defined on a bounded domain. Let $N \geq 3$ and Ω be a bounded domain of $\mathbb{R}^N$, containing the origin. Then, Hardy's inequality (on a bounded domain), takes the form

$$\lim_{\varepsilon \to 0} \left(I_{B_\varepsilon^c}[u] - \Lambda_\varepsilon(u) \right) > 0, \tag{7}$$

for any function $u \not\equiv 0, u \in H$. With Λ_ε we denote the quantity:

$$\Lambda_\varepsilon(u) = \frac{N-2}{2}\, \varepsilon^{-1} \int_{S_\varepsilon} u^2 \, dS\,, \tag{8}$$

where dS denotes the surface measure. Actually the left hand side of (7) represents the norm of $H(\Omega)$. As we discuss in Sect. 2, Λ_ε may have a bad behavior; oscillating or tending to infinity. In these cases, the Hardy functional I_ε has the same behavior with Λ_ε, so that the sum of them to become a positive real number.

2. Next we consider the case of the Hardy inequality (1) defined on an exterior domain. Let $N \geq 3$ and $\Omega = \mathbb{R}^N \backslash B_1(0)$ be an exterior domain. We note that the inverse square potential corresponds to singular phenomena also at infinity. We consider the Hilbert space $H(\Omega)$ as the completion of the $C_0^\infty(\Omega)$ functions under the norm (4). Then, Hardy's inequality (on an exterior domain), takes the form

$$\lim_{\varepsilon \to 0} \left(I_{B_{1/\varepsilon}^c}[w] + \Lambda_{1/\varepsilon}(w) \right) > 0, \tag{9}$$

for any function $u \not\equiv 0$, $u \in H$. Actually the left hand side of (9) represents the norm of $H(\Omega)$. The Hardy functional posed in the exterior domain is not necessarily a positive quantity; functions which belong in H and behaving at infinity like $|x|^{-(N-2)/2}$ may be negative; for an example see [53]. Thus, the validity of (9), is actually based on $\Lambda_{1/\varepsilon}$.

3. For the case of the whole space Ω, where $\Omega = \mathbb{R}^N$, the Hardy inequality is sharp; we cannot expect a Hardy–Poincaré inequality to hold, for any smooth function.

To overcome this difficulty, the authors in [55] made use of the similarity variables. They introduced the following weighted Hardy inequality:

$$I_K[w] \geq 0, \tag{10}$$

for any $C_0^\infty(\mathbb{R}^N)$ function, where

$$I_K[w] := \int_{\mathbb{R}^N} K\,|\nabla w|^2\,dy - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} K\,\frac{w^2}{|y|^2}\,dy. \tag{11}$$

and $K(|y|) = \exp(|y|^2/4)$. Also, in this case,

$$\frac{(N-2)^2}{4},$$

is the best constant for (11).
As above, we introduce the Hilbert space $H(K)$ as the completion of the space of $C_0^\infty(\mathbb{R}^N)$ functions under the norm

$$||\phi||^2_{H(K)} = I_K[\phi], \quad \phi \in C_0^\infty(\mathbb{R}^N). \tag{12}$$

Then, this weighted Hardy's inequality, takes the form

$$\lim_{\varepsilon\to 0}\left(I_{K,B_\varepsilon^c}[w] - \Lambda_{K,\varepsilon}(w)\right), \tag{13}$$

for any function $u \not\equiv 0$, $u \in H(K)$, where $\Lambda_{K,\varepsilon}$ is defined as:

$$\Lambda_{K,\varepsilon}(w) := \frac{N-2}{2}\,\varepsilon^{-1}\int_{S_\varepsilon} K\,w^2\,dS, \tag{14}$$

For the details we refer to [54].

4. The following inequality is derived from the previous one, by replacing

$$K(|y|) = \exp(|y|^2/4) \qquad \text{with} \qquad \tilde{K}(|y|) = \exp(1/(4|y|^2)).$$

The weighted Hardy functional is now considered:

$$I_{\tilde{K}}[\tilde{w}] := \int_{\mathbb{R}^N} \tilde{K}\,|\nabla \tilde{w}|^2\,d\tilde{y} - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \tilde{K}\,\frac{\tilde{w}^2}{|\tilde{y}|^2}\,d\tilde{y}. \tag{15}$$

What is interesting here is that $I_{\tilde{K}}$ is not necessarily a positive quantity; functions which behave at infinity like $|y|^{-(N-2)/2}$ might be negative. However, in this case, we may prove that this weighted Hardy's inequality, takes the form

$$\lim_{\varepsilon\to 0}\left(I_{\tilde{K},\varepsilon}[\tilde{w}] + \Lambda_{\tilde{K},1/\varepsilon}(\tilde{w})\right) + \frac{N-2}{2}\,||\tilde{w}||^2_{L^2(\tilde{K}\,|\tilde{y}|^{-4})} \geq 0, \tag{16}$$

for any function belonging to the corresponding space and $\Lambda_{\tilde{K},1/\varepsilon}$ is given by (14). We emphasize the existence in (16), of the $L^2(\tilde{K}\,|\tilde{y}|^{-4})$ norm. It turns out that it might be crucial for the validity of (16). For an example, see [54, pp. 5477–5478].

5. Next, we consider an improved Hardy inequality. Consider the weights

$$V_k(x) = \frac{1}{4}\sum_{i=1}^{k} \frac{1}{|x|^2} X_1^2\left(\frac{|x|}{D}\right) X_2^2\left(\frac{|x|}{D}\right)\dots X_i^2\left(\frac{|x|}{D}\right), \qquad k = 1, 2, \dots \quad (17)$$

with $D > D_0 := \sup\{|x|, x \in \Omega\}$ and

$$X_1(t) = (1 - \log t)^{-1}, \quad X_k(t) = X_1(X_{k-1}(t)), \quad k = 2, 3, \dots .$$

This study is motivated by the work [30], where the authors have provided an answer to a question raised in [16] concerning the improvements of the Hardy inequality. They proved that the Hardy inequality has an infinite series improvement, such that the k-improved Hardy functional (kIHT)

$$\begin{aligned} I_k[u] = \int_\Omega |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,dx \\ - \frac{1}{4}\sum_{i=1}^{k}\int_\Omega \frac{1}{|x|^2} X_1^2 X_2^2 \dots X_i^2 u^2\,dx. \end{aligned} \quad (18)$$

is positive for any $u \in C_0^\infty(\Omega)$ and any $k = 1, 2, \dots$. Related topics concerning improvements of the Hardy inequality are discussed in the sequel.
We introduce the Hilbert space $H_k(K)$ as the completion of the space of $C_0^\infty(\Omega)$-functions, Ω is a bounded domain, under the norm

$$||\phi||_{H_k}^2 = I_k[\phi], \quad \phi \in C_0^\infty(\Omega). \quad (19)$$

Thus, this improved Hardy's inequality, takes the form

$$\lim_{\varepsilon\to 0}\left(I_{k,B_\varepsilon^c}[u] - \Lambda_{k,\varepsilon}(u)\right), \quad (20)$$

for any function $u \not\equiv 0$, $u \in H(K)$. With $\Lambda_{K,\varepsilon}$ we denote the quantity:

$$\Lambda_{k,\varepsilon}(u) = -\frac{1}{2}\int_{S_\varepsilon} \phi_k^{-1}\,\phi_k'\,u^2\,ds, \quad (21)$$

where

$$\phi_k(|x|) = |x|^{-(N-2)}\prod_{i=1}^{k} X_i^{-1}, \quad (22)$$

and

$$\frac{\phi_k'}{\phi_k} = -\frac{1}{r}\left[(N-2) + \sum_{i=1}^{k} X_1 \cdots X_i\right], \quad (23)$$

As we discuss in Sect. 2, $\Lambda_{k,\varepsilon}$ may have a bad behavior; oscillating or tending to infinity. In these cases, the Hardy functional $I_{k,\varepsilon}$ becomes negative and accepts the same behavior with $\Lambda_{k,\varepsilon}$, so that the sum of them becomes a positive real number.

6. Finally, we explore the existence of an analogue of the k-Hardy singularity energy for problems posed in exterior domains. Consider the weights

$$\tilde{V}_k(|y|) = \frac{1}{4}\sum_{i=1}^{k} \frac{1}{|y|^2} X_1^2\left(\frac{1}{D\,|y|}\right) X_2^2\left(\frac{1}{D\,|y|}\right)\dots X_i^2\left(\frac{1}{D\,|y|}\right), \quad k = 1, 2, \dots \tag{24}$$

with $D > \delta$, $c_* = (N-2)^2/4$ is the critical coefficient, $B_\delta^c = \mathbb{R}^N \backslash B_\delta(0)$ is the standard exterior domain and $\delta > 0$. Without loss of generality, we set $\delta = 1$. We introduce the Hardy type functional

$$I_{k,B_1^c(0)}[\phi] = \int_{\mathbb{R}^N \backslash B_1(0)} |\nabla\phi|^2\, dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N \backslash B_1(0)} \frac{\phi^2}{|x|^2}\, dx - \sum_{i=1}^{k} \tilde{V}_k(|x|)\,\phi^2\, dx, \tag{25}$$

which is positive for any compactly supported function $\phi \in C^\infty(B_1^c(0))$ that vanishes on the boundary. We denote by $I_{k,\varepsilon}$ and $I_{k,1/\varepsilon}$, the Hardy type functional defined on $B_1(0)\backslash B_\varepsilon$ and $B_{1/\varepsilon}(0)\backslash B_1(0)$, respectively.

We consider the Hilbert space $H_k(K)$ as the completion of the space of $C_0^\infty(\mathbb{R}^N)$-functions under the norm

$$||\phi||_{H_k}^2 = I_{k,B_1^c(0)}[\phi], \quad \phi \in C_0^\infty(\mathbb{R}^N). \tag{26}$$

Then, this improved Hardy's inequality, takes the form

$$\lim_{\varepsilon\to 0} \left(I_{k,1/\varepsilon}[w] + \Lambda_{k,1/\varepsilon}(w)\right). \tag{27}$$

Recall that $\Lambda_{k,\varepsilon}(u)$ is given by (21). For more details, we refer to [54].

We also mention the recent results obtained in [22], where analogous results where obtained for the Hardy inequality, defined on a bounded domain and the singularity being at the boundary.

The proof of the above results was based mainly on a more convenient variable by means of the formulay means of the formula

$$u(x) = |x|^{-(N-2)/2}\, v(x). \tag{28}$$

We will consider the transformation as $u = \mathcal{T}(v)$. Clearly, this is an isometry from the space $X = L^2(\Omega)$ into the space $\widetilde{X} = L^2(d\mu, \Omega)$, with $d\mu = |x|^{2-N} dx$. This transformation (28), was first used in [16] and from then, it is a basic tool in the

study of Hardy's inequalities. The great advantage of this formula is that it simplifies $I_\Omega(u)$, at least for smooth functions, such that

$$I_1(v) := \int_\Omega |x|^{-(N-2)}|\nabla v|^2\,dx\,. \tag{29}$$

It is easily checked that $I_\Omega(u) = I_1(v)$ for functions $u \in C_0^\infty(\Omega)$ and the equivalence fails for functions with a singularity of the type $|x|^{-(N-2)/2}$ at the origin. It is clear, that this change of variables relates the study of Hardy inequality with the critical case of the Caffarelli–Kohn–Nirenberg Inequalities (see [19, 21]).

Moreover, Hardy and Hardy type inequalities might also be connected with the Sobolev inequality in $\mathbb{R}^N$;

Proposition 1 *For some radial function u we set*

$$w(t) = |x|^{\frac{N-2}{2}}u(|x|), \qquad t = \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{1}{N-2}}. \tag{30}$$

Then, $u \in H_r(B_R)$, the radial subspace of H, if and only if $w \in D_r^{1,2}(\mathbb{R}^N)$ and

$$||u||^2_{H_r(B_R)} = (N-2)^{-1}\,||w||^2_{D_r^{1,2}(\mathbb{R}^N)}, \tag{31}$$

where $D_r^{1,2}(\mathbb{R}^N)$ is the radial subspace of $D^{1,2}(\mathbb{R}^N)$, which is defined as the closure of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$||\phi||_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla\phi|^2\,dx.$$

For more details about this space, we refer to the classical book [1].

Two consequences of this relation are the existence of non H_0^1 minimizers and the formulation of improved Hardy–Sobolev inequalities.

Nonexistence of H_0^1 minimizers was implied in [16], where they had calculated exactly the first eigenpair of the problem (5). However, a general proof was given for the first time in [30] for the minimizing problem $\min_{u\in H}||u||^2_H/||u||^2_{L^2_V}$, in the case of certain weights V. The connection of Hardy and Hardy type inequalities with the Sobolev inequality, enable us to provide a much more easier proof, which applies also to the problem $\min_{u\in H}||u||^2_H/||u||^2_{L^p}$, $1 < p < 2^*$, as well as, to more general Hardy-type inequalities. Moreover, we may obtain the exact behavior of the minimizer at the singularity. For example, in the case of problems $\min_{u\in H}||u||^2_H/||u||^2_{L^p}$, $1 < p < 2^*$, their behavior at the origin is exactly $|x|^{-(N-2)/2}$. However, in case 5, the minimizers are more singular; as k grows they are getting slightly more singular. It is interesting that contrary to the simple case, the k-improved Hardy functional for these functions is not well defined. Their behavior at the origin is precisely $|x|^{-(N-2)/2}\prod_{i=1}^k X_i^{-1/2}$. We discuss about these results in Sect. 3.

In the following, we consider *improved Hardy–Sobolev inequalities (IHS)*. In the last years much attention was given for the study of various versions of improved

Hardy and Hardy type inequalities. Their applications extend from the stability of solutions of elliptic and parabolic equations in the asymptotic behavior, the controllability of solutions of heat equations with singular potentials, and the stability of eigenvalues in elliptic problems. For some of these results one is referred to [3, 5–7, 9–14, 20, 22–25, 27–35, 41–43, 46–48, 50–57].

In the case of the critical Sobolev exponent, the following inequality

$$I[u] \geq \int_{\Omega} |u|^{\frac{2N}{N-2}}\, dx, \tag{32}$$

cannot hold for any $u \in C_0^{\infty}(\Omega)$, where Ω is bounded. For example, take a radial function which behaves at the origin like $|x|^{-(N-2)/2}$. It is clear from the previous discussion that the Hardy functional $I[u]$ is well defined and it is finite as a principal value. On the other hand, the right hand side of (32) is infinite.

However, in [30] the following IHS inequality was proved: Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, containing the origin, $D_0 = \sup_{x\in\Omega}|x|$ and $D > D_0$, then the following inequality

$$\int_{\Omega} |\nabla u|^2\, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2}\, dx + C_{HS}(\Omega) \left(\int_{\Omega} |u|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{D}\right)\right)^{-\frac{2(N-1)}{N-2}}\, dx\right)^{\frac{N-2}{N}} \tag{33}$$

holds for any $u \in C_0^{\infty}(\Omega\backslash\{0\})$. We note that (33) is sharp in the sense that $X^{1+\frac{N}{N-2}}$ cannot be replaced by a smaller power of X. From the discussion in [30, 47], it is clear that the nature of (33) depends on the distance of D from D_0, for instance in the case where $D = D_0$. R. Musina [47] proved that the inequality cannot hold if one considers nonradial functions.

On the other hand, as it is shown in [57], inequality (33) holds in the case where $D = D_0$

$$\int_{B_R} |\nabla u(|x|)|^2\, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{u^2(|x|)}{|x|^2}\, dx + C_{HS} \left(\int_{B_R} |u(|x|)|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2(N-1)}{N-2}}\, dx\right)^{\frac{N-2}{N}}, \tag{34}$$

in the radial case, i.e., where B_R is the open ball in $\mathbb{R}^N$, $N \geq 3$, of radius R centered at the origin and $u \in C_0^{\infty}(B_R\backslash\{0\})$ is a radially symmetric function. This was done using transformation (30). It is interesting to mention that (34) cannot have a minimizer (for a proof see Sect. 3). However, the minimization problem

$$||u||_{H(B_R)} dx \geq C_{HS} \left(\int_{B_R} |u(|x|)|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2(N-1)}{N-2}}\, dx\right)^{\frac{N-2}{N}}, \tag{35}$$

accepts a solution, which behaves at the origin like $|x|^{-(N-2)/2}$. More precisely, the minimizers of (35) are

$$u_{m,n}(|x|) = |x|^{-\frac{N-2}{2}} \left(\mu^2 + \nu^2 \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2}{N-2}}\right)^{-\frac{N-2}{2}}, \tag{36}$$

for nonzero μ and ν.

We note that the best constant of (33), was obtained in [6], using basically transformation (30) and the connection of (33) with the Sobolev inequality in a bounded domain. The best constant of (33), in the radial case, was obtained independently from [6], in [57], using transformation (30) and the connection of (33) with the Sobolev inequality in $\mathbb{R}^N$.

The arguments of [57] may be applied to more general cases; the difficulty in these cases is to find the proper weight function that makes such an inequality to hold. Transformation (30) may provide us with an answer. For instance in the case 3; we have to consider the singularity at zero and the behavior at infinity. In the bounded domain case, the weight function was a logarithm; in the case of $\mathbb{R}^N$, the proper function turns to be the *exponential integral* $E(r)$. More precisely, we have

Theorem 1 *Let $N \geq 3$ and $\alpha > 0$ be an arbitrary real number. For any $w \in C_0^\infty(\mathbb{R}^N)$, the following inequality holds*

$$\int_{\mathbb{R}^N} K\, |\nabla w|^2\, dy - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} K\, \frac{w^2}{|y|^2}\, dy \geq c \left(\int_{\mathbb{R}^N} K^{-1} \left(\frac{1}{2} E\left(\frac{|y|^2}{4}\right) + \alpha\right)^{-\frac{2(N-1)}{N-2}} |w|^{\frac{2N}{N-2}}\, dy\right)^{\frac{N-2}{N}}. \tag{37}$$

The best constant is

$$C_{HS} := S(N)\,(N-2)^{-2(N-1)/N}, \qquad \textit{if} \quad \alpha \geq \frac{1}{N-2} \tag{38}$$

and

$$\alpha^{\frac{2(N-1)}{N}}\, S(N), \qquad \textit{if} \quad 0 < \alpha < \frac{1}{N-2}, \tag{39}$$

where $S(N)$ is the best constant in the Sobolev inequality and there exists no minimizer.

In order to clarify the use of (28) in obtaining improved Hardy–Sobolev inequalities, we state Lemma 1 in Sect. 3 and an application for the case 5.

Finally, we make a reference to works studying applications of the Hardy inequality in pde's. First we note that c_*, which is the best constant in the inequality,

is also critical for the basic theory of the evolution equation. Indeed, the usual variational theory applies to the subcritical cases: $u_t = \Delta u + c\, u/|x|^2$ with $c < c_*$, using the standard space $H_0^1(\Omega)$, and a global in time solution is then produced. On the other hand, there are no positive solutions of the equation for $c > c_*$ (instantaneous blow-up), [8, 18, 37]. In the critical case we still get existence but the functional framework changes; this case serves as an example of interesting functional analysis and more complex evolution. Problems with inverse square appear in Schrödinger equations and in combustion theory (See for e.g., [3, 4, 8, 13, 14, 16–18, 20, 22, 29, 32–43, 48, 50, 52–56] and the references therein).

2 Hardy and Hardy Type Inequalities

In this section we make some comments concerning the cases 1–6. The proof of these inequalities might be found in [53, 54] and actually is based on the transformation (30). More precisely, we consider the weighted space $\widetilde{\mathcal{H}} = W_0^{1,2}(d\mu, \Omega)$, which is the completion of the space of $C_0^\infty(\Omega)$-functions under the norm

$$||v||^2_{\widetilde{\mathcal{H}}} = \int_\Omega |x|^{-(N-2)}\, |\nabla v|^2\, dx\,. \tag{40}$$

We may prove that the space of $C_0^\infty(\Omega\backslash\{0\})$-functions is dense in $\widetilde{\mathcal{H}}$. Next, we introduce the space $\mathcal{H}$ as the isometric space of $\widetilde{\mathcal{H}} = W_0^{1,2}(|x|^{-(N-2)}dx, \Omega)$ under the transformation $\mathcal{T}$ given by (28). In other words, $\mathcal{H}$ is defined as the completion of the set

$$\left\{u = |x|^{-\frac{N-2}{2}}\, v, \quad v \in C_0^\infty(\Omega)\right\} = \mathcal{T}(C_0^\infty(\Omega)),$$

under the norm $N(u) = \|u\|_{\mathcal{H}}$ defined by

$$||u||^2_{\mathcal{H}} = \int_\Omega |x|^{-(N-2)}\, |\nabla \left(|x|^{\frac{N-2}{2}}\, u\right)|^2\, dx. \tag{41}$$

Then, we are able to prove that the spaces $\mathcal{H}$ and H are actually the same space and the norm of H is defined by

$$||u||^2_H = \lim_{\varepsilon \to 0} \left(I_{B_\varepsilon^c}[u] - \Lambda_\varepsilon(u)\right). \tag{42}$$

This is exactly inequality (7). As it follows from (42), inequality (7) is sharp concerning the behavior at the singularity.

For this, we explain next the connection of the norm of space H with the Hardy functional (2). We distinguish the following four cases:

- If $u \in H_0^1(\Omega)$, then $u \in \mathcal{H}$ and we have

$$\Lambda(u) := \lim_{\varepsilon \to 0} \Lambda_\varepsilon(u) = 0,$$

Note that the converse is not true; If $\Lambda(u) = 0$, it does not imply that $u \in H_0^1(\Omega)$. For example, take a function u such that v behaves at zero like $(-\log|x|)^{-1/2}$.

- If $v \in \widetilde{\mathcal{H}}$ is such that $\lim_{|x|\to 0} v^2(x) = v^2(0)$ exists as a real positive number; then it follows that $u \in \mathcal{H}$ but $u \notin H_0^1(\Omega)$. In this case

$$\Lambda(u) = \frac{N(N-2)}{2}\,\omega_N\, v^2(0), \tag{43}$$

where ω_N denotes the Lebesgue measure of the unit ball in $\mathbb{R}^N$. $\Lambda(u)$ is then a well-defined positive number. We note that this is the case of the principal eigenfunction and the case of the minimizer of the improved Hardy–Sobolev inequality, see [57], in the radial case. Actually, this is the case for the minimizers of

$$\min_{u \in H} \frac{||u||_H^2}{||u||_{L^p}^p}, \qquad 1 \le p < \frac{2N}{N-2}. \tag{44}$$

- If $v \in \widetilde{\mathcal{H}}$ is such that v at zero is bounded but the $\lim_{x\to 0} v^2(x)$ does not exist, i. e., v oscillates near zero. For example, let

$$v \sim \sin\left((-\log|x|)^a\right), \quad |x| \to 0.$$

Then, v belongs in $\widetilde{\mathcal{H}}$ if $0 < a < 1/2$, thus $u = |x|^{-(N-2)}v \in \mathcal{H}$. In this case, the limit $L(u)$ does not exist, since it oscillates, and we have that the same holds true for the Hardy functional, in the sense that

$$\lim_{\varepsilon\to 0}\left(I_{B_\varepsilon^c}[u] - \Lambda_\varepsilon(u)\right) = ||v||^2_{\widetilde{\mathcal{H}}}. \tag{45}$$

- If $v \in \widetilde{\mathcal{H}}$ is such that $\lim_{x\to 0} v^2(x) = \infty$. For example, let

$$v \sim (-\log|x|)^a, \quad |x| \to 0.$$

Then, v belongs to $\widetilde{\mathcal{H}}$ if $0 < a < 1/2$, thus $u = |x|^{-(N-2)}v \in \mathcal{H}$. It is clear that $\Lambda(u) = \infty$, and we have that the same holds true for the Hardy functional, in the sense that (45) holds.

Note that in all these cases, Λ_ε is a nonnegative quantity, for every $\varepsilon > 0$ and so is $I_{B_\varepsilon^c}[u]$. As a consequence, we obtain a generalized form of the Hardy inequality valid in the limiting case of (45), when the Hardy functional is not defined or it is infinite.

The other cases (2–4) are similar to the above discussion and we refer to [53, 54]. The cases that are more delicate are the fifth and the sixth.

By k-improved Hardy functional, we refer to $I_k(u)$ defined in (18) with limits taken in the sense of principal value if the integrals diverge. Denote by B_ε, the ball centered at the origin with radius ε, and by B_ε^c, its complement in Ω. Assume now that $u \in \mathcal{H}_k$, so that $v = \phi_k^{-1/2} u \in \widetilde{\mathcal{H}}_k$. Then, we have that

$$I_{k,B_\varepsilon^c}[u] = \int_{B_\varepsilon^c} |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_{B_\varepsilon^c} \frac{u^2}{|x|^2}\,dx - \frac{1}{4}\sum_{i=1}^{k} \int_{B_\varepsilon^c} \frac{1}{|x|^2} X_1^2 X_2^2 \ldots X_i^2 u^2\,dx.$$

Applying change of variables and integration by parts, the following remarkable formula is obtained:

$$I_{k,B_\varepsilon^c}[u] = ||v||^2_{\widetilde{\mathcal{H}}_k(B_\varepsilon^c)} - \frac{1}{2}\int_{S_\varepsilon} \phi_k^{-1}\, \phi_k'\, u^2\, dS, \tag{46}$$

where dS is the surface measure. From this definition, we obtain the connection of $\Lambda_{k,\varepsilon}(u)$ with $\Lambda_\varepsilon(u)$, see (8), which for some fixed $v \in C_0^\infty(\Omega)$, is given by

$$\Lambda_{k,\varepsilon}(u_1) = \Lambda_\varepsilon\left(\prod_{i=1}^{k} X_i^{-1/2}\, u_2\right) + \text{ lower order terms}, \tag{47}$$

as $\varepsilon \downarrow 0$, where $v = \mathcal{T}(u_2)$ and $v = \mathcal{T}_k(u_1)$. While for a fixed $u \in \mathcal{H}_k$, holds that

$$\Lambda_{k,\varepsilon}(u) = \Lambda_\varepsilon(u) + \text{ lower order terms}, \tag{48}$$

as $\varepsilon \downarrow 0$. It is also clear that

$$\lim_{\varepsilon\to 0} ||v||^2_{\widetilde{\mathcal{H}}_k(B_\varepsilon^c)} = ||v||^2_{\widetilde{\mathcal{H}}_k}.$$

In order to take the limit $\varepsilon \to 0$, in (46) we distinguish the following cases:

- If $u \in H_0^1(\Omega)$, then $u \in \mathcal{H}_k$ and we have

  $$\Lambda_k(u) := \lim_{\varepsilon\to 0} \Lambda_{k,\varepsilon}(u) = 0,$$

  thus the limit as $\varepsilon \to 0$, in (46), implies the well-known formula

  $$I_{k,\Omega}[u] = ||v||^2_{\widetilde{\mathcal{H}}_k} = N_k^2(u),$$

  which holds for any $u \in H_0^1(\Omega)$. Note that the converse is not true; If $\Lambda_k(u) = 0$, it does not imply that $u \in H_0^1(\Omega)$. For example, assume a function v that behaves at zero like $\prod_{i=1}^k X_i$.
- If u behaves at zero like $c\,|x|^{-(N-2)}$, which means that $v \sim c\, \prod_{i=1}^k X_i^{1/2}$, we have that $u \in \mathcal{H}(K)$. In this case

  $$\Lambda_k(u) = \frac{N(N-2)}{2}\,\omega_N\, c^2,$$

  $\Lambda_k(u)$ is a well-defined positive number and (46) implies that

  $$I_\Omega[u] = ||v||^2_{\widetilde{\mathcal{H}}_k} + \Lambda_k(u).$$

  Note that, in terms of u, this is exactly the same as in the simple Hardy case. However, in the case of k-improved Hardy we must have $v(0) = 0$.

- If $v \in \widetilde{\mathcal{H}}_k$ is such that $\prod_{i=1}^{k} X_i^{-1/2}\, v$ at zero is bounded but the

$$\lim_{|x|\to 0} \prod_{i=1}^{k} X_i^{-1/2}\, v^2(x)$$

  does not exist, i.e., v oscillates near zero. For example, let

$$v \sim \prod_{i=1}^{k} X_i^{1/2} \, \sin\left(X_{k+1}^{-a}\right), \quad |x| \to 0.$$

  Then, v belongs to $\widetilde{\mathcal{H}}_k$ for some $0 < a < 1/2$. In this case, the limit $\Lambda_k(u)$ does not exist, since it oscillates, and from (46) we have that the same happens to the (kIHT), in the sense that

$$\lim_{\varepsilon\to 0} \left(I_{k,B_\varepsilon^c}[u] - \Lambda_{k,\varepsilon}(u) \right) = ||v||^2_{\widetilde{\mathcal{H}}_k}. \tag{49}$$

- If $v \in C_0^\infty(\Omega)$ is such that $v(0) = 1$. Then, v belongs to $\widetilde{\mathcal{H}}_k$ and

$$\lim_{\varepsilon\to 0} \Lambda_{k,\varepsilon}(u) = \infty.$$

  From (46) we have that the same happens to the k-improved Hardy functional, in the sense that (49) holds. We emphasize that, in contrast with Λ_ε, we can find $v \in \widetilde{\mathcal{H}}_k$, such that $v(0) = 0$ and $\Lambda_{k,\varepsilon} \to \infty$. For example let $v \sim \prod_{i=1}^{k} X_i^{1/4}$, at the origin.
  Moreover, this last case applies for certain minimizers, see the next section; These not only fail to be in H_0^1, but also fail to have a finite k-improved Hardy functional, as a principal value, contrary to the case 1. More precisely, they behave at the origin like $|x|^{-(N-2)/2} \prod_{i=1}^{k} X_i^{-1/2}$. In addition, as k grows, the minimizers are getting slightly more singular.

Note that in all cases, $\Lambda_{k,\varepsilon}$ is a positive quantity, for every $\varepsilon > 0$ and so is $I_{k,B_\varepsilon^c}[u]$. As a consequence, we obtain a generalized form of the k-improved Hardy inequality in the limiting case of (49), when the k-improved Hardy functional is not defined or is infinite.

Finally, we give the inclusion between the spaces $\mathcal{H}_k$;

$$H_0^1(\Omega) \subset H \subset \mathcal{H}_1 \subset \ldots \mathcal{H}_k \subset \mathcal{H}_{k+1} \ldots \subset \cap_{1\le q<2} W^{1,q}(\Omega). \tag{50}$$

Note that, every one of each imbedding is dense and strict.

3 Critical Inequalities and the Sobolev Inequality on $\mathbb{R}^N$

In this section, we discuss some applications of transformation (30) concerning nonexistence of H_0^1-minimizers and the formulation of improved Hardy–Sobolev inequalities. As already stated in the introduction, with the use of (30), Hardy and

Hardy type inequalities are related with the Sobolev inequality in $\mathbb{R}^N$, in the radial case.

The best constant in the Sobolev inequality in $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq S \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\, dx \right)^{\frac{N-2}{N}}, \tag{51}$$

as it is well known, is

$$S(N) = \frac{N(N-2)}{4}\, |\mathbb{S}_N|^{2/N} = 2^{2/N}\, \pi^{1+1/N}\, \Gamma\left(\frac{N+1}{2}\right)^{-2/N},$$

where $\mathbb{S}_N$ is the area of the N-dimensional unit sphere and the extremal functions are

$$\psi_{\mu,\nu}(|x|) = \left(\mu^2 + \nu^2 |x|^2\right)^{-(N-2)/2},$$

for $\mu \neq 0$, and $\nu \neq 0$.

Nonexistence of H_0^1-Minimizers Transformation (30) provides us with an extra argument concerning the nonexistence of H_0^1-minimizers. In fact, we are able to obtain the exact behavior of these minimizers at the singularity. We shall prove that these minimizers belong to H, they do not belong to H_0^1 and their behavior at the origin is exactly $|x|^{-(N-2)/2}$.

Assume on the contrary that $u \in H_0^1$ is a minimizer of the problem

$$\min_{u \in H} \frac{||u||_H^2}{||u||_{L^2}^2}.$$

Then, u may be chosen to be a nonnegative and radial function, i.e., satisfying $u(x) = u(r) \geq 0$. Let w be the transformation of u, through (30). Since $u \in H_0^1$, we obtain that

$$w(0) = 0. \tag{52}$$

Moreover, we have that $w \in D^{1,2}(\mathbb{R}^N)$ is a minimizer of

$$\frac{1}{(N-2)^2}\, \frac{\int_{\mathbb{R}^N} |\nabla w|^2\, dx}{\int_{\mathbb{R}^N} V(|x|)\, w^2\, dx}, \tag{53}$$

where $V(|x|) = |x|^{-2(N-1)}\, e^{-2\,|x|^{-(N-2)}}$. Note that if we set $V(0) = 0$, V is a continuous function. Then, w should be a nonnegative solution of the Euler–Lagrange equation corresponding to (53):

$$-\Delta w = c(N)\, V(|x|)\, w, \quad w \in D^{1,2}(\mathbb{R}^N).$$

However, application of the maximum principle contradicts (52), hence (5) does not admit an H_0^1-minimizer. This argument might be applied to more general problems;

Proposition 2 *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Then, minimizers of*

$$\min_{u \in H} \frac{||u||^2_{H(\Omega)}}{\int_\Omega |u|^q \, dx}, \qquad 1 \leq q < \frac{2N}{N-2}, \tag{54}$$

do not exist in $H^1_0(\Omega)$.

The case $q = \frac{2N}{N-2}$, as we know from (36) has the same quantitative behavior (in the radial case) and this maybe also obtained following the same argument. Moreover, the principal eigenvalue and the minimizer of the improved Hardy–Sobolev inequality (in the radial case) behave at the origin like $|x|^{-(N-2)/2}$. Then, the Hardy functional for these functions is a well-defined positive number, although it does not represent their H-norm. These functions do not belong to the "worst" cases, where I_Ω is not well defined or is infinite. As a corollary of the previous argument, we have that the same happens to every minimizer $u_{\Omega,q}$ of (54).

Corollary 1 *Every minimizer $u_{\Omega,q}$ of (54) behaves at the origin like $|x|^{-(N-2)/2}$.*

In the cases of Hardy type inequalities, similar results may be obtained, except the case 5 where the minimizers not only fail to be in H^1_0, but also fail to have a finite k-improved Hardy functional, as a principal value. More precisely, they behave at the origin like $|x|^{-(N-2)/2} \prod_{i=1}^k X_i^{-1/2}$, as we will see in the case of the minimizer of the k-Improved Hardy–Sobolev inequality (radial case). Their norm given by (20) is such that both

$$I_{k,B^c_\varepsilon} \to \infty \quad \text{and} \quad \Lambda_{k,\varepsilon} \to \infty,$$

as $\varepsilon \to 0$. Moreover, as k grows, the minimizers are getting slightly more singular. We consider the minimization problems

$$\min_{u \in \mathcal{H}_k} \frac{||u||^2_{\mathcal{H}_k(\Omega)}}{\left(\int_\Omega |u|^q \, dx\right)^{2/q}}, \qquad 1 \leq q < \frac{2\,N}{N-2}. \tag{55}$$

Proposition 3 *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Then the minimizers of (55) cannot exist in $H^1_0(\Omega)$. Moreover, every minimizer $u_{k,q}$ of (55) behaves at the origin like $|x|^{-(N-2)/2} \prod_{i=1}^k X_i^{-1/2}$.*

For the proof of the above results, we refer to [53, 54].

Improved Hardy–Sobolev Inequalities Transformation (30) applies also to improved Hardy–Sobolev inequalities. More precisely, it might give us the formulation of the inequality, providing us with the proper weight function, such that the inequality holds.

The key result is the following Lemma, which actually relates critical inequalities with the Sobolev inequality on the space $\mathbb{R}^N$. Then, the best constants and the minimizers are related with the ones of the Sobolev inequality. All the arguments considered the radial case since this case is the delicate one. With the exception of

inequality (37), we state also an inequality related to the case 5. For further details, one is referred to [54, 57].

Lemma 1 *Let $a \in (0, \infty]$ be fixed and $K(r)$, $r \in (0, a)$, a positive function. Assume that the function $E(r)$, with*

$$E'(r) = r^{-1}\, K^{-1}(r),$$

is a well-defined negative function. Moreover, we assume that

$$\lim_{r\to 0} E(r) = -\infty \qquad \textit{and} \qquad \lim_{r\to a} E(r) = 0.$$

Then, inequality

$$\int_0^a r\, K\, (v')^2\, dr \le c \left(\int_0^a r^{-1}\, K^{-1}\, (-E(r))^{-\frac{2(N-1)}{N-2}}\, |v|^{\frac{2\,N}{N-2}}\, dr \right)^{\frac{N-2}{N}}, \tag{56}$$

is equivalent to the inequality

$$\int_0^\infty t^{N-1}\, (w')^2\, dt \le c\, (N-2)^{-\frac{2(N-1)}{N}} \left(\int_0^\infty t^{N-1}\, |w|^{\frac{2\,N}{N-2}}\, dt \right)^{\frac{N-2}{N}}, \tag{57}$$

with the use of transformation

$$w(t) = v(r), \qquad t = (-E(r))^{-\frac{1}{N-2}}.$$

It is clear that the best constant in (56) is

$$c = S(N)\, (N-2)^{-2(N-1)/N},$$

and the minimizers are

$$\psi_{\mu,\nu}\left((-E(r))^{-\frac{1}{N-2}} \right),$$

where $S(N)$ and $\psi_{\mu,\nu}$ are the best constant and the minimizers, respectively, of the Sobolev inequality in $\mathbb{R}^N$.

Next, we state the k-improved Hardy Sobolev inequality (kIHS) in the radial case. In the general case, this inequality was proved in [30] and the best constant was obtained in [6]. For the radial case, we consider almost the same inequality, with a small difference, to than in [30, Lemma 7.1], following the procedure followed in [57]. For the sake of the representation, we assume that $\Omega = B_1$, the unit sphere on R^N, and in the definition of the X_i's we take $D = 1$.

Lemma 2 *For any radial function $h \in C_0^\infty(B_1)$, the following inequality holds*

$$\int_0^1 r \prod_{i=1}^k X_i^{-1} |h'|^2\, dr \ge c \left(\int_0^1 r^{-1} \prod_{i=1}^k X_i\ (X_{k+1} - 1)^{\frac{2(N-1)}{N-2}}\, |h|^{\frac{2\,N}{N-2}}\, dr \right)^{\frac{N-2}{N}}. \tag{58}$$

The best constant is given in (38) *and it is achieved by*

$$h_{\mu,\nu}(r) = \psi_{\mu,\nu}\left((X_{k+1}(r)-1)^{\frac{1}{N-2}}\right), \tag{59}$$

where $\psi_{\mu,\nu}$ *are the minimizers of the Sobolev inequality in* $\mathbb{R}^N$.

Proof we set

$$h(r) = \tilde{h}(t), \qquad t = (X_{k+1}(r)-1)^{\frac{1}{N-2}} = (-\log X_k)^{-\frac{1}{N-2}}.$$

Using the fact that

$$(X_{k+1})' = r^{-1}\prod_{i=1}^{k} X_i\, X_{k+1}^2,$$

we have

$$dt = \frac{1}{N-2}\, t^{N-1}\, r^{-1}\prod_{i=1}^{k} X_i\, dr.$$

Then, (58) is equivalent to

$$\frac{1}{N-2}\int_{\mathbb{R}^N} |\nabla \tilde{h}|^2\, dy \geq c\left((N-2)\int_{\mathbb{R}^N} |\tilde{h}|^{\frac{2N}{N-2}}\, dy\right)^{\frac{N-2}{N}},$$

and the result follows.

As a consequence of the above lemma, we obtain the following (kIHS) inequality in the radial case.

Theorem 2 *For any radial function* $u \in \mathcal{H}_k(B_1)$*, the following inequality holds*

$$||u||_{\mathcal{H}_k(B_1)} \geq c\left(\int_{B_1}\prod_{i=1}^{k} X_i^{\frac{2(N-1)}{N-2}}\ (X_{k+1}(r)-1)^{\frac{2(N-1)}{N-2}}\, |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{2N}}. \tag{60}$$

The best constant is given in (38) *and it is achieved by*

$$h_{\mu,\nu}(|x|) = |x|^{-\frac{N-2}{2}}\prod_{i=1}^{k} X_i^{-\frac{1}{2}}\ \psi_{\mu,\nu}\left((X_{k+1}(r)-1)^{\frac{1}{N-2}}(|x|)\right), \tag{61}$$

where $\psi_{\mu,\nu}$ *are the minimizers of the Sobolev inequality in* $\mathbb{R}^N$.

Note that, $h_{\mu,\nu}$ not only fail to be in H_0^1 but also fail to have a well-defined k-improved Hardy functional, as a principal value. As we saw in Sect. 2, this is the case for certain minimizers in $\mathcal{H}_k$. In this sense, inequality (60) is different from the inequality

$$I_k(u) \geq c\left(\int_{B_1}\prod_{i=1}^{k} X_i^{\frac{2(N-1)}{N-2}}\ (X_{k+1}(r)-1)^{\frac{2(N-1)}{N-2}}\, |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{2N}}.$$

The latter cannot hold as an equality for some radial function in $\mathcal{H}_k(B_1)$.

Nonexistence of Minimizers for Inequality (34) Finally, we provide a nonexistence result for inequality (34). We emphasize that inequality (35) has a minimizer and is given by (36). The difference of these two inequalities is actually the norm of H, which is given by (42) and the fact that the minimizers of (35) have a singularity at the origin of the type $|x|^{-(N-2)/2}$. The procedure here is based on this fact.

Theorem 3 *A minimizing sequence for* (34) *is*

$$\phi_n(|x|) = |x|^{-\frac{N-2}{2}} \psi_n\left(\left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{1}{N-2}}\right), \quad x \in B_R\backslash\{0\}, \quad \phi_n|_{\partial B_R} = 0. \tag{62}$$

where

$$\psi_n(|x|) = \left(\mu_n^2 + \nu^2 |x|)^2\right)^{-(N-2)/2}, \quad \mu_n \to \infty, \quad \nu \neq 0,$$

is for each n, the extremal of the Sobolev inequality and there exists no minimizer.

Proof of Theorem 3 We define the functionals $I : H(B_R) \to \mathbb{R}$ and $J : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ as follows

$$I(u) := \int_{B_R} |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{u^2}{|x|^2}\,dx$$
$$- C_{HS}\left(\int_{B_R} |u|^{\frac{2N}{N-2}} \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2(N-1)}{N-2}} dx\right)^{\frac{N-2}{N}}$$

and

$$J(w) := \int_{\mathbb{R}^N} (\nabla w(t))^2\,dt - C\,\left(\int_{\mathbb{R}^N} |w(t)|^{\frac{2N}{N-2}}\,dt\right)^{\frac{N-2}{N}} + \frac{N(N-2)^2}{2}\omega_N w^2(0).$$

By direct calculation we get that $I_{C_1}(u) \geq 0$ if and only if $J_C(r^{-(N-2/2)}u) \geq 0$, and $I_{C_1}(u) = 0$ if and only if $J_C(r^{-(N-2/2)}u) = 0$, with $C_1 = C\,(N-2)^{-2(N-1)/N}$. It is clear now that the best constant for J_C to be positive is $S(N)$; assume that for some $C > S(N)$, $J_C(w) \geq 0$, for any w. Then, $J_C(\psi) \geq 0$, for ψ an extreme of the Sobolev inequality. This implies that

$$-(C-S)\left(\int_{\mathbb{R}^N} |\psi(t)|^{\frac{2N}{N-2}}\,dt\right)^{\frac{N-2}{N}} + \frac{N(N-2)}{2}\omega_N \psi^2(0) \geq 0.$$

or

$$c_1\,\left(\int_0^\infty t^{N-1}|\psi(t)|^{\frac{2N}{N-2}}\,dt\right)^{\frac{N-2}{N}} \leq c_2\,\psi^2(0). \tag{63}$$

Let $\psi(t) = (\mu^2 + \nu^2\,t^2)^{-(N-2)/2}$ for some μ and b. We will prove that (63) cannot hold for every ψ i.e., we will find some μ and b such that (63) is not satisfied. From

(63) we compute the value of

$$c_1 \int_0^\infty t^{N-1} \left(\mu^2 + \nu^2\, t^2\right)^{-N} dt \le c_2 \mu^{-2(N-2)}.$$

We compute the first integral by setting $t = \frac{\mu}{\nu} \tan\omega$ and we obtain that

$$c_1 \frac{1}{\nu^N} L \le c_2 \mu^{-3N+2},$$

where

$$L = \int_0^{\pi/2} (\tan\omega)^{N-1} (\cos\omega)^{2\,N}\, d\omega = \int_0^{\pi/2} (\sin\omega)^{N-1} (\cos\omega)^{N-1}\, d\omega$$
$$= c \int_0^{\pi} (\sin\zeta)^{N-1}\, d\zeta > 0,$$

and it is independent of μ and ν. Thus, we can find a ψ such that (63) is not satisfied and the best constant for J to be positive is $S(N)$. Then, the best constant for (34) is given by (38). In this case, one minimizing sequence for $J_S \to 0$ is ψ_n and there exists no minimizer for J_S and so for $I_{C_{HS}}$. Thus the proof is complete. ■

Remark 1 It is clear that ψ_n are minimizers of J_S in the level sets $w(0) = c$, $c > 0$ fixed number. This implies that these solve the corresponding Euler–Lagrange equation

$$-\Delta w(t) = (N-2)^2\, w^{\frac{N+2}{N-2}}(t), \quad t \in \mathbb{R}_+. \tag{64}$$

In this direction, ϕ_n may be seen as the minimizers of $I_{C_{HS}}$ in the level set with $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u = c$, $c > 0$ fixed number, so these satisfy the Euler–Lagrange equation

$$-\Delta u - \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} = \left(-\log\left(\frac{|x|}{R}\right)\right)^{-\frac{2(N-1)}{N-2}} u^{\frac{N+2}{N-2}} \tag{65}$$
$$u|_{\partial\Omega} = 0.$$

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